ENCE 201 Solutions to Final Exam

Question 1: 20 points

Recall from our class lectures that if $[A]$ is an $(n \times n)$ matrix then, in general, it can be factored into a product of lower and upper triangular matrices, i.e., $[A] = [L]\,[U]$ where $[L]$ and $[U]$ are also $(n \times n)$ matrices. Our examples in class assumed that upper diagonal elements would be unity (i.e., $U_{ii} = 1$), but this is only one way of enabling the factorization. A second possibility is to set the lower diagonal elements to unity (i.e., $L_{ii} = 1$). The key point here is that any set of constraints that reduces the total number of unknowns from $\left(n^2 + n\right)$ to n^2 might work.

Part [1a] (10 pts). Calculate the LU decomposition for the matrix

$$[A] = \begin{bmatrix} 3 & -1 & 5 \\ 1 & 2 & -3 \\ 4 & 1 & a^2 - 14 \end{bmatrix}. \tag{1}$$

by assuming that $L_{ii} = 1$. Show all of your working ...

Solution: With $L_{ii} = 1$, the LU decomposition can be written:

$$\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 5 \\ 1 & 2 & -3 \\ 4 & 1 & a^2 - 14 \end{bmatrix}. \tag{2}$$

We have 9 matrix terms and 9 unknowns.

Matching terms in the first row of L and U gives:

$$1U_{11} + 0U_{21} + 0U_{31} = 3, \longrightarrow U_{11} = 3. \tag{3}$$

$$1U_{12} + 0U_{22} + 0U_{32} = -1, \longrightarrow U_{12} = -1. \tag{4}$$

$$1U_{13} + 0U_{23} + 0U_{33} = 5, \longrightarrow U_{13} = 5. \tag{5}$$

Matching terms in the second row of L and U gives:

$$L_{21}U_{11} + L_{22}U_{21} + L_{23}U_{31} = 1, \longrightarrow L_{21} = 1/3. \tag{6}$$

$$L_{21}U_{12} + L_{22}U_{22} + L_{23}U_{32} = 2, \longrightarrow U_{22} = 7/3. \tag{7}$$

$$L_{21}U_{13} + L_{22}U_{23} + L_{23}U_{33} = -3, \longrightarrow U_{23} = -14/3. \tag{8}$$

Matching terms in the third row of L and U gives:

$$L_{31}U_{11} + L_{32}U_{21} + L_{33}U_{31} = 4, \longrightarrow L_{31} = 4/3. \tag{9}$$

$$L_{31}U_{12} + L_{32}U_{22} + L_{33}U_{32} = 1, \longrightarrow L_{32} = 1. \tag{10}$$

$$L_{31}U_{13} + L_{32}U_{23} + L_{33}U_{33} = a^2 - 14, \longrightarrow U_{33} = a^2 - 16. \tag{11}$$

Collecting terms gives:

$$\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 4/3 & 1 & 1 \end{bmatrix} \tag{12}$$

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & -1 & 5 \\ 0 & 7/3 & -14/3 \\ 0 & 0 & a^2 - 16 \end{bmatrix} \tag{13}$$

Part [1b] (3 pts). (3 pts). Hence, write down the det[A]?. Note: Do not calculate the determinate by the method of cofactors - there is a much faster way that is a one line calculation!

Solution: det(A) = det(L).det(U) = 7(a^2-16) = 0.

Part [1c] (7 pts). Use forward and backward substitution to show that the general solution to $[L]\,[U]\,[x] = [2, 4, a+2]^T$. can be written:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (8a+25)/(7a+28) \\ (10a+54)/(7a+28) \\ 1/(a+4) \end{bmatrix}. \tag{14}$$

This is a hand calculation, so show all of your working.

Solution: Two steps:

Forward Substitution: Solve Lz = B, i.e.,

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 4/3 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ a+2 \end{bmatrix} \longrightarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 10/3 \\ a-4 \end{bmatrix}. \tag{15}$$

Backward Substitution: Solve Ux = Z, i.e.,

$$\begin{bmatrix} 3 & -1 & 5 \\ 0 & 7/3 & -14/3 \\ 0 & 0 & a^2-16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (8a+25)/(7a+28) \\ (10a+54)/(7a+28) \\ 1/(a+4) \end{bmatrix}. \tag{16}$$

Question 2: 20 points

This question covers function interpolation with the methods of divided differences and lagrange interpolation, and curve fitting with the method of least squares.

The whole question is motivated by the small dataset:

```
   x |  0.0 |   2.0 |   3.0 |   5.0
----------------------------------
f(x) | 36.0 |  32.0 |  27.0 |  11.0
```

Part [2a] (10 pts). Use the method of **divided differences** to find a polynomial of lowest order that will fit the dataset. Be sure to show all of your working.

Solution: We seek a third order polynomial p(x):

$$p(x) = f[x_0]+f[x_o,x_1](x-x_o)+f[x_o,x_1,x_2](x-x_o)(x-x_1)+f[x_o,x_1,x_2,x_3](x-x_o)(x-x_1)(x-x_2). \tag{17}$$

where

x_i	$f[x_i] = f(x_i)$	$f[,]$	$f[,,]$	$f[,,,]$
0.0	36	$f[x_0,x_1]$	$f[x_0,x_1,x_2]$	$f[x_0,x_1,x_2,x_3]$
2.0	32	$f[x_1,x_2]$	$f[x_1,x_2,x_3]$	
3.0	27	$f[x_2,x_3]$		
5.0	11			

Elements in the divided difference table are as follows:

$$f[x_0,x_1] = \left[\frac{f(x_1)-f(x_0)}{x_1-x_0}\right] = \left[\frac{32-36}{2-0}\right] = -2. \tag{18}$$

$$f[x_1,x_2] = \left[\frac{f(x_2)-f(x_1)}{x_2-x_1}\right] = \left[\frac{27-32}{3-2}\right] = -5. \tag{19}$$

$$f[x_2,x_3] = \left[\frac{f(x_2)-f(x_1)}{x_2-x_1}\right] = \left[\frac{11-27}{5-3}\right] = -8. \tag{20}$$

The second column of computations:

$$f[x_0, x_1, x_2] = \left[\frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}\right] = \left[\frac{-5+2}{3-0}\right] = -1.0. \quad (21)$$

$$f[x_1, x_2, x_3] = \left[\frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}\right] = \left[\frac{-8+5}{5-2}\right] = -1.0. \quad (22)$$

The third column computations:

$$f[x_0, x_1, x_2, x_3] = \left[\frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_3]}{x_3 - x_0}\right] = \left[\frac{0-0}{5-0}\right] = 0.0 \quad (23)$$

Thus, the interpolated polynomial is:

$$\begin{aligned} f(x) &= f[x_0] + f[x_o, x_1](x - x_o) + f[x_0, x_1, x_2](x - x_o)(x - x_1) \\ &= 36 + -2(x-0) - 1(x-0)(x-2) + 0(x-0)(x-2)(x-3) \\ &= 36 - x^2 \end{aligned}$$

Part [2b] (10 pts). Check your answer in part 2a by computing the functional form via the method of **Lagrange Interpolation**. Be sure to show all of your working.

Solution: For the given dataset,

$$f(x) = f(x_o)p_o(x) + f(x_1)p_1(x) + f(x_2)p_2(x) + f(x_3)p_3(x) \quad (24)$$

where

$$p_o(x) = \frac{(x-2)(x-3)(x-5)}{(0-2)(0-3)(0-5)} = \left[\frac{-x^3 + 10x^2 - 31x + 30}{30}\right]. \quad (25)$$

$$p_1(x) = \frac{(x-0)(x-3)(x-5)}{(2-0)(2-3)(2-5)} = \left[\frac{x^3 - 8x^2 + 15x}{6}\right]. \quad (26)$$

$$p_2(x) = \frac{x(x-2)(x-5)}{3(3-2)(3-5)} = \left[\frac{-x^3 + 7x^2 - 10x}{6}\right]. \quad (27)$$

$$p_3(x) = \frac{x(x-2)(x-3)}{5(5-2)(5-3)} = \left[\frac{x^3 - 5x^2 + 6x}{30}\right]. \quad (28)$$

Also note: f(x_o) = 36, f(x_1) = 32, f(x_2) = 27 and f(x_3) = 11.

Substitute equations 25 through 28 into equation 24. Notice that the constant term in equation 25 is one, and the cubic and linear terms cancel out completely. This leaves:

$$f(x) = 36 - x^2. \quad (29)$$

Question 3: 10 points

Figure 1 shows an enclosed region A-B-C-D-E-F-G-H whose boundary is defined by two curves:

$$f(x) = \left[x^2 - 16\right]. \tag{30}$$

and

$$g(x) = \left[\frac{x^3}{8} - 2x + 5\right]. \tag{31}$$

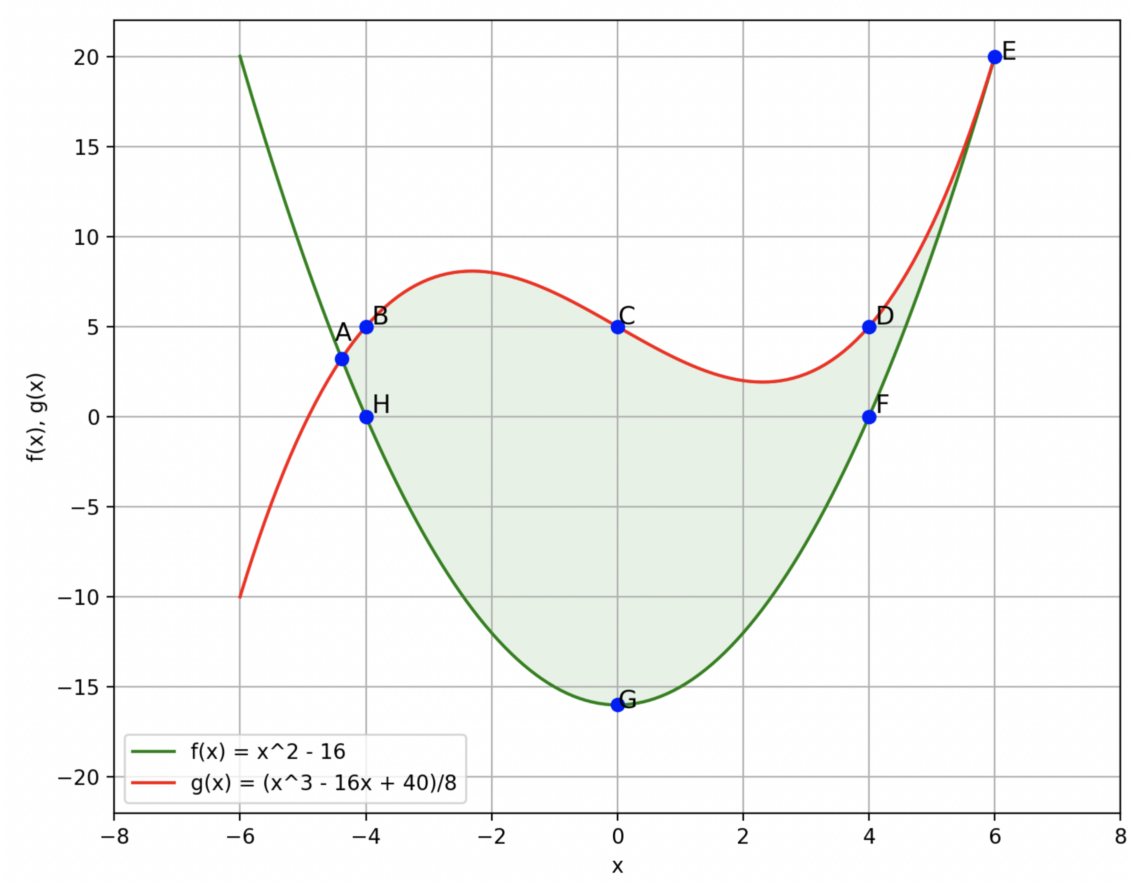

Figure 1. Filled region A-B-C-D-E-F-G-H.

From the graphic we see that curves f(x) and g(x) intersect at points A and E (i.e., in the interval $[-4, -5]$ and again $[5, 7]$).

Part [3a] (3 pts). Show that coordinate points A and E are defined by solutions to the cubic equation:

$$x^3 - 8x^2 - 16x + 168 = 0. \tag{32}$$

This is a hand calculation, so show all of your working.

Solution. At the point of intersection, $f(x)$ = $g(x)$, i.e.,

$$\left[x^2 - 16\right] = \left[\frac{x^3}{8} - 2x + 5\right] \tag{33}$$

Rearranging equation 33 gives the required result. Equation 32 can be factorized:

$$(x - 6)\left(x^2 - 2x - 28\right) = 0. \tag{34}$$

Thus, Point A has coordinates: (x,y) = (1 - $\sqrt{29}$, 14-2$\sqrt{29}$). Point E has coordinates: (x,y) = (6,20).

Part [3b] (2 pts). (2 pts). Using calculus, or otherwise, show that the area of the shaded region is $\left[781 + 145\sqrt{29}\right]/12 \approx 130.154$.

Solution. Notice that within the interval [1 - $\sqrt{29}$, 6], g(x) > f(x). If we let s(x) = g(x) - f(x), then the area of the shaded region is:

$$\begin{aligned}
\text{Shaded Area} &= \int_{1-\sqrt{29}}^{6} s(x)dx \\
&= \left[\frac{x^4}{32} - \frac{x^3}{3} - x^2 + 21x\right]_{1-\sqrt{29}}^{6} \\
&= \frac{1}{12}\left[781 + 145\sqrt{29}\right] \approx 130.154.
\end{aligned}$$

Part [3c] (3 pts). (3 pts). Demonstrate how you can use **one step** of Simpson's Rule, to obtain a high-accuracy estimate the area of region A-B-C-D-E-F-G-H.

Solution. Using one step of Simpson's Rule:

$$I = \int_a^b s(x)dx = \frac{h}{3}\left[s(a) + 4s(m) + s(b)\right] \tag{35}$$

where point a = (1 - $\sqrt{29}$), point b = 6, h = (b-a)/2, and m is the midpoint m = (a+b)/2 = (7 - $\sqrt{29}$)/2. Also, notice that by design, that s(a) = s(b) = 0.0. Hence, equation 35 simplfies to:

$$I = \int_a^b s(x)dx = \frac{4h}{3}s(m) = \frac{1}{12}\left[781 + 145\sqrt{29}\right] \approx 130.154. \tag{36}$$

Part [3d] (2 pts). What is the expected error with Simpson's Rule? Briefly compare and discuss the analytical and numerical results.

Solution. The error estimate for Simpson's Rule is as follows:

$$I = \int_a^b s(x)dx = S_1 - \frac{|s^{''''}(\xi)|}{180}h^4(b-a). \tag{37}$$

Given that s(x) is a cubic function, the fourth derivative will be zero. Hence, we expect one step of Simpson's Rule to provide an exact answer.

Question 4. (10 pts.)

Consider the integral:

$$I = \int_0^{2\pi} |\sin(x) - \sqrt{3}\cos(x)|\, dx. \tag{38}$$

and the plot shown in Figure 2:

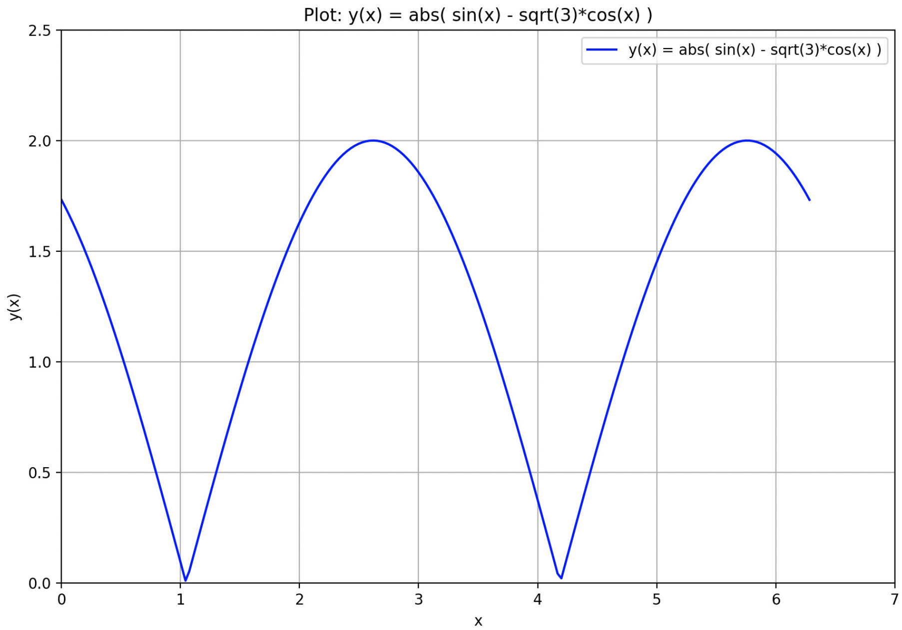

Figure 2. Graph of y(x) = $|\sin(x) - \sqrt{3}\cos(x)|$ from 0 to 2π.

Notice that y(x) = $|\sin(x) - \sqrt{3}\cos(x)|$ is periodic. The values of y(x) can be summarized as follows:

x	0.0	pi/6	pi/3	pi/2	4pi/6	5pi/6
y(x)	1.7321	1.0000	0.0000	1.0000	1.7321	2.0000

x	pi	7pi/6	8pi/6	9pi/6	10pi/6	11pi/6
y(x)	1.7321	1.0000	0.0000	1.0000	1.7321	2.0000

x	2pi
y(x)	1.7321

Part [4a] (2 pts). Briefly explain how you can use the periodic nature of y(x) to simplify the evaluation of equation 38.

Solution: Notice that y(x) = | $\sin(x) - \sqrt{3}\cos(x)$ | = 0 at x = $\pi/3$ and $4\pi/3$, and that the y(x) is periodic. Dividing the interval [0, 2π] into segments:

$$I = \int_0^{2\pi} y(x)dx = \int_0^{pi/3} y(x)dx + \int_{\pi/3}^{4pi/3} y(x)dx + \int_{4\pi/3}^{6pi/3} y(x)dx = 2\int_{\pi/3}^{4pi/3} y(x)dx. \tag{39}$$

Part [4b] (3 pts). Compute the analytical solution to equation 38. This is a hand calculation, so show all of your working.

Solution: Integrating equation 39:

$$I = 2\Big[-\cos(x) - \sqrt{3}\sin(x)\Big]_{\pi/3}^{4\pi/3} = 8.0. \tag{40}$$

Part [4c] (5 pts). Now suppose that equation 38 is evaluated using only two steps of the Trapezoid Rule. With your answer to part [4a] in mind, what is the maximum error that will occur with this numerical approximation? Is the actual error within this bound?

Solution: Using two steps of trapezoid on the interval [$\pi/3, 4\pi/3$] gives:

$$T_2 = 2 \cdot \frac{\pi}{4}\left[f(\pi/3) + 2f(5\pi/6) + f(4pi/3)\right] = \pi f(5\pi/6) = 2\pi. \tag{41}$$

Differenting y(x) twice gives: $y^{''}(x) = -sin(x) + \sqrt{3}cos(x)$, which has a maximum value when x = π (i.e., sin(x) = 0 and cos(x) = -1). The error estimate is as follows:

$$I = \int_a^b f(x)dx = T_2 - \frac{|y^{''}(\xi)|}{12}h^2(b-a). \tag{42}$$

Here, h = $\pi/2$, b-a = 2π and $|f^2(\xi)|$ = $\sqrt{3}$. Thus, we predict that the actual error will be less than:

$$\text{Error estimate} = \frac{\sqrt{3}\pi^3}{24} = 2.237. \tag{43}$$

The actual numerical error is 8 - 2π = 1.714. It works!!!

Question 5: 10 points

Theoretical considerations indicate that:

$$\int_0^4 x^3 \left[16 - x^2\right] dx = \frac{1024}{3} \approx 341.33. \tag{44}$$

Part [5a] (5 pts). Use the method of Romberg integration to obtain an $O(h^6)$ accurate estimate of equation 44. Be sure to show all steps in your working.

Solution: To get an $O(h^6)$ accurate estimate we need three levels of refinement with Trapezoid (i.e., h = 4, h = 2 and h = 1).

```
--- Inputs:
---   a =    0.0000 ...
---   b =    4.0000 ...
---   no intervals = 3 ...
---
--- Execution:
---   Compute trapezoid rule for first column ...
---   Extrapolation for column  2 ...
---   Extrapolation for column  3 ...

Matrix: Romberg Integration Table (instantiated)
  0.00000000e+00   0.00000000e+00   0.00000000e+00
  1.92000000e+02   2.56000000e+02   0.00000000e+00
  3.00000000e+02   3.36000000e+02   3.41333333e+02

---  Integral I = 341.3333333333 ...  <--- Numerical integration is exact ...
```

Part [5b] (5 pts). Evaluate equation 44 using 3-pt Gauss Quadrature. Be sure to show all steps in your working.

Solution: Integrating f1(x) with 3-point quadrature ...

```
---   w0 =   5.555556e-01, u0 =  -7.745967e-01, x0 =   4.508067e-01 ...
---   w1 =   8.888889e-01, u1 =   0.000000e+00, x1 =   2.000000e+00 ...
---   w2 =   5.555556e-01, u2 =   7.745967e-01, x2 =   3.549193e+00 ...
---
---   f1(x0) =   1.447236e+00 ...
---   f1(x1) =   9.600000e+01 ...
---   f1(x2) =   1.521528e+02 ...
---
---   Integral I = 3.41333333e+02 ... <--- Numerical integration is exact ...
```