Department of Civil and Environmental Engineering, Spring Semester, 2024

ENCE 201 Final Exam, Open Notes and Open Book

Name : Austin

E-mail (print neatly!): austin@umd.edu.

Exam Format and Grading. This take home final exam is open notes and open book. You need to comply with the university regulations for academic integrity.

There are five questions. Partial credit will be given for partially correct answers, so please show all your working.

IMPORTANT: Please see the **class web page for instructions on how to submit your exam paper**. Also, before submitting your exam, check that **every page has been scanned correctly**.

Question	Points	Score
1	20	
2	20	
3	10	
4	10	
5	10	
Total	70	

Question 1: 20 points.

Consider the family of matrix equations Ax = b, where:

$$\begin{bmatrix} 3 & -1 & 5 \\ 1 & 2 & -3 \\ 4 & 1 & a^2 - 14 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ a+2 \end{bmatrix}. \tag{1}$$

[1a] (4 pts). Derive an expression for the det(A) as a function of 'a'. Determine the values of 'a' for which matrix A will be singular.

$$\det(A) = 3 \det\begin{bmatrix} 2 & -3 \\ 1 & a^2-14 \end{bmatrix} + 1 \det\begin{bmatrix} 1 & -3 \\ 4 & a^2-14 \end{bmatrix} + 5 \det\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}.$$

$$= 3[2a^2 - 28 + 3] + [a^2 - 2] - 35$$

$$= 7a^2 - 112$$

$\det(A) = 0$ when $a^2 = \frac{112}{7} = \pm 4.$

[1b] (6 pts). Based on your solutions to part 1a, what can you say about: (1) the rank of the system of equations, and (2) the number of solutions to the matrix equations?

(1) $\det(A) \neq 0$ when $a \neq \pm 4$. Rank$(A) = 3$.

(2) $a = 4$

$$[A \,|\, b] = \left[\begin{array}{ccc|c} 3 & -1 & 5 & 2 \\ 1 & 2 & -3 & 4 \\ 4 & 1 & 2 & 6 \end{array}\right] \xrightarrow{r_3 \leftarrow r_3 - r_1 - r_2} \left[\begin{array}{ccc|c} 3 & -1 & 5 & 2 \\ 1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank$(A \,|\, b)$ = Rank$(A) = 2$ → Infinite Sol'ns.

(3) Let $a = -4$

$$\left[\begin{array}{ccc|c} 3 & -1 & 5 & 2 \\ 1 & 2 & -3 & 4 \\ 4 & 1 & 2 & -2 \end{array}\right] \xrightarrow{r_3 \leftarrow r_3 - r_1 - r_2} \left[\begin{array}{ccc|c} 3 & -1 & 5 & 2 \\ 1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 8 \end{array}\right]$$

Rank$(A \,|\, b) \neq$ Rank(A) → No Solutions.

[1c] (10 pts). Use the method of Gauss Elimination (i.e., row operations followed by back substitution) to show that solutions to matrix equation 1 can be written:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (8a+25)/(7a+28) \\ (10a+54)/(7a+28) \\ 1/(a+4) \end{bmatrix}. \tag{2}$$

This is a hand calculation, so show all of your working.

Row Operations.

$$\left[\begin{array}{ccc|c} 3 & -1 & 5 & 2 \\ 1 & 2 & -3 & 4 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right]$$

$$\xrightarrow[r_3 \leftarrow r_3 - 4r_1]{r_2 \leftarrow r_2 - 3r_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right]$$

$$\xrightarrow[r_3 \leftarrow r_3/-7]{r_2 \leftarrow r_2/-7} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 1 & \frac{a^2-2}{-7} & \frac{a-14}{-7} \end{array}\right]$$

$$\xrightarrow[r_3 \leftarrow 7r_3]{r_3 \leftarrow r_3 - r_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right]$$

Back Substitution.

$$x_3 = \frac{a-4}{a^2-16} = \frac{1}{(a+4)} \checkmark$$

Question [1c] continued:

$$x_2 - 2x_3 = \frac{10}{7} \Rightarrow x_2 = \frac{10}{7} + \frac{2}{(a+4)}$$

$$= \frac{10a + 54}{7(a+4)} \checkmark$$

$$x_1 = 4 - 2x_2 + 3x_3$$

$$= 4 - \frac{2(10a+54)}{7(a+4)} + \frac{3}{(a+4)}$$

$$= \left[\frac{8a + 25}{7(a+4)} \right] \checkmark$$

Question 2: 20 points.

This question covers numerical integration of quadratic and cubic functions with the Trapezoid Rule and Simpson's Rule.

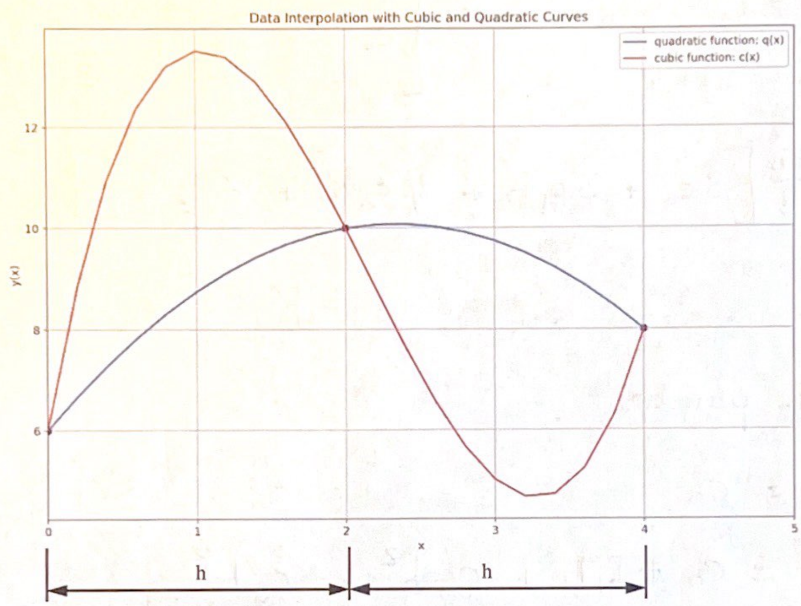

Figure 1. Interpolation of three data points with quadratic and cubic functions.

To motivate the problem setup, Figure 1 shows a scenario where quadratic (i.e., q(x) = $q_o + q_1 x + q_2 x^2$) and cubic (i.e., c(x) = $c_o + c_1 x + c_2 x^2 + c_3 x^3$) curves interpolate three equally spaced data points.

[2a] (4 pts). Derive an expression for the integral:

$$I_q = \int_0^{2h} q(x)dx \tag{3}$$

in terms of q_o, q_1, q_2 and h.

$$I_q = \int_0^{2h} (q_0 + q_1 x + q_2 x^2)\,dx = \left[q_0 x + \frac{q_1 x^2}{2} + \frac{q_2 x^3}{3} \right]_0^{2h}$$

$$\left.\begin{aligned} &= 2q_0 h + 2q_1 h^2 + \frac{8}{3} q_2 h^3 \\ &= \frac{2h}{3}\left[3q_0 + 3q_1 h + 4q_2 h^2\right] \end{aligned}\right\} \longrightarrow A$$

[2b] (4 pts). Derive an expression for the integral:

$$I_c = \int_0^{2h} c(x)dx \tag{4}$$

in terms of c_o, c_1, c_2, c_3 and h.

$$= \left[c_o x + \frac{c_1 x^2}{2} + \frac{c_2 x^3}{3} + \frac{c_3 x^4}{4} \right]_0^{2h}$$

$$I_c = \frac{2h}{3}\left[3c_o + 3c_1 h + 4c_2 h^2 + 6c_3 h^3 \right] \longrightarrow \text{(B)}$$

[2c] (6 pts). Show that one step of Simpson's Rule integrates the cubic equation exactly !!!

Three data points:

$$c(0) = c_o$$

$$c(h) = c_o + c_1 h + c_2 h^2 + c_3 h^3.$$

$$c(2h) = c_o + c_1(2h) + c_2(2h)^2 + c_3(2h)^3$$

Simpson's Rule:

$$I = \frac{h}{3}\left[c(0) + 4c(h) + c(2h) \right]$$

$$= \frac{2h}{3}\left[3c_o + 3c_1 h + 4c_2 h^2 + 6c_3 h^3 \right] \longrightarrow \text{(C)}$$

(B) and (C) are the same. Simpson's Rule is exact!

[2d] (6 pts). Now suppose that equation 3 is approximated by one step of the Trapezoid Rule. Develop a symbolic expression for the actual error. Compute the error estimate (which we have covered in class). Is the latter less than in actual error?

One step of Trapezoid:

$$T_o = \frac{2h}{2}\left[q(0) + q(2h)\right]$$

$$= \frac{2h}{3}\left[3q_0 + 3q_1h + 6q_2h^2\right]. \quad \text{— D}$$

From [2a]:

$$I_q = \frac{2h}{3}\left[3q_0 + 3q_1h + 4q_2h^2\right]. \quad \text{— E}$$

$$\text{Actual Error} = T_o - I_q = \frac{4q_2h^3}{3}. \quad \text{— F}$$

$$\text{Error estimate} \leq \frac{f''(\xi)\,h^2\,(b-a)}{12} = \frac{2q_2(2h)^2(2h)}{12}$$

Here, $q(x) = q_0 + q_1x + q_2x^2$

$q''(x) = 2q_2.$

$b - a = 2h$

step length $= 2h.$

$$= \frac{16}{12}q_2h^3$$

$$= \frac{4}{3}q_2h^3 \quad \text{— G}$$

The actual error and the error estimate are identical.

Question 3: 10 points

The family of equations:

$$f(x) = [x - 2]^2. \tag{5}$$

$$g(x) = [x + 2]^2. \tag{6}$$

$$h(x) = \left[\frac{x^2 - 8}{2}\right]. \tag{7}$$

defines the region A-B-C-D shown in Figure 2.

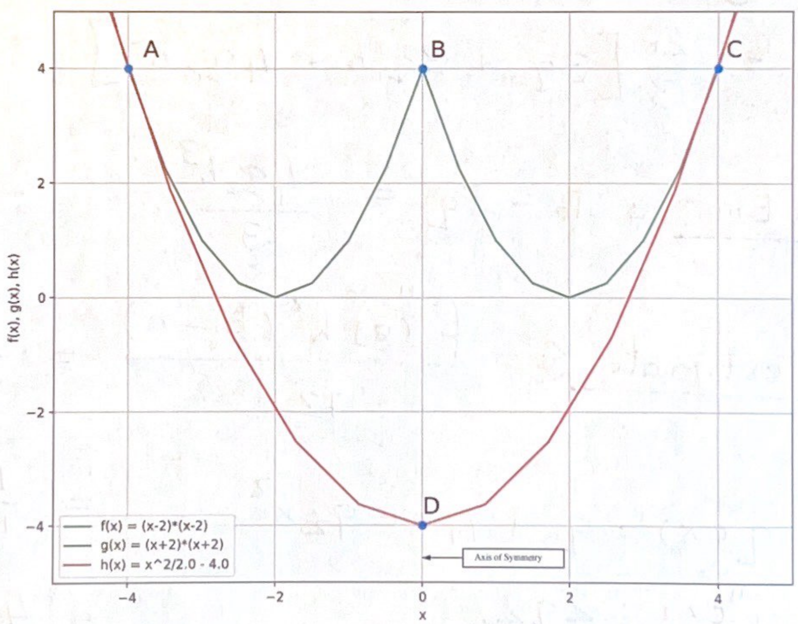

Figure 2. Region A-B-C-D.

Theoretical considerations indicate that equation h(x) is tangent to f(x) and g(x) at their intersection (i.e., at points A and C). Also notice an axis of symmetry in region A-B-C-D along the line B-D – this observation can be used to simplify the analysis.

[3a] (3 pts). Using calculus, or otherwise, show that the area of region A-B-C-D is 64/3.

$$\text{Area A-B-C-D} = 2\int_0^4 (f(x) - h(x))\,dx.$$

$$= 2\int_0^4 \left(\frac{x^2}{2} - 4x + 8\right) dx$$

$$= \frac{64}{3}.$$

[3b] (4 pts). Demonstrate how you can use **one step** of Simpson's Rule, to obtain a high-accuracy estimate the area of region A-B-C-D.

Apply Simpsons to half of shape (exploit axis of symmetry).

x	0	2	4
$f(x)$	4	0	4
$h(x)$	-4	-2	4
$f(x)-h(x) = f^*(x)$	8	2	0

$$\text{Area B-D-C} = \frac{h}{3}\left[f^*(0) + 4f^*(2) + f^*(4)\right]$$

$$= \frac{2}{3}\left[8 + 8 + 0\right] = \frac{32}{3}$$

$$\text{Area A-B-C-D} = 2 \times \frac{32}{3} = \frac{64}{3}.$$

[3c] (3 pts). What is the expected error with Simpson's Rule? Briefly compare and discuss the analytical and numerical results.

<u>Note</u>: $f(x) - h(x)$ is a quadratic function.

$\Rightarrow$ Integration with Simpson's Rule is exact!

If you try to apply Simpson's on the whole region (i.e., $h = 4$)

$$\text{Area A-B-C-D} = \frac{4}{3}(4 \times 8) = \frac{128}{3} \longleftarrow$$

Doesn't work. Why? Function is no longer a quadratic....

Question 4. (10 pts.)

Consider the integral:

$$I = \int_0^{2\pi} |\ \sin(x) - \cos(x)\ |\ dx. \tag{8}$$

and the plot shown in Figure 3:

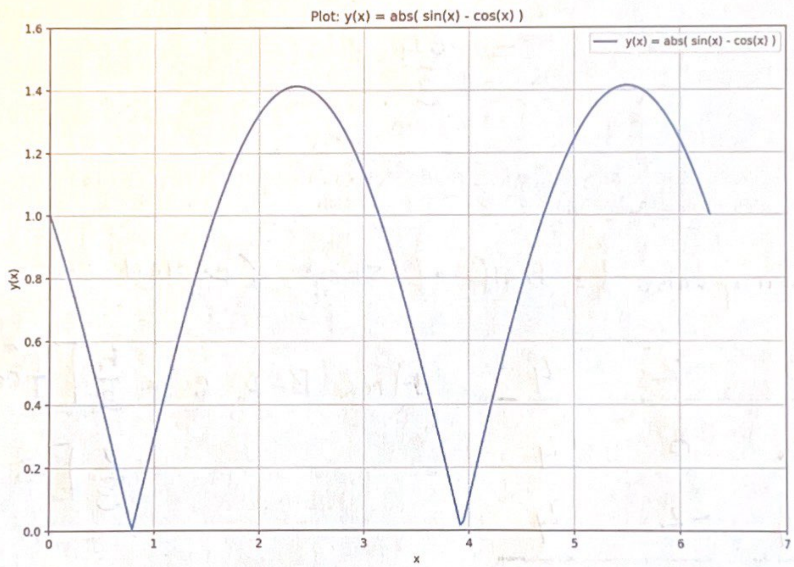

Figure 3. Graph of y(x) $|\ \sin(x) - \cos(x)\ |$ from 0 to 2π.

Notice that y(x) = $|\ \sin(x) - \cos(x)\ |$ is periodic. The values of y(x) can be summarized as follows:

x	0.0	pi/8	pi/4	3pi/8	pi/2	5pi/8
y(x)	1.0000	0.5412	0.0000	0.5412	1.0000	1.3066

x	3pi/4	7pi/8	pi	9pi/8	5pi/4	11pi/8
y(x)	1.4142	1.3066	1.0000	0.5412	0.0000	0.5412

x	6pi/4	13pi/8	7pi/4	15pi/8	2pi
y(x)	1.0000	1.3066	1.4142	1.3066	1.0000

[4a] (2 pts). Briefly explain how you can use the periodic nature of y(x) to simplify the evaluation of equation 8.

Note: $|\sin(x) - \cos(x)| = 0$ at $x = \frac{\pi}{4}$ and $x = \frac{5}{4}\pi$.

Periodic nature implies:

$$\int_0^{2\pi} |\sin(x) - \cos(x)|\, dx = 2\int_{\pi/4}^{5/4\pi} |\sin(x) - \cos(x)|\, dx.$$

[4b] (3 pts). Compute the analytical solution to equation 8 This is a hand calculation, so show all of your working.

First, let's ignore periodic nature:

$$I = \int_0^{\pi/4} \cos(x) - \sin(x)\, dx + \int_{\pi/4}^{5\pi/4} \sin(x) - \cos(x)\, dx + \int_{5\pi/4}^{2\pi} \cos(x) - \sin(x)\, dx$$

$$= \Big[\sin(x) + \cos(x)\Big]_0^{\pi/4} + \Big[-\cos(x) - \sin(x)\Big]_{\pi/4}^{5\pi/4} + \Big[\sin(x) + \cos(x)\Big]_{5\pi/4}^{2\pi}$$

$$= \left(\frac{2}{\sqrt{2}} - 1\right) + \left(\frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}\right) + \left(1 + \frac{2}{\sqrt{2}}\right) = \frac{8}{\sqrt{2}} = 4\sqrt{2}.$$

Accounting for periodic nature:

$$I = 2\int_{\pi/4}^{5\pi/4} (\sin(x) - \cos(x))\, dx = 2\Big[-\cos(x) - \sin(x)\Big]_{\pi/4}^{5\pi/4}$$

$$= 2\left[\frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}\right] = 4\sqrt{2}.$$

[4c] (5 pts). Now suppose that equation 8 is evaluated using only four steps of the Trapezoid Rule. With your answer to part [4a] in mind, what is the maximum error that will occur with this numerical approximation? Is the actual error within this bound?

We will take advantage of the periodic nature of the indegral and evaluate (then double):

$$I = 2\int_{\pi/4}^{5\pi/4} (\sin(x) - \cos(x))\, dx.$$

Using 4 intervals of Trapezoid:

$$T_4 = \frac{\pi/4}{2}\left[f(\pi/4) + 2\left(f(\tfrac{\pi}{2}) + f(\tfrac{3\pi}{4}) + f(\pi)\right) + f(\tfrac{5\pi}{4})\right]$$

$$\doteq \frac{\pi}{8}\left[0 + 2(1 + 1.4142 + 1) + 0\right]$$

$$\doteq \frac{3.4142\,\pi}{4}$$

$$\Rightarrow I_{estimate} = 2\left[\frac{3.4142\,\pi}{4}\right] = 5.3630.$$

$$I_{analytical} = 4\sqrt{2} = 5.6568.$$

Max Error = 0.2938.

$$\text{Error estimate} \leq \frac{|f(\xi)|_{Max}\, h^2 (b-a)}{12}$$

Can show $f''(x) = -f(x)$, with max value 1.42 (taken from Fig 3). But strictly speaking $|f(x)|$ is not continuosly differentiable, so error formula doesn't work.

Question 5: 10 points

Consider the family of integration problems:

$$I_n = \int_0^{\pi/2} \left[\frac{\sin^n(x)}{\sin^n(x) + \cos^n(x)}\right] dx \qquad (9)$$

where n is an integer. When $n = 2$, for example, we have:

$$I_2 = \int_0^{\pi/2} \left[\frac{\sin^2(x)}{\sin^2(x) + \cos^2(x)}\right] dx = \int_0^{\pi/2} \sin^2(x) dx = \frac{\pi}{4}. \qquad (10)$$

You can check this result via integration by parts (calculus), or Wolfram Alpha, or ChatGPT 3.5/4.

That was a little too easy – how about we set n = 5. The values for $f(x) = \sin^5(x)/(\sin^5(x) + \cos^5(x))$ can be summarized as follows:

x	0.0	pi/8	pi/4	3pi/8	pi/2
f(x)	0.0	0.012046	0.5	0.987954	1.0

[5a] (3 pts). Use the method of Romberg integration to obtain an $O(h^4)$ accurate estimate of equation 9.

Trapezoid : $h = \pi/2$.

$$T_1 = \frac{h}{2}[f(0) + f(\pi/2)] = \frac{\pi}{4}[0 + 1] = \frac{\pi}{4}$$

Trapezoid : $h = \pi/4$

$$T_2 = \frac{h}{2}[f(0) + 2f(\pi/4) + f(\pi/2)]$$

$$= \frac{\pi}{8}[0 + 2 \times 0.5 + 1] = \frac{\pi}{4}.$$

Romberg Integration :

$$R_{21} = \frac{4T_2 - T_1}{(4-1)} = \frac{\pi/4 \cdot 3}{(4-1)} = \frac{\pi}{4}.$$

[5b] (4 pts). Evaluate equation 9 using 2-pt Gauss Quadrature. Be sure to show all steps in your working.

$$I = \int_0^{\pi/2} \left[\frac{\sin^5(x)}{\sin^5(x) + \cos^5(x)} \right] dx.$$

Map domain $[0, \pi/2] \longrightarrow [-1, 1]$.

Let $x = \frac{\pi}{4}[1 + u] \Rightarrow dx = \frac{\pi}{4} du$.

$$\Rightarrow I = \frac{\pi}{4} \int_{-1}^{1} \left[\frac{\sin^5(\frac{\pi}{4}(1+u))}{\sin^5(\frac{\pi}{4}(1+u)) + \cos^5 \frac{\pi}{4}(1+u)} \right] du.$$

Two point quadrature: $w_0 = w_1 = 1$

$u_0 = \frac{-1}{\sqrt{3}}, \quad u_1 = \frac{1}{\sqrt{3}}$

$$I = \frac{\pi}{4} \left\{ \left[\frac{\sin^5(\frac{\pi}{4}(1 - \frac{1}{\sqrt{3}}))}{\sin^5(\frac{\pi}{4}(1 - \frac{1}{\sqrt{3}})) + \cos^5(\frac{\pi}{4}(1 - \frac{1}{\sqrt{3}}))} \right] + \left[\frac{\sin^5(\frac{\pi}{4}(1 + 1/\sqrt{3}))}{\sin^5(\frac{\pi}{4}(1 + \frac{1}{\sqrt{3}})) + \cos^5(\frac{\pi}{4}(1 + \frac{1}{\sqrt{3}}))} \right] \right\}$$

$= 0.785398$

Note: Analytic sol'n $= \frac{\pi}{4} = 0.785398$.

Sometimes in calculus a simple change of variables (or adjustment in your point of view) can transform a problem from one that seems impossible into something quite simple. With that in mind:

[5c] (3 pts). Show that:

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx. \tag{11}$$

Hence, or otherwise, prove:

$$I_n = \int_0^{\pi/2} \left[\frac{\sin^n(x)}{\sin^n(x)+\cos^n(x)}\right] dx = \frac{\pi}{4}. \tag{12}$$

for **any integer** n. Show all of your working.

Consider equation 11 -- $I = \int_0^a f(a-x)dx$.

Let $u = a - x$. Then $\frac{du}{dx} = -1$. Also, when $x = 0$, $u = a$, and when $x = a$, $u = 0$. Hence.

$$I = \int_a^0 -f(u)\,du = \int_0^a f(u)\,du = \int_0^a f(x)\,dx \checkmark.$$

Consider equation 12: Using the hint: let $a = \pi/2$.

$$I_n = \int_0^{\pi/2} \frac{\sin^n(x)}{\sin^n(x)+\cos^n(x)}\,dx = \int_0^{\pi/2} \frac{\sin^n(\pi/2 - x)}{\sin^n(\pi/2-x)+\cos^n(\pi/2-x)}\,dx$$

But $\sin(\pi/2 - x) = \cos(x)$; $\cos(\pi/2 - x) = \sin(x)$.

$$= \int_0^{\pi/2} \frac{\cos^n(x)}{\cos^n(x)+\sin^n(x)}\,dx.$$

Add!!

$$2I_n = \int_0^{\pi/2} \frac{\sin^n(x)+\cos^n(x)}{\cos^n(x)+\sin^n(x)}\,dx = \int_0^{\pi/2} 1.\,dx = \frac{\pi}{2}$$

$$\Rightarrow I_n = \pi/4 \checkmark\checkmark.$$