Numerical Solution of Ordinary Differential Equations

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Overview

Ordinary Differential Equations

Ordinary Differential Equations

Example 1: Oscillatory mechanical systems:

$$[m]\, y''(t) + [c]\, y'(t) + [k]\, y(t) = f(t). \tag{1}$$

Example 2: Basic equation for beam deflection:

$$[EI]\, \frac{d^4 y}{dx^4} + w(x) = 0. \tag{2}$$

Example 3: Equation for cable suspension:

$$\frac{d^2 y}{dx^2} = \frac{w}{H} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{3}$$

Ordinary Differential Equations

Initial Value Problems (IVP): First order equation:

$$\frac{dy}{dx} = f(x, y) \quad \text{with} \quad y(x_o) = y_o. \tag{4}$$

Note: In general, an n-th order differential equation, i.e.,

$$Ay^n(t) + By^{n-1}(t) + \cdots = f(t). \tag{5}$$

can be written as a family of 1st order differential equations:

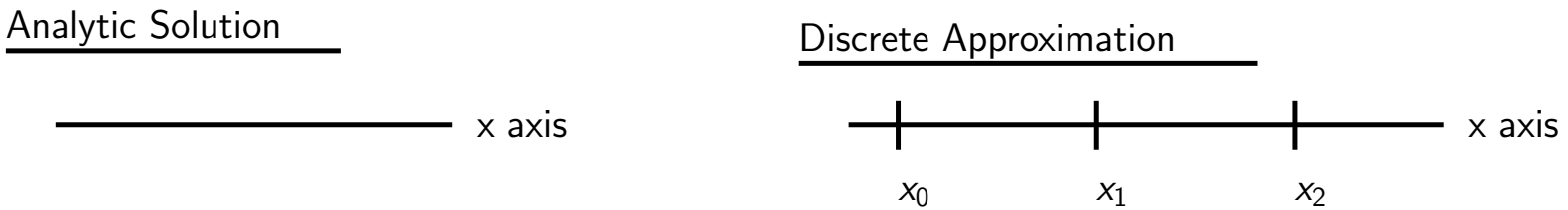
$$\left.\begin{aligned} y_1'(x) &= f_1(x, y_1, y_2, \cdots, y_n) \\ y_2'(x) &= f_2(x, y_1, y_2, \cdots, y_n) \\ &\cdots \\ y_n'(x) &= f_n(x, y_1, y_2, \cdots, y_n) \end{aligned}\right\}. \tag{6}$$

Ordinary Differential Equations

Key Challenge: Sometimes analytic solutions are unobtainable, even for seemingly simple problems:

$$\frac{dy}{dx} = x^2 + y^2. \tag{7}$$

Discrete Approximation: Abandon idea of computing solution for all values of x.



Computational Strategy: Compute solutions to equation 4 at a discrete number of points.

Ordinary Differential Equations

Numerical Solution of Initial Value Problem:

$$\frac{dy}{dx} = f(x, y) \Longleftrightarrow y(x_{i+1}) = y(x_i) + \int_{x_i}^{x_{i+1}} f(x, y(x))dx. \quad (8)$$

Can approximate integral with Trapezium Rule:

$$y(x_{i+1}) = y(x_i) + \frac{h}{2}\left[f(x_i, y_i(x_i)) + f(x_{i+1}, y(x_{i+1}))\right]. \quad (9)$$

where $x_{i+1} = x_i + h$.

Equation 9 is a discrete approximation, implicit in the value of $y(x_{i+1})$.

Euler Method

Euler Method

Mathematical Theory. Consider the IVP:

$$\begin{cases} \dfrac{dy}{dx} = f(x, y) \\ y(0) = y_o. \end{cases} \tag{10}$$

Let:

$$\frac{dy}{dx} = \left[\frac{y_{i+1} - y_i}{h}\right] = f(y_i, x_i). \tag{11}$$

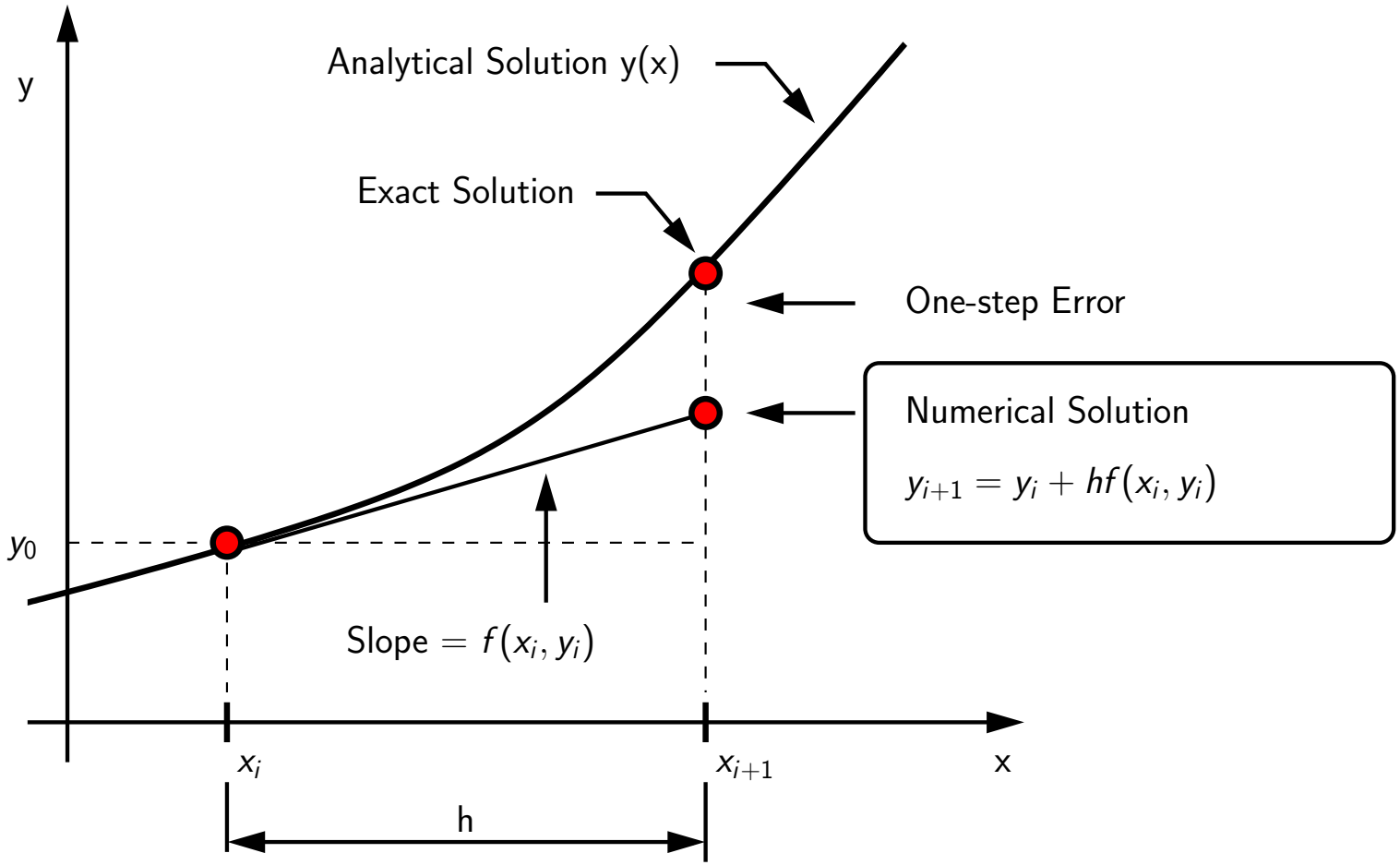
Rearranging equation 11 gives the Euler update:

$$y_{i+1} = y_i + hf(y_i, x_i). \tag{12}$$

One-step truncation error: $O(h^2)$.

Euler Method

Discrete Update for Euler's Method:



Euler Method

Example 1. Let:

$$\frac{dy}{dx} = f(x, y) = x + y \quad \text{with} \quad y(0) = 0. \tag{13}$$

Analytic Solution:

$$y(x) = e^x - x - 1. \tag{14}$$

Large Step Size: h = 0.5

```
x coordinate      Euler Method       Exact y(x)
=============================================
0.0000000e+00    0.0000000e+00    0.0000000e+00
5.0000000e-01    0.0000000e+00    1.4872127e-01
1.0000000e+00    2.5000000e-01    7.1828183e-01
=============================================
```

Poor accuracy ...

Euler Method

Small Step Size: h = 0.10

```
x coordinate       Euler Method        Exact y(x)
=============================================
0.0000000e+00    0.0000000e+00    0.0000000e+00
1.0000000e-01    0.0000000e+00    5.1709181e-03
2.0000000e-01    1.0000000e-02    2.1402758e-02
3.0000000e-01    3.1000000e-02    4.9858808e-02
4.0000000e-01    6.4100000e-02    9.1824698e-02
.....
9.0000000e-01    4.5794769e-01    5.5960311e-01
1.0000000e+00    5.9374246e-01    7.1828183e-01
=============================================
```

Still not very accurate ...

Modified Euler Method

Modified Euler Method

Computational Strategy: Base update on average of gradients at successive points:

Prediction Step:

$$y^*_{i+1} = y_i + hf(x_i, y_i) \tag{15}$$

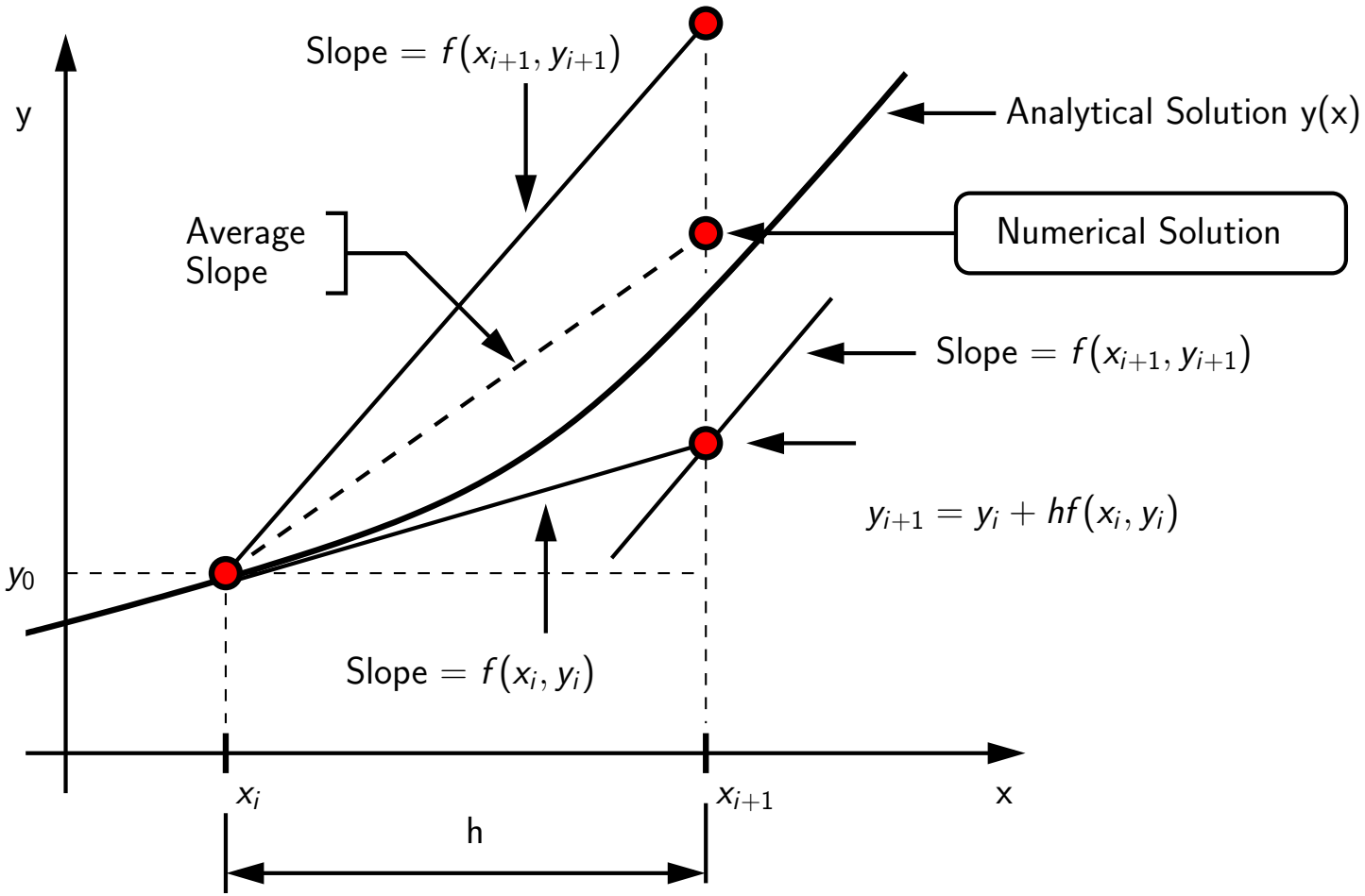
Correction Step:

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y^*_{i+1})\right] \tag{16}$$

One-step truncation error: $O(h^3)$.

Modified Euler Method

Discrete Update for Modified Euler's Method:



Modified Euler Method

Example 1. Let:

$$\frac{dy}{dx} = f(x, y) = x + y \quad \text{with} \quad y(0) = 0. \tag{17}$$

Analytic Solution:

$$y(x) = e^x - x - 1. \tag{18}$$

Predictor-Corrector Algorithm:

$$y^*_{i+1} = y_i + h\left[x_i + y_i\right] \tag{19}$$

$$y_{i+1} = y_i + \frac{h}{2}\left[x_i + y_i + x_{i+1} + y^*_{i+1}\right] \tag{20}$$

Modified Euler Method

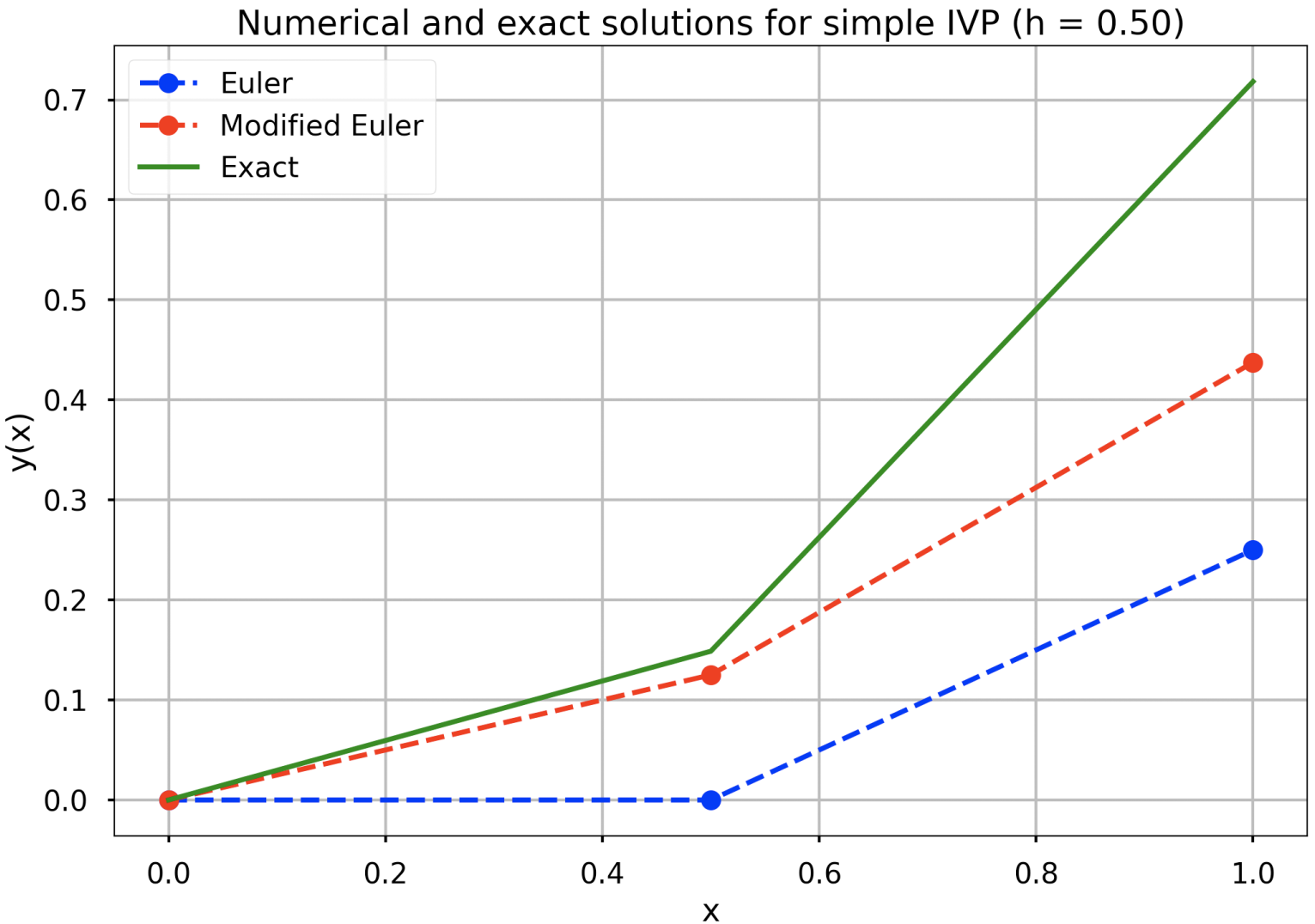
Large Step Size: h = 0.5

X coordinate	Euler Method	Modified Euler	Exact y(x)
0.0000000e+00	0.0000000e+00	0.0000000e+00	0.0000000e+00
5.0000000e-01	0.0000000e+00	1.2500000e-01	1.4872127e-01
1.0000000e+00	2.5000000e-01	4.3750000e-01	7.1828183e-01

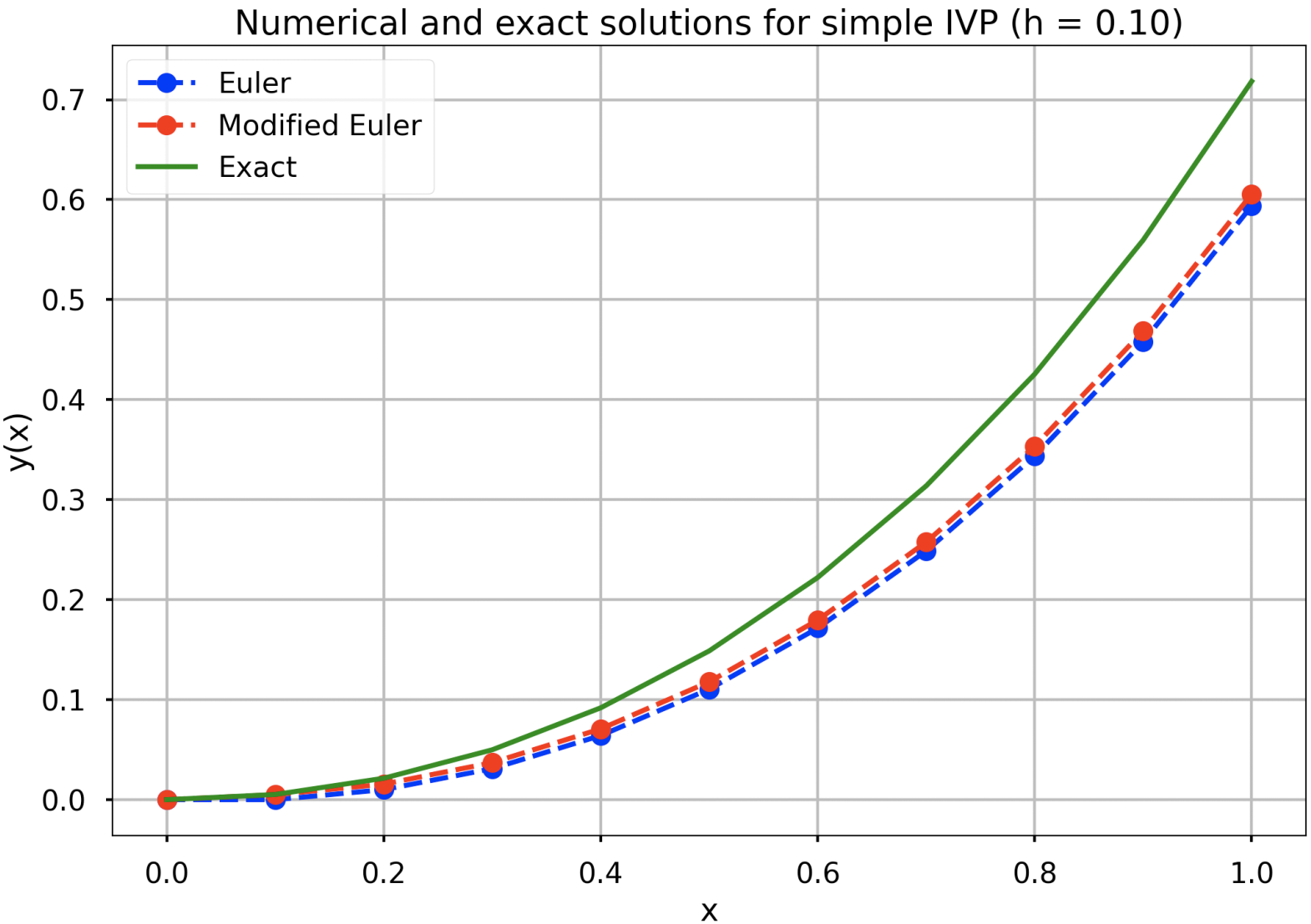
Small Step Size: h = 0.1

X coordinate	Euler Method	Modified Euler	Exact y(x)
0.0000000e+00	0.0000000e+00	0.0000000e+00	0.0000000e+00
1.0000000e-01	0.0000000e+00	5.0000000e-03	5.1709181e-03
2.0000000e-01	1.0000000e-02	1.5500000e-02	2.1402758e-02
3.0000000e-01	3.1000000e-02	3.7050000e-02	4.9858808e-02
4.0000000e-01	6.4100000e-02	7.0755000e-02	9.1824698e-02
.....			
9.0000000e-01	4.5794769e-01	4.6866564e-01	5.5960311e-01
1.0000000e+00	5.9374246e-01	6.0553220e-01	7.1828183e-01

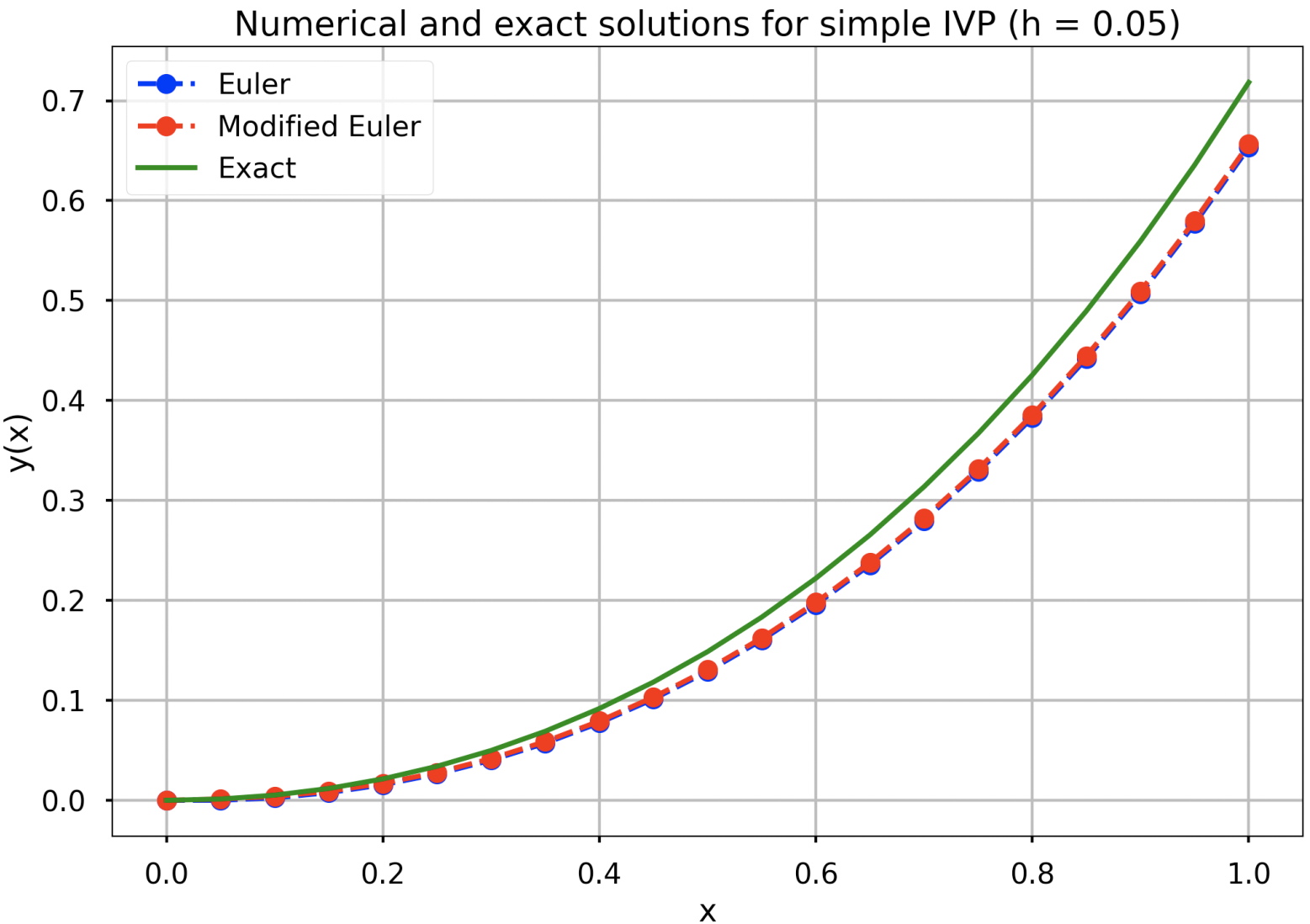
Modified Euler Method



Modified Euler Method



Modified Euler Method



Python Code Listings

Code 1: Modified Euler Method

```
1   # ========================================================================
2   # TestOdeModifiedEuler01.py: Use Modified Euler Method to solve IVP.
3   #
4   # Written By: Mark Austin                                        July 2023
5   # ========================================================================
6
7   import math
8   import numpy as np
9   import matplotlib.pyplot as plt
10
11  import LinearMatrixEquations as lme
12
13  # Define mathematical functions ...
14
15  def f1(x,y):
16      return x + y
17
18  def f2(x):
19      return math.exp(x) - x - 1.0
20
21  # main method ...
22
23  def main():
24      print("--- Enter TestOdeModifiedEuler01.main()        ... ");
25      print("--- ========================================== ... ");
26
27      # Define parameters
```

Code 1: Modified Euler Method

```
29      s0 = 0.0                        # <-- Initial Condition
30      h  = 0.50                       # <-- Step size
31      x  = np.arange(0, 1 + h, h) # <-- Create numerical grid on [0,1] ...
32
33      # Setup array for numerical and exact solutions ...
34
35      euler01 = np.zeros(len(x))  # <-- store euler method ...
36      euler02 = np.zeros(len(x))  # <-- store improved euler method ..
37      exact01 = np.zeros(len(x))  # <-- store exact solution ...
38
39      # Initialize array values ...
40
41      euler01[0] = s0             # <-- Euler estimate ....
42      euler02[0] = s0             # <-- Modified Euler estimate ...
43      exact01[0] = s0             # <-- Exact solution ...
44
45      # Compute iterations of Euler and Modified Euler, Exact solution ...
46
47      for i in range(0, len(x) - 1):
48         euler01[i+1] = euler01[i] + h*f1( x[i], euler01[i] )
49         euler02[i+1] = euler01[i] + (h/2)*( f1( x[i], euler01[i] ) + f1( x[i+1], euler01[
50         exact01[i+1] = f2( x[i+1] )
51
52      # Print x, euler, improved euler, and y(x) vectors ...
53
54      lme.printvector("(x coordinate)", x )
55      lme.printvector("(euler method)",          euler01 )
56      lme.printvector("(improved euler method)", euler02 )
```

Code 1: Modified Euler Method

```
57        lme.printvector("y(x) (exact solution)",    exact01 )
58
59        # Plot output ...
60
61        plt.style.use('seaborn-poster')
62        plt.figure(figsize = (12, 8))
63        plt.plot( x, euler01,'bo--', label='Euler')
64        plt.plot( x, euler02,'ro--', label='Modified Euler')
65        plt.plot( x, exact01,   'g', label='Exact')
66        plt.title('Numerical and exact solutions for simple IVP (h = 0.50)')
67        plt.xlabel('x')
68        plt.ylabel('y(x)')
69        plt.grid()
70        plt.legend(loc='upper left')
71        plt.show()
72
73        print("--- ======================================== ... ");
74        print("--- Leave TestOdeModifiedEuler01.main()      ... ");
75
76    # call the main method ...
77
78    main()
```