Linear Matrix Equations – Part 2

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Overview

Numerical Solution of Linear Matrix Equations

Mathematical Viewpoint

Mathematical Viewpoint

Solutions to linear equations is given by theoretical results from linear algebra, namely:

- A unique solution $\{X\} = \left[A^{-1}\right] \cdot \{B\}$ exists when $\left[A^{-1}\right]$ exists (i.e., $\det[A] \neq 0$).
- The equations are inconsistent when $[A]$ is singular and $\text{rank}\,[A|B] \neq \text{rank}\,[A]$.
- If $\text{rank}\,[A|B]$ equals $\text{rank}\,[A]$, then there are an infinite number of solutions.

We can use these results to guide the design of numerical algorithms for computing solutions to linear equations.

Engineering Viewpoint

Direct Methods:

- Transform the original equations into equivalent equations that can be solved more easily.
- Exact answer results if there are no round-off errors. Finite work. Use on dense matrices containing few zeros.
- Gausss Elimination, LU Decomposition.

Iterative Methods:

- Start with a guess of the the solution and repeatedly refine the solution until convergence criteria are achieved.
- Needs infinite work to get an exact answer. No round-off errors. Use on sparse, large order systems.
- Gauss-Seidel Iteration, Jacobi's Iteration Method.

Engineering Viewpoint

Evaluation Criteria:

- Robustness (i.e., Do the solvers work? When will they work?)
- Accuracy and Efficiency (i.e., How does computational effort vary as a function of problem size?).
- Ease of Implementation (i.e., implementation speed and cost).

Common Problems:

- Is A singular (or not)?
- Matrix A might be nearly singular. This is important for finite-precision arithmetic.

The pathway from mathematical analysis to a software implementation requires consideration of ill-conditioned equations and finite precision arithmetic.

Ill-Conditioned Equations

Definition. A system of equations is said to be ill-conditioned if small changes to one or more coefficients of $[A][X] = [B]$ causes large deviations in the solution.

Example: The following pairs of almost parallel equations:

$$\begin{bmatrix} 1.00 & 2.00 \\ 0.48 & 0.99 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3.00 \\ 1.47 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (1)$$

Let's increment coefficient a_{21} by 2%:

$$\begin{bmatrix} 1.00 & 2.00 \\ 0.49 & 0.99 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3.00 \\ 1.47 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}. \quad (2)$$

Finite-Precision Arithmetic

Example 1. Compute a numerical solution to:

$$\begin{bmatrix} 1.0 \times 10^{-3} & 1.0 \\ 1.0 & 1.0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 2.0 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \approx \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix}. \quad (3)$$

with a computer having 2 digits of precision (mantissa + floating point + rounded).

Method 1: Transform the matrix equations into echelon form.

Step 1: Compute row multiplier m_{21}:

$$m_{21} = \left[\frac{a_{21}}{a_{11}} \right] = \frac{1}{1.0 \times 10^{-3}} = 1.0 \times 10^{3}. \quad (4)$$

Finite-Precision Arithmetic

Step 2: Compute row operation: $r_2 \rightarrow r_2 - m_{21} r_1$.

$$
\begin{aligned}
(1 - 1000 \times 1.0) x_2 &= (2.0 - 1.0 \times 1000) \\
-999 x_2 &= -998 \\
-1.0 \times 10^3 x_2 &= -1.0 \times 10^3 \\
x_2 &= 1.0 \\
x_1 &= 0.0
\end{aligned}
$$

This is completely the wrong answer! The source of the problem is the large multiplier m_{12}.

Finite-Precision Arithmetic

Method 2: Idea: Swap rows 1 and 2. Then transform the matrix equations into echelon form.

Step 1: Compute row multiplier m_{21}:

$$m_{21} = \left[\frac{a_{21}}{a_{11}}\right] = \frac{1.0 \times 10^{-3}}{1.0} = 10^{-3}. \tag{5}$$

Step 2: Compute row operation: $r_2 \rightarrow r_2 - m_{21} r_1$.

$$\begin{aligned}(1 - 0.001 \times 1.0)x_2 &= (1.0 - 0.0001 \times 2.0) \\ 0.999x_2 &= 0.998 \\ 1.0x_2 &= 1.0 \\ x_2 &= 1.0\end{aligned}$$

Finite-Precision Arithmetic

Step 3: Backsubstitution:

$$x_1 = 1.0 \tag{6}$$

To two decimal places of accuracy $[x_1, x_2] = [1.0, 1.0]$ is correct.

The absolutely correct answer is $[x_1, x_2] = [1.001, 0.999]$.

Recommended Strategy

- Rearrange rows so that the largest coefficient in absolute value in the column is in the pivotal position.
- This process is called partial pivoting.
- Can also row equilibrate – scale individual rows so maximum element value is 1.

Gauss Elimination

(No Pivoting)

Gauss Elimination (No Pivoting)

Elementary Row Operations

- Interchange any two rows.
- Multiply any row by a non-zero number.
- Add to one equation a non-zero multiple of another equation.

Step-by-Step Procedure

- Write $[A][x] = [B]$ in augmented matrix form.
- Apply a sequence of row operations to transform the augmented matrix into echelon form.
- Use back substitution to compute the solution vector.
- Validate answer.

Gauss Elimination (No Pivoting)

Example 1. Consider the family of matrix equations:

$$\begin{bmatrix} 3 & -6 & 7 \\ 9 & 0 & -5 \\ 5 & -8 & 6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix} \tag{7}$$

Step 1. Augmented Matrix Form

$$\left[\begin{array}{ccc|c} 3 & -6 & 7 & 3 \\ 9 & 0 & -5 & 3 \\ 5 & -8 & 6 & -4 \end{array}\right] \tag{8}$$

Gauss Elimination (No Pivoting)

Step 2. Apply Row Operations

Step 2.1: Divide Row 1 by 3 ($r_1 \rightarrow r_1/3$)

$$\left[\begin{array}{ccc|c} 3 & -6 & 7 & 3 \\ 9 & 0 & -5 & 3 \\ 5 & -8 & 6 & -4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 7/3 & 1 \\ 9 & 0 & -5 & 3 \\ 5 & -8 & 6 & -4 \end{array}\right] \tag{9}$$

Step 2.2: Subtract 9 times row 1 from row 2 ($r_2 \rightarrow r_2 - 9r_1$) Step 2.3: Subtract 5 times row 1 from row 3 ($r_3 \rightarrow r_3 - 5r_1$)

$$\left[\begin{array}{ccc|c} 1 & -2 & 7/3 & 1 \\ 9 & 0 & -5 & 3 \\ 5 & -8 & 6 & -4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 7/3 & 1 \\ 0 & 18 & -26 & -6 \\ 0 & 2 & -5.667 & -9 \end{array}\right] \tag{10}$$

Gauss Elimination (No Pivoting)

Step 2.4: Divide row 2 by 18 ($r_2 \rightarrow r_2/18$)

Step 2.5: Subtract 2 times row 2 from row 3 ($r_3 \rightarrow r_3 - 2r_2$)

Step 2.6: Divide row 3 by 2.777 ($r_3 \rightarrow r_3/2.777$)

$$\left[\begin{array}{ccc|c} 1 & -2 & 7/3 & 1 \\ 0 & 18 & -26 & -6 \\ 0 & 2 & -5.667 & -9 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 7/3 & 1 \\ 0 & 1 & -26/18 & -1/3 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad (11)$$

Equation 11 is now in echelon form.

Gauss Elimination (No Pivoting)

Step 3. Compute Solution via Back Substitution

Step 3.1: Row 3: $x_3 = 3.0 \rightarrow x_3 = 3.0$.

Step 3.2: Row 2: $x_2 - 26/18x_3 = -1/3 \rightarrow x_2 = 4.0$.

Step 3.3: Row 1: $x_1 - 2x_2 + 7/3x_3 = 1.0 \rightarrow x_1 = 2.0$.

Step 4. Validate the Answer, $\mathrm{X} = [2, 4, 3]$.

$$\begin{bmatrix} 3 & -6 & 7 \\ 9 & 0 & -5 \\ 5 & -8 & 6 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}. \tag{12}$$

Gauss Elimination (No Pivoting)

Python Source Code: Step-by-Step Row Reduction

```
1   import math
2   import numpy as np
3   import LinearMatrixEquations as lme
4
5   def main():
6       print("--- Step 1: Create (3x3) matrix A, (3x1) matrix B ... ");
7
8       A = np.array([ [ 3, -6,  7],
9                      [ 9,  0, -5],
10                     [ 5, -8,  6] ]);
11      B = np.array([ [ 3],
12                     [ 3],
13                     [-4] ]);
14
15      lme.printmatrix("A", A);
16      lme.printmatrix("B", B);
17
18      print("--- Step 2: Create (3x4) augmented matrix [ A | B ] (join on vertical axis) .
19
20      AB = np.concatenate((A, B), axis=1)
21      lme.printmatrix("Augmented Matrix [A|B] ...", AB );
22
23      print("--- Step 3.1: Scale row 1 by 1/3 ... ");
24
25      gauss01 = lme.rowscale(AB, 0, 1.0/3.0)
26      lme.printmatrix("Matrix gauss01 ...", gauss01 );
```

Gauss Elimination (No Pivoting)

Python Source Code: Step-by-Step Row Reduction

```
28      print("--- Step 3.2: Subtract 9 times row 1 from row 2 ... ");
29
30      gauss02 = lme.rowadd( gauss01, 0, 1, -9.0)
31      lme.printmatrix("Matrix gauss02 ...", gauss02 );
32
33      print("--- Step 3.3: Subtract 5 times row 1 from row 3 ... ");
34
35      gauss03 = lme.rowadd( gauss02, 0, 2, -5.0)
36      lme.printmatrix("Matrix gauss03 ...", gauss03 );
37
38      print("--- Step 3.4: Scale row 2 by 1/18 ... ");
39
40      gauss04 = lme.rowscale(gauss03, 1, 1.0/18.0)
41      lme.printmatrix("Matrix gauss04 ...", gauss04 );
42
43      print("--- Step 3.5: Subtract 2 times row 2 from row 3 ... ");
44
45      gauss05 = lme.rowadd( gauss04, 1, 2, -2.0)
46      lme.printmatrix("Matrix gauss05 ...", gauss05 );
47
48      print("--- Step 3.6: Scale row 3 by -1/2.77777778 ... ");
49
50      echelon01 = lme.rowscale(gauss05, 2, -1.0/2.77777778 )
51
52      print("--- Step 4.0: Print matrix [A|B] in Echelon Form ... ");
53
54      lme.printmatrix("Matrix [A|B] in Echelon Form ...", echelon01 );
```

Gauss Elimination (No Pivoting)

Abbreviated Output: See TestLinearMatrixEquations01.py

```
--- Step 1: Create (3x3) matrix A, (3x1) matrix B ...

Matrix: A
   3.0000000e+00   -6.0000000e+00    7.0000000e+00
   9.0000000e+00    0.0000000e+00   -5.0000000e+00
   5.0000000e+00   -8.0000000e+00    6.0000000e+00

Matrix: B
   3.0000000e+00
   3.0000000e+00
  -4.0000000e+00

--- Step 2: Create (3x4) augmented matrix [ A | B ] (join on vertical axis) ...

Matrix: Augmented Matrix [A|B] ...
   3.0000000e+00   -6.0000000e+00    7.0000000e+00    3.0000000e+00
   9.0000000e+00    0.0000000e+00   -5.0000000e+00    3.0000000e+00
   5.0000000e+00   -8.0000000e+00    6.0000000e+00   -4.0000000e+00
```

Gauss Elimination (No Pivoting)

Abbreviated Output: Continued ...

```
--- Step 3.1: Scale row 1 by 1/3 ...

Matrix: Matrix gauss01 ...
   1.0000000e+00   -2.0000000e+00    2.3333333e+00    1.0000000e+00
   9.0000000e+00    0.0000000e+00   -5.0000000e+00    3.0000000e+00
   5.0000000e+00   -8.0000000e+00    6.0000000e+00   -4.0000000e+00

--- Step 3.2: Subtract 9 times row 1 from row 2 ...
--- Step 3.3: Subtract 5 times row 1 from row 3 ...
--- Step 3.4: Scale row 2 by 1/18 ...
--- Step 3.5: Subtract 2 times row 2 from row 3 ...
--- Step 3.6: Scale row 3 by -1/2.77777778 ...
--- Step 4.0: Print matrix [A|B] in Echelon Form ...

Matrix: Matrix [A|B] in Echelon Form ...
   1.0000000e+00   -2.0000000e+00    2.3333333e+00    1.0000000e+00
   0.0000000e+00    1.0000000e+00   -1.4444444e+00   -3.3333333e-01
  -0.0000000e+00   -0.0000000e+00    1.0000000e+00    3.0000000e+00
```

Gauss Elimination

(with Partial Pivoting)

Gauss Elimination (with Partial Pivoting)

Step-by-Step Procedure:

- Find largest element in absolute value in 1st column. Interchange rows so that it is the pivotal equation.
- Compute multipliers: $m_{i1} = [a_{i1}/a_{11}]$ for i = 2, 3, $\cdots$ n.
- Compute new coefficients in rows 2, 3, $\cdots$ n.

  For example, $a'_{22} = a_{22} - m_{21}a_{12}$, $a'_{23} = a_{23} - m_{21}a_{13}$, $a'_{32} = a_{32} - m_{31}a_{12}$, etc.

  The partially transform matrix is:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \end{bmatrix}. \qquad (13)$$

Gauss Elimination (with Partial Pivoting)

Step-by-Step Procedure:

- Repeat process for 2nd column.
- After (n-1) column calculations the A matrix will be in upper-triangular (echelon) form.
- Solve for unknowns through backsubstitution.

Key Points:

- If the pivotal coeffient is small, then the row multiplier will be very large, and errors magnified due to finite arithmetic.
- Row reduction and back substitution require $O(n^3/3)$ and $O(n^2)$ operations, respectively.

Gauss Elimination (with Partial Pivoting)

Example 2. Consider the following set of equations:

$$\begin{aligned} 2x_1 + 6x_2 - x_3 &= -12 \\ 5x_1 - x_2 + 2x_3 &= 29 \\ -3x_1 - 4x_2 + x_3 &= 5 \end{aligned}$$

Steps 1 and 2: Augmented matrix form, then row equilibration.

$$\left[\begin{array}{rrr|r} 2 & 6 & -1 & -12 \\ 5 & -1 & 2 & 29 \\ -3 & -4 & 1 & 5 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 2/6 & 1 & -1/6 & -2.0 \\ 1 & -1/5 & 2/5 & 29/5 \\ -3/4 & -1 & 1/4 & 5/4 \end{array}\right]. \tag{14}$$

Gauss Elimination (with Partial Pivoting)

Step 3: Partial pivoting (swap rows 1 and 2):

$$\left[\begin{array}{rrr|r} 1 & -1/5 & 2/5 & 29/5 \\ 2/6 & 1 & -1/6 & -2.0 \\ -3/4 & -1 & 1/4 & 5/4 \end{array}\right] \tag{15}$$

Step 4: Compute row multipliers:

$$m_{21} = [a_{21}/a_{11}] = 2/6, \quad m_{31} = [a_{31}/a_{11}] = -3/4. \tag{16}$$

Step 5: Compute new coefficients for rows 2 and 3:

$$\left[\begin{array}{rrr|r} 1 & -1/5 & 2/5 & 29/5 \\ 0 & 32/30 & -9/30 & -118/30 \\ 0 & -23/20 & 11/20 & 118/20 \end{array}\right] \tag{17}$$

Gauss Elimination (with Partial Pivoting)

Step 6: Compute multiplier for row 3, columm 2.

$$m_{32} = [a_{32}/a_{22}] = -64/69. \tag{18}$$

Step 7: Eliminate elements below diagonal in column 2.

$$\left[\begin{array}{ccc|c} 1 & -1/5 & 2/5 & 29/5 \\ 0 & 32/30 & -9/30 & -118/30 \\ 0 & 0 & 145/690 & 870/690 \end{array}\right] \tag{19}$$

Step 8: Solve for x_1 through x_3 via back substitution.

$$x_3 = [870/690] \,/\, [145/690] = 6.0, \;\; x_2 = -2.0, \;\; x_1 = 3.0. \tag{20}$$

Step 9: Check solution by substitution.

Gauss Elimination (with Partial Pivoting)

Python Source Code: Row Reduction + Back Substitution

```
1   # ============================================================================
2   # TestLinearMatrixEquations02.py: Gauss Elimination with row reduction, partial
3   # pivoting, and back substitution.
4   # ============================================================================
5
6   import math
7   import numpy as np
8   import LinearMatrixEquations as lme
9
10  def main():
11      n = 3  # <--- size of the matrix equations ...
12
13      print("--- Problem 1: Create (3x3) matrix A, (3x1) matrix B ... ");
14
15      A = np.array([ [ 3, -6,  7],
16                     [ 9,  0, -5],
17                     [ 5, -8,  6] ]);
18
19      B = np.array([ [ 3], [ 3], [-4] ]);
20
21      lme.printmatrix("A", A);
22      lme.printmatrix("B", B);
23
24      print("--- Step 2: Compute augmented matrix equations ... ");
25
26      AB = np.concatenate((A, B), axis=1)
27      lme.printmatrix("Augmented Matrix [A|B] ...", AB );
```

Gauss Elimination (with Partial Pivoting)

Python Source Code: Continued ...

```
28
29      print("--- Step 3: Compute row reduction operations to echelon form ... ");
30
31      echelon01 = lme.rowreduction( AB );
32      lme.printmatrix("Matrix echelon01 ...", echelon01 );
33
34      print("--- Step 4: Split echelon01 back to (nxn) piece and (nx1) pieces ... ");
35
36      B_reduced = echelon01[:,n:n+1]
37      A_reduced = echelon01[:,0:n]
38
39      lme.printmatrix("Matrix A (reduced) ...", A_reduced );
40      lme.printmatrix("Matrix B (reduced) ...", B_reduced );
41
42      print("--- Step 5: Compute back substitution ... ");
43
44      soln01 = lme.backsubstitution( A_reduced, B_reduced)
45      lme.printmatrix("Matrix X (solution) ...", soln01 );
46
47      print("--- Problem 2: Create (3x3) matrix A, (3x1) matrix B ... ");
48
49      A = np.array([ [  2,  6, -1],
50                     [  5, -1,  2],
51                     [ -3, -4,  1] ]);
52
53      B = np.array([ [-12], [ 29], [  5] ]);
54
55      lme.printmatrix("A", A);
56      lme.printmatrix("B", B);
```

Gauss Elimination (with Partial Pivoting)

Python Source Code: Continued ...

```
58      print("--- Step 2: Compute augmented matrix equations ... ");
59
60      AB = np.concatenate((A, B), axis=1)
61      lme.printmatrix("Augmented Matrix [A|B] ...", AB );
62
63      print("--- Step 3: Compute row reduction operations to echelon form ... ");
64
65      echelon01 = lme.rowreduction( AB );
66      lme.printmatrix("Matrix echelon01 ...", echelon01 );
67
68      print("--- Step 4: Split echelon01 back to (nxn) piece and (nx1) pieces ... ");
69
70      B_reduced = echelon01[:,n:n+1]
71      A_reduced = echelon01[:,0:n]
72
73      lme.printmatrix("Matrix A (reduced) ...", A_reduced );
74      lme.printmatrix("Matrix B (reduced) ...", B_reduced );
75
76      print("--- Step 5: Compute back substitution ... ");
77
78      soln01 = lme.backsubstitution( A_reduced, B_reduced)
79      lme.printmatrix("Matrix X (solution) ...", soln01 );
80
81  # call the main method ...
82
83  main()
```

Gauss Elimination (with Partial Pivoting)

Abbreviated Output: Problem 1 ...

```
Matrix: A                                                Matrix: B
   3.0000000e+00   -6.0000000e+00    7.0000000e+00        3.0000000e+00
   9.0000000e+00    0.0000000e+00   -5.0000000e+00        3.0000000e+00
   5.0000000e+00   -8.0000000e+00    6.0000000e+00       -4.0000000e+00

--- Step 2: Compute augmented matrix equations ...

Matrix: Augmented Matrix [A|B] ...
   3.0000000e+00   -6.0000000e+00    7.0000000e+00    3.0000000e+00
   9.0000000e+00    0.0000000e+00   -5.0000000e+00    3.0000000e+00
   5.0000000e+00   -8.0000000e+00    6.0000000e+00   -4.0000000e+00

--- Step 3: Compute row reduction operations to echelon form ...

Matrix: Matrix echelon01 ...
   1.0000000e+00   -2.0000000e+00    2.3333333e+00    1.0000000e+00
   0.0000000e+00    1.0000000e+00   -1.4444444e+00   -3.3333333e-01
  -0.0000000e+00   -0.0000000e+00    1.0000000e+00    3.0000000e+00
```

Gauss Elimination (with Partial Pivoting)

Abbreviated Output: Problem 1 continued ...

```
--- Step 4: Split echelon01 back to (nxn) piece and (nx1) pieces ...

Matrix: Matrix A (reduced) ...

   1.0000000e+00   -2.0000000e+00    2.3333333e+00
   0.0000000e+00    1.0000000e+00   -1.4444444e+00
  -0.0000000e+00   -0.0000000e+00    1.0000000e+00

Matrix: Matrix B (reduced) ...
   1.0000000e+00
  -3.3333333e-01
   3.0000000e+00

--- Step 5: Compute back substitution ...

Matrix: Matrix X (solution) ...
   2.0000000e+00
   4.0000000e+00
   3.0000000e+00
```

Gauss Elimination (with Partial Pivoting)

Abbreviated Output: Problem 2 ...

```
Matrix: A                                                Matrix: B
   2.0000000e+00    6.0000000e+00   -1.0000000e+00       -1.2000000e+01
   5.0000000e+00   -1.0000000e+00    2.0000000e+00        2.9000000e+01
  -3.0000000e+00   -4.0000000e+00    1.0000000e+00        5.0000000e+00

--- Step 2: Compute augmented matrix equations ...

Matrix: Augmented Matrix [A|B] ...
   2.0000000e+00    6.0000000e+00   -1.0000000e+00   -1.2000000e+01
   5.0000000e+00   -1.0000000e+00    2.0000000e+00    2.9000000e+01
  -3.0000000e+00   -4.0000000e+00    1.0000000e+00    5.0000000e+00

--- Step 3: Compute row reduction operations to echelon form ...

Matrix: Matrix echelon01 ...
   1.0000000e+00    3.0000000e+00   -5.0000000e-01   -6.0000000e+00
  -0.0000000e+00    1.0000000e+00   -2.8125000e-01   -3.6875000e+00
   0.0000000e+00    0.0000000e+00    1.0000000e+00    6.0000000e+00
```

Gauss Elimination (with Partial Pivoting)

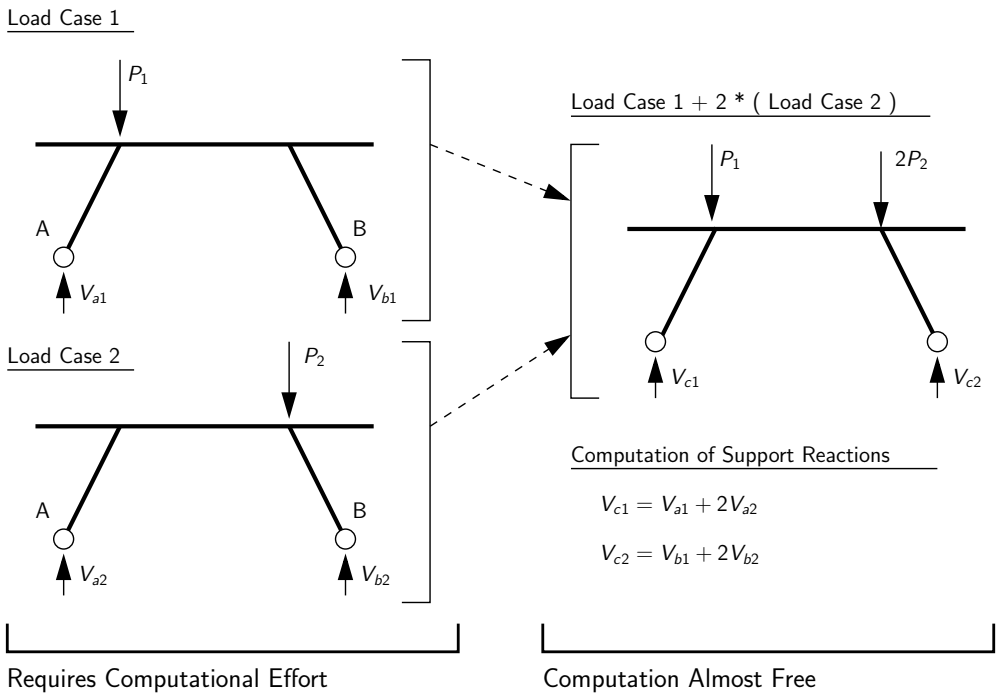
Abbreviated Output: Problem 2 continued ...

```
--- Step 4: Split echelon01 back to (nxn) piece and (nx1) pieces ...

Matrix: Matrix A (reduced) ...
   1.0000000e+00    3.0000000e+00   -5.0000000e-01
  -0.0000000e+00    1.0000000e+00   -2.8125000e-01
   0.0000000e+00    0.0000000e+00    1.0000000e+00

Matrix: Matrix B (reduced) ...
  -6.0000000e+00
  -3.6875000e+00
   6.0000000e+00

--- Step 5: Compute back substitution ...

Matrix: Matrix X (solution) ...
   3.0000000e+00
  -2.0000000e+00
   6.0000000e+00
```

LU Decomposition

LU Decomposition

Motivation. Suppose AX = B needs to be solved for a multiplicity of B vectors (e.g., A $X_1 = B_1$, A $X_2 = B_2$, etc).



Solving AX = B requires $O(n^3)$ computational work. Can we do better? ...

LU Decomposition

Objective. We want to solve $[A]\,[X] = [B]$, where A, X, and B are $(n \times n)$, $(n \times 1)$ and $(n \times n)$ matrices, respectively.

Idea. Factor $[A]$ into the product:

$$[A] = [L]\,[U] \rightarrow [L]\,[U]\,[X] = [B]\,. \tag{21}$$

and then solve in two steps.

Step 1: Foward Substitution to get $\mathrm{Z} = [z_1, z_2, \cdots, z_n]$.

$$\begin{bmatrix} L_{11} & 0 & \cdots & 0 \\ L_{21} & L_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}. \tag{22}$$

LU Decomposition

Step 2: Back Substitution to get $X = [x_1, x_2, \cdots, x_n]$.

$$\begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1n} \\ 0 & U_{22} & \cdots & U_{2n} \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & U_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}. \tag{23}$$

Observation: There are n^2 elements in matrix A, and $n(n+1)$ unknowns in equations 22 and 23.

We need to impose n additional constraints to make the method work.

LU Decomposition

Row Reduction Strategies:

- Assume upper diagonal elements are unity, i.e., $U_{ii} = 1$ (Crout Reduction).
- Assume $L_{ii} = \mathrm{I}$ (Doolittle's Method).
- Assume $U_{ii} = L_{ii}$ (Cholesky Decomposition).

Step-by-Step Solution Procedure:

- Compute $\mathrm{A} = \mathrm{LU}$ (this requires $\mathrm{O}(n^3)$ work).
- Solve $\mathrm{LU}\ X_1 = B_1$ (only requires $\mathrm{O}(n^2)$ work).
- Solve $\mathrm{LU}\ X_2 = B_2$ (only requires $\mathrm{O}(n^2)$ work).
- Solve $\mathrm{LU}\ X_3 = B_3$ (only requires $\mathrm{O}(n^2)$ work), etc ...

LU Decomposition

General Equations. For a $(n \times n)$ set of equations $[L][U] = [A]$,

$$a_{ij} = \sum_{k=min(i,j)}^{n} (L_{ik} U_{kj}) . \tag{24}$$

Three cases:

$$i \leq j \quad U_{ij} = \left[A_{ij} - \sum_{k=1}^{i-1} L_{ik} U_{kj} \right] / U_{ii}$$

$$i = j \quad U_{jj} = \left[A_{ij} - \sum_{k=1}^{j-1} L_{ik} U_{kj} \right] / L_{jj}$$

$$i \geq j \quad L_{ij} = \left[A_{ij} - \sum_{k=1}^{j-1} L_{ik} U_{kj} \right] / U_{jj}$$

LU Decomposition

Example 1. Consider the matrix equations:

$$\begin{bmatrix} 3 & -6 & 7 \\ 9 & 0 & -5 \\ 5 & -8 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}. \tag{25}$$

Assume $[L][U] = [A]$ with values of unity along the upper diagonal (Crout Reduction).

$$\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -6 & 7 \\ 9 & 0 & -5 \\ 5 & -8 & 6 \end{bmatrix}. \tag{26}$$

LU Decomposition

Equating terms (1st row of L):

$$\begin{aligned} L_{11} \cdot 1 = 3 \quad &\rightarrow \quad L_{11} = 3. \\ L_{11} \cdot U_{12} = -6 \quad &\rightarrow \quad U_{12} = -2. \\ L_{11} \cdot U_{13} = -7 \quad &\rightarrow \quad U_{13} = 7/3. \end{aligned}$$

Equating terms (2nd row of L):

$$\begin{aligned} L_{21} \cdot U_{11} = 9 \quad &\rightarrow \quad L_{21} = 9. \\ L_{21} \cdot U_{12} + L_{22} \cdot 1 = 0 \quad &\rightarrow \quad L_{22} = 18. \\ L_{21} \cdot U_{13} + L_{22} \cdot U_{23} = 0 \quad &\rightarrow \quad U_{23} = -13/9. \end{aligned}$$

LU Decomposition

Equating terms (3rd row of L):

$$\begin{aligned} L_{31} \cdot U_{11} = 5 \quad &\rightarrow \quad L_{31} = 5. \\ L_{31} \cdot U_{12} + L_{32} = -8 \quad &\rightarrow \quad L_{32} = 2. \\ L_{31} \cdot U_{13} + L_{32} \cdot U_{23} + L_{33} = 6 \quad &\rightarrow \quad L_{33} = -25/9. \end{aligned}$$

Hence:

$$\begin{bmatrix} 3 & 0 & 0 \\ 9 & 18 & 0 \\ 5 & 2 & -25/9 \end{bmatrix} \begin{bmatrix} 1 & -2 & 7/3 \\ 0 & 1 & -13/9 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -6 & 7 \\ 9 & 0 & -5 \\ 5 & -8 & 6 \end{bmatrix}. \tag{27}$$

LU Decomposition

Forward Substitution:

$$\begin{bmatrix} 3 & 0 & 0 \\ 9 & 18 & 0 \\ 5 & 2 & -25/9 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix} \rightarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1.0 \\ -1/3 \\ 3.0 \end{bmatrix}. \tag{28}$$

Backward Substitution:

$$\begin{bmatrix} 1 & -2 & 7/3 \\ 0 & 1 & -13/9 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1/3 \\ 3.0 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2.0 \\ 4.0 \\ 3.0 \end{bmatrix}. \tag{29}$$

LU Decomposition

Example 2. Consider $[L][U] = [A]$ with values of unity along the lower diagonal (Doolittle's Method).

$$\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = \begin{bmatrix} 3 & -6 & 7 \\ 9 & 0 & -5 \\ 5 & -8 & 6 \end{bmatrix}. \quad (30)$$

Can show:

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5/3 & 1/9 & 1 \end{bmatrix} \begin{bmatrix} 3 & -6 & 7 \\ 0 & 18 & -26 \\ 0 & 0 & -25/9 \end{bmatrix} = \begin{bmatrix} 3 & -6 & 7 \\ 9 & 0 & -5 \\ 5 & -8 & 6 \end{bmatrix}. \quad (31)$$

Working Examples

Example 3: LU Decomposition with Symbolic Python

Problem Statement: Let's try to use SymPy to compute algebraic representations for LU decomposition for matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = LU, \tag{32}$$

and

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = LU. \tag{33}$$

Then, we will check the results by assigning numerical values to the symbols to match equation 25. Note: det(A) = det(L).det(U) = -150.

Example 3: LU Decomposition with Symbolic Python

Python Source Code: Compute A = LU.

```
1  # ============================================================
2  # TestMatrixLUDecomposition01.py: Compute symbolic descriptions
3  # of matrix LU decomposition ....
4  # ============================================================
5
6  import sympy as sp
7  from sympy import Integral, Matrix, pi, pprint
8
9  def main():
10     # Define symbolic representation of matrix ...
11
12     print("--- Part 1: Compute LU decomposition of 2x2 matrix equations ...");
13
14     a, b, c, d, e, f = sp.symbols('a,b,c,d,e,f')
15
16     A = sp.Matrix(([a,b],[c,d]))
17     B = sp.Matrix([e,f])
18
19     pprint(A)
20
21     L, U, P = A.LUdecomposition()
22
23     print("--- Lower triangular matrix ...\n");
24
25     pprint(L)
26
27     print("--- Upper triangular matrix ...\n");
```

Example 3: LU Decomposition with Symbolic Python

Python Source Code: Continued ...

```
28
29      pprint(U)
30
31      print("--- Check matrix product L*U ...\n");
32
33      pprint(L*U)
34
35      print("--- Part 2: Symbolic LU decomposition of 3x3 matrix equations ...");
36
37      g, h, i = sp.symbols('g,h,i')
38      j, k, l = sp.symbols('j,k,l')
39
40      A = sp.Matrix(([a,b,c],[d,e,f], [g,h,i] ))
41      B = sp.Matrix([j,k,l])
42
43      pprint(A)
44
45      L, U, P = A.LUdecomposition()
46
47      print("--- Lower triangular matrix ...\n");
48
49      pprint(L)
50
51      print("--- Upper triangular matrix ...\n");
52
53      pprint(U)
```

Example 3: LU Decomposition with Symbolic Python

Python Source Code: Continued ...

```
55      print("--- Check matrix product L*U ...\n");
56
57      pprint(L*U)
58
59      print("--- Part 3: LU decomposition for sample 3x3 matrix equations ...");
60
61      print("--- Set values in L ...");
62      print("---   L = L.subs( {a:3, b:-6, c:7, d:9, e:0, f:-5, g:5, h:-8, i:6 } ) \n")
63
64      L = L.subs( {a:3, b:-6, c:7, d:9, e:0, f:-5, g:5, h:-8, i:6 } )
65
66      pprint(L)
67
68      print("--- L.det() = {:s} ...".format( str( sp.det(L) ) ))
69
70      print("--- Set values in U ...");
71      print("---   U = U.subs( {a:3, b:-6, c:7, d:9, e:0, f:-5, g:5, h:-8, i:6 } ) \n")
72
73      U = U.subs( {a:3, b:-6, c:7, d:9, e:0, f:-5, g:5, h:-8, i:6 } )
74
75      pprint(U)
76
77      print("--- U.det() = {:s} ...".format( str( sp.det(U) ) ))
78
79  # call the main method ...
80
81  main()
```

Example 3: LU Decomposition with Symbolic Python

Abbreviated Output: Part 1: decomposition of 2x2 matrices

```
| a  b |
|      |
| c  d |
```

```
--- Lower triangular matrix
```

```
| 1  0 |
|      |
| c    |
| -  1 |
| a    |
```

```
--- Upper triangular matrix
```

```
| a      b    |
|             |
|        b.c  |
| 0  d - ---  |
|        a    |
```

```
--- Check matrix product L*U ...
```

```
| a  b |
|      |
| c  d |
```

Example 3: LU Decomposition with Symbolic Python

Abbreviated Output: Part 2: decomposition of 3x3 matrices

```
| a  b  c |
|         |
| d  e  f |
|         |
| g  h  i |
```

```
--- Lower triangular matrix
```

```
| 1     0     0 |
|               |
| d             |
| -     1     0 |
| a             |
|               |
|       b.g     |
|   h - ---     |
| g       a     |
| -  -------  1 |
| a     b.d     |
|   e - ---     |
|       a       |
```

```
--- Upper triangular matrix ...
```

```
| a     b                    c                  |
|                                                |
|       b.d                   c.d                |
| 0  e - ---              f - ---                |
|       a                     a                  |
|                                                |
|                 (    c.d) (    b.g)            |
|                 (f - ---).(h - ---)            |
|                 (     a ) (     a )   c.g      |
| 0     0     i - ------------------ - ---       |
|                         b.d             a      |
|                    e - ---                     |
|                         a                      |
```

Example 3: LU Decomposition with Symbolic Python

Abbreviated Output: Part 2: decomposition of 3x3 matrices

```
| a  b  c |  # <--- Check matrix product L*U ...
|         |
| d  e  f |
|         |
| g  h  i |

--- Check results with sample (3x3) matrix (see equation 25) ...

--- Set values in L ...
--- L = L.subs( { a:3, b:-6,  c:7,
                  d:9,  e:0,  f:-5,
                  g:5, h:-8,  i:6 } )

| 1    0    0 |
|               |
| 3    1    0 |
|               |
| 5/3  1/9  1 |

--- L.det() = 1,

--- Set values in U ...
--- U = U.subs( { a:3, b:-6,  c:7,
                  d:9,  e:0,  f:-5,
                  g:5, h:-8,  i:6 } )

| 3   -6     7    |
|                 |
| 0   18    -26   |
|                 |
| 0    0   -25/9 |

--- U.det() = -150
```

Example 3: LU Decomposition with Symbolic Python

Points to note:

- Theoretical analysis indicates:

$$det(A) = det(L)det(U) = 1 * -150 = -150. \quad (34)$$

  The symbolic analysis works!
- The analysis only works when the parameter "a" is non-zero.
- To avoid division-by-zero, we need a way to shuffle the matrix rows.

Example 4: LU Decomposition with Row Permutations

Problem Statement: Compute LU decomposition with row permutation matrix P, i.e.,

$$A = PLU, \tag{35}$$

then solve matrix equations 25:

Note: Here, P is an orthogonal matrix (i.e., $P^T = P^{-1}$). Thus, solving AX = B translates to LUX = P^T B.

Solution Procedure:

- Create matrices A and B.
- Use scipy.linalg to compute A = PLU.
- Solve LY = P^T B (forward substitution).
- Solve UX = Y (backward substitution).

Example 4: LU Decomposition with Row Permutations

Python Source Code:

```
1   # ============================================================================
2   # TestLUDecomposition01.py: Use scipy.linalg to compute LU decomposition with
3   # row permutation matrix P.
4   # ============================================================================
5
6   import math
7   import numpy as np
8   import scipy.linalg as la
9   import LinearMatrixEquations as lme
10
11  def main():
12      print("--- Problem 1: Create (3x3) matrix A, (3x1) matrix B ... ");
13
14      print("--- Step 1: Define A and B matrices ... ");
15
16      A = np.array([ [ 3, -6,  7], [ 9,  0, -5], [ 5, -8,  6] ]);
17      B = np.array([ [3], [3], [-4] ]);
18
19      lme.printmatrix("A", A);
20      lme.printmatrix("B", B);
21
22      print("--- Step 2: Compute A = PLU Decomposition ... ");
23
24      P, L, U = la.lu(A)
25
26      lme.printmatrix("P", P);
27      lme.printmatrix("L", L);
```

Example 4: LU Decomposition with Row Permutations

Python Source Code: Continued ...

```
28      lme.printmatrix("U", U);
29
30      print("--- Step 3: Check decomposition A = PLU ... ");
31
32      lme.printmatrix("np.dot(P,np.dot(L,U))", np.dot(P,np.dot(L,U)) );
33
34      print("--- Step 4: Compute Z = P^T.B (note: matrix P is orthogonal) ... ");
35
36      Z = np.dot(P.T,B);
37      lme.printmatrix("Z", Z);
38
39      print("--- Step 5: Compute forward substitution ... ");
40
41      Y = la.solve_triangular(L, Z, lower=True)
42
43      print("--- Step 6: Compute backward substitution ... ");
44
45      X = la.solve_triangular(U, Y, lower=False)
46
47      lme.printmatrix("Solution: X", X);
48
49  # call the main method ...
50
51  main()
```

Example 4: LU Decomposition with Row Permutations

Abbreviated Output: ...

```
--- Problem 1: Create (3x3) matrix A, (3x1) matrix B ...
--- Step 1: Define A and B matrices ...

Matrix: A                                Matrix: B
   3.0e+00 -6.0e+00  7.0e+00                3.0e+00
   9.0e+00  0.0e+00 -5.0e+00                3.0e+00
   5.0e+00 -8.0e+00  6.0e+00               -4.0e+00

--- Step 2: Compute A = PLU Decomposition ...

Matrix: P
   0.0e+00  0.0e+00  1.0e+00
   1.0e+00  0.0e+00  0.0e+00
   0.0e+00  1.0e+00  0.0e+00

Matrix: L                                Matrix: U
   1.0e+00  0.0e+00  0.0e+00                9.0e+00  0.0e+00 -5.0e+00
   5.5e-01  1.0e+00  0.0e+00                0.0e+00 -8.0e+00  8.7e+00
   3.3e-01  7.5e-01  1.0e+00                0.0e+00  0.0e+00  2.0e+00
```

Example 4: LU Decomposition with Row Permutations

Abbreviated Output: Continued ...

```
--- Step 3: Check decomposition A = PLU ...

Matrix: np.dot(P,np.dot(L,U))
   3.0000000e+00   -6.0000000e+00    7.0000000e+00    <---  It works !!!
   9.0000000e+00    0.0000000e+00   -5.0000000e+00
   5.0000000e+00   -8.0000000e+00    6.0000000e+00

--- Step 4: Compute Z = P^T.B (note: matrix P is orthogonal) ...

Matrix: Z
   3.0000000e+00
  -4.0000000e+00
   3.0000000e+00

--- Steps 5-6: Compute forward/backward substitution ...

Matrix: Solution: X   <---  It works !!!
   2.0000000e+00
   4.0000000e+00
   3.0000000e+00
```

Python Code Listings

Code 1: Matrix Row Operations for Gauss Elimination

```
1   # ========================================================================
2   # LinearMatrixEquations.py: Functions to compute operations on linear matrix
3   #                           equations.
4   # ========================================================================
5
6   import math
7   import numpy as np
8
9   # ========================================================================
10  # LinearMatrixEquations.printmatrix(): Print two-dimensional matrices.
11  #
12  # Args: name: string description of matrix.
13  #       A (nxn) matrix.
14  #
15  # Returns: void.
16  # ========================================================================
17
18  def printmatrix(name, a):
19      print("");
20      print("Matrix: {:s} ".format(name) );
21      for row in a:
22          for col in row:
23              print("{:16.7e}".format(col), end=" ")
24          print("")
25
26  # ========================================================================
27  # LinearMatrixEquations.rowswap(): Create duplicate of matrix, then
28  #                                  swap rows.
```

Code 1: Matrix Row Operations for Gauss Elimination

```
29  #
30  # Args: A (nxn) numpy array.
31  #       k: row to be swapped.
32  #       l: row to be swapped.
33  #
34  # Returns: B (nxn) copy of A with rows swapped.
35  # ================================================================
36
37  def rowswap(A,k,l):
38      m = A.shape[0]   # m is number of rows in A
39      n = A.shape[1]   # n is number of columns in A
40
41      B = np.copy(A).astype('float64')
42
43      for j in range(n):
44          temp = B[k][j]
45          B[k][j] = B[l][j]
46          B[l][j] = temp
47
48      return B
49
50  # ================================================================
51  # LinearMatrixEquations.rowscale(): Create duplicate of matrix, then
52  #                                   scale specified row.
53  #
54  # Args: A (nxn) numpy array.
55  #     : k (int: is matrix row number to be scaled.
56  #     : scale (double): scale factor.
```

Code 1: Matrix Row Operations for Gauss Elimination

```
57  #
58  # Returns: B (nxn) copy of A with row k multiplied by scale ...
59  # ================================================================
60
61  def rowscale(A,k,scale):
62
63      m = A.shape[0]  # m is number of rows in A
64      n = A.shape[1]  # n is number of columns in A
65
66      B = np.copy(A).astype('float64')
67
68      for j in range(n):
69          B[k][j] *= scale
70
71      return B
72
73  # ================================================================
74  # LinearMatrixEquations.rowadd(): Add multiple of a row to another row.
75  #
76  # Args: A (nxn) numpy array.
77  #     : k (int: is matrix row number to be added to.
78  #     : l (int: is matrix row number to be scaled.
79  #     : scale (double): scale factor.
80  #
81  # Returns: B (nxn) copy of A with row k multiplied by scale ...
82  # ================================================================
83
84  def rowadd(A,k,l,scale):
85      m = A.shape[0]  # m is number of rows in A
86      n = A.shape[1]  # n is number of columns in A
```

Code 1: Matrix Row Operations for Gauss Elimination

```
88        B = np.copy(A).astype('float64')
89
90        for j in range(n):
91            B[l][j] += B[k][j]*scale
92
93        return B
```

Code 2: Gauss Elimination with Partial Pivoting

```
1   # ========================================================================
2   # LinearMatrixEquations.py: Functions to compute operations on linear matrix
3   #                           equations.
4   # ========================================================================
5
6   import math
7   import numpy as np
8
9   # ========================================================================
10  # LinearMatrixEquations.rowreduction(): Computes row reduction to echelon form.
11  #
12  # Args: A: an augmented matrix of dimension n x (n+1) associated with a
13  #          linear system.
14  #
15  # Returns: B, a numpy array that represents the row echelon form of A.
16  #
17  # Note: RowReduction may not return correct results if the the matrix A does
18  # not have a pivot in each column (i.e., the matrix is rank deficient).
19  # ========================================================================
20
21  def rowreduction(A):
22
23      m = A.shape[0]  # A has m rows
24      n = A.shape[1]  # It is assumed that A has m+1 columns
25
26      B = np.copy(A).astype('float64')
27
28      # For each step of elimination, we find a suitable pivot, move it into
```

Code 2: Gauss Elimination with Partial Pivoting

```
29        # position and create zeros for all entries below.
30
31        for k in range(m):
32            # Set pivot as (k,k) entry
33            pivot = B[k][k]
34            pivot_row = k
35
36            # Find a suitable pivot if the (k,k) entry is zero
37
38            while(pivot == 0 and pivot_row < m-1):
39                pivot_row += 1
40                pivot = B[pivot_row][k]
41
42            # Swap row if needed
43
44            if (pivot_row != k):
45                B = rowswap(B,k,pivot_row)
46
47            # If pivot is nonzero, carry on with elimination in column k
48
49            if (pivot != 0):
50                B = rowscale(B,k,1./B[k][k])
51                for i in range(k+1,m):
52                    B = rowadd(B,k,i,-B[i][k])
53            else:
54                print("Pivot could not be found in column",k,".")
55
56        return B
```

Code 2: Gauss Elimination with Partial Pivoting

```
57
58  # ================================================================================
59  # LinearMatrixEquations.backsubstitution(): Returns an (nx1) vector that is the
60  # solution to the matrix system UX = B.
61  #
62  # Args: U: A numpy array that represents an upper triangular square mxm matrix.
63  #       B: A numpy array that represents an nx1 vector
64  #
65  # Returns: X, A (nx1) numpy array representing the solution to UX = B.
66  # ================================================================================
67
68  def backsubstitution(U,B):
69
70      # m is number of rows and columns in U
71
72      m = U.shape[0]
73      X = np.zeros((m,1))
74
75      # Calculate entries of X backward from m-1 to 0
76
77      for i in range(m-1,-1,-1):
78          X[i] = B[i]
79          for j in range(i+1,m):
80              X[i] -= U[i][j]*X[j]
81          if (U[i][i] != 0):
82              X[i] /= U[i][i]
83          else:
84              print("Zero entry found in U pivot position",i,".")
85      return X
```

Code 2: Gauss Elimination with Partial Pivoting

```
86
87  # ============================================================================
88  # LinearMatrixEquations.backsubstitution(): Returns an (nx1) vector that is the
89  # solution to the matrix system AX = B.
90  #
91  # Args: A: A numpy array that represents a matrix of dimension n x n.
92  #       B: A numpy array that represents a matrix of dimension n x 1.
93  #
94  # Returns: X, A (nx1) numpy array representing the solution to AX = B.
95  #
96  # Note: Computational procedure will fail if AX = B does not have a
97  #       unique solution.
98  # ============================================================================
99
100 def solvesystem(A,B):
101
102     # Step 1: Check shape of A
103
104     if (A.shape[0] != A.shape[1]):
105         print("--- ERROR: solvesystem accepts only square arrays ...")
106         return
107     n = A.shape[0]  # n is number of rows and columns in A
108
109     # Step 2: Join A and B to make the augmented matrix
110
111     A_augmented = np.hstack((A,B))
112
113     # Step 3: Carry out elimination
```

Code 2: Gauss Elimination with Partial Pivoting

```
114
115         R = rowreduction(A_augmented)
116
117         # Step 4: Split R back to nxn piece and nx1 piece
118
119         B_reduced = R[:,n:n+1]
120         A_reduced = R[:,0:n]
121
122         # Step 5: Compute back substitution
123
124         X = backsubstitution( A_reduced, B_reduced)
125
126         return X
```