Department of Civil and Environmental Engineering, Fall Semester, 2024

ENCE 201: Solutions to Midterm 2

Question 1: 15 points.

Consider the family of matrix equations Ax = b, where:

$$\begin{bmatrix} 3 & -1 & 5 \\ 1 & 2 & -3 \\ 4 & 1 & a^2-14 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ a+2 \end{bmatrix}. \tag{1}$$

Part [1a] Derive an expression for the det(A) as a function of 'a'. Determine the values of 'a' for which matrix A will be singular.

Solution: We begin with:

$$det(A) = 3 \cdot det \begin{bmatrix} 2 & -3 \\ 1 & a^2-14 \end{bmatrix} + 1 \cdot det \begin{bmatrix} 1 & -3 \\ 4 & a^2-14 \end{bmatrix} + 5 \cdot det \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} = 7a^2 - 112. \tag{2}$$

Setting det(A) = $7a^2 - 112$ = 0 gives a = -4 or a = 4.

Part [1b] Based on your solutions to part `1a`, what can you say about: (1) the rank of the system of equations, and (2) the number of solutions to the matrix equations?

Solution: Three key points:

- det(A) $\neq$ 0 when $a \neq \pm 4$. rank(A) = 3.
- When a = 4:

$$\left[\begin{array}{ccc|c} 3 & -1 & 5 & 2 \\ 1 & 2 & -3 & 4 \\ 4 & 1 & 2 & 6 \end{array}\right] \xleftarrow{r_3 \leftarrow r_3 - r_1 - r_2} \left[\begin{array}{ccc|c} 3 & -1 & 5 & 2 \\ 1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{3}$$

  rank(A) = rank(A||b) = 2 $\longrightarrow$ Infinite solutions.

- When a = -4:

$$\left[\begin{array}{ccc|c} 3 & -1 & 5 & 2 \\ 1 & 2 & -3 & 4 \\ 4 & 1 & 2 & -2 \end{array}\right] \xleftarrow{r_3 \leftarrow r_3 - r_1 - r_2} \left[\begin{array}{ccc|c} 3 & -1 & 5 & 2 \\ 1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 8 \end{array}\right] \tag{4}$$

rank(A) $\neq$ rank(A$\|$b) $\longrightarrow$ No solutions.

Part [1c] Use the method of Gauss Elimination (i.e., row operations followed by back substitution) to show that solutions to matrix equation 1 can be written:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (8a+25)/(7a+28) \\ (10a+54)/(7a+28) \\ 1/(a+4) \end{bmatrix}. \tag{5}$$

This is a hand calculation, so show all of your working.

Solution: We begin with the augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & -1 & 5 & 2 \\ 1 & 2 & -3 & 4 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right] \tag{6}$$

Design sequence of row operations:

- Swap rows 1 and 2 (i.e., $r_1 \longleftrightarrow r_2$).
- Replace row 2 by row 2 minus 3 times row 1 (i.e., $r_2 \longleftarrow r_2 - 3r_1$).
- Replace row 3 by row 3 minus 4 times row 1 (i.e., $r_3 \longleftarrow r_3 - 4r_1$).
- Normalize row 2 (i.e., $r_2 \longleftarrow r_2/-7$).
- Normalize row 3 (i.e., $r_3 \longleftarrow r_2/-7$).
- Replace row 3 by row 3 minus row 2 (i.e., $r_3 \longleftarrow r_3 - r_2$).
- Multiply row 3 by 7 (i.e., $r_3 \longleftarrow 7r_3$).

Equation 6 in echelon form is:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right] \tag{7}$$

Apply back substitution:

$$x_3 = \left[\frac{a-4}{a^2-16}\right] = \left[\frac{1}{a+4}\right]. \tag{8}$$

$$x_2 - 2x_3 = 10/7 \quad \longrightarrow \quad x_2 = \left[\frac{10a+54}{7(a+4)}\right]. \tag{9}$$

$$x_1 + 2x_2 - 3x_3 = 4 \quad \longrightarrow \quad x_1 = \left[\frac{8a+25}{7(a+4)}\right]. \tag{10}$$

Question 2: 15 points.

This question covers numerical solutions to the root of the equation:

$$f(x) = 1 - e^{-(x-1)^2} = 0. \tag{11}$$

at x = 1. Figure 1 plots f(x) over the range $[-2, 4]$.

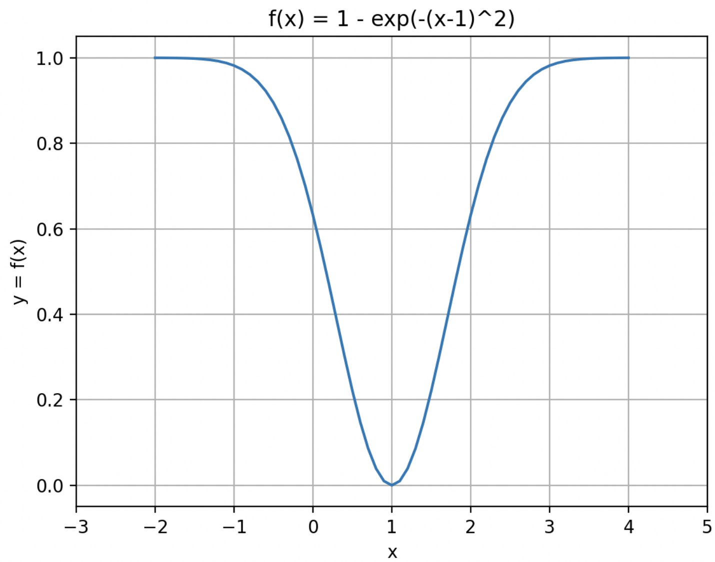

Figure 1. Plot y = f(x) vs x.

Part [2a]. Show that the Newton-Raphson update formula for solutions to equation 11 can be written:

$$x_{n+1} = x_n - \left[\frac{1 - e^{-(x_n-1)^2}}{(2x_n - 2) \cdot e^{-(x_n-1)^2}} \right]. \tag{12}$$

State all of your assumptions and show all of your working.

Solution: The standard form for Newton-Raphson update is:

$$x_{n+1} = x_n - \left[\frac{f(x_n)}{f'(x_n)} \right] \tag{13}$$

where,

$$f(x_n) = 1 - e^{-(x_n-1)^2} \tag{14}$$

and

$$f'(x_n) = (2x_n - 2)\, e^{-(x_n-1)^2} \tag{15}$$

Plugging equations 14 and 15 into 13 and rearranging terms gives the required result.

Part [2b]. Briefly explain why iterations of equation 12 will struggle to converge to the root at x = 1. Be specific.

Solution: As x_n tends toward 1, $f'(x_n)$ tends toward 0.

Part [2c]. Derive a formula for numerical solutions to equation 11 using Modified Newton-Raphson. **Note:** The answer is a bit long, so I suggest you simply state formulae for the various pieces of the update and how they fit together.

Solution: The update formula for Modified Newton-Raphson is:

$$x_{n+1} = x_n - \left[\frac{f(x_n)f'(x_n)}{f'(x_n)f'(x_n) - f(x_n)f''(x_n)} \right] \tag{16}$$

The second derivative of f is:

$$f''(x_n) = -2\left(2x_n^2 - 4x_n + 1\right) e^{-(x_n-1)^2} \tag{17}$$

Equations 14, 15 and 17 are plugged into equation 16.

Part [2d]. Use a starting value $x_o = 2$ and the Modified Newton Raphson Formula to find an improved estimate of the root of the polynomial. Do no more than 1 iteration !!.

Solution: With x_o = 2, equation 14 evaluates to:

$$f(2) = 1 - e^{-(2-1)^2} = 1 - \frac{1}{e} = 0.632. \tag{18}$$

Equation 15 evaluates to:

$$f'(2) = (4-2)\, e^{-(2-1)^2} = \frac{2}{e} = 0.735. \tag{19}$$

And equation 17 evaluates to:

$$f''(2) = -2\,(2 \cdot 4 - 4 \cdot 2 + 1)\, e^{-(2-1)^2} = \frac{-2}{e} = -0.735. \tag{20}$$

Substituting equations 18 – 20 into 16 gives:

$$x_1 = 2 - \left[\frac{0.632 \cdot 0.735}{0.735 \cdot 0.735 - 0.632 \cdot -0.735}\right] = 1.537. \tag{21}$$

Question 3: 10 points.

This question covers linear algebra.

Part [3a]. Show that for all values of x, y and $a \geq 0$ the triple product of matrices:

$$\begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1+a \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \geq 0 \tag{22}$$

Solution. Multiplying out the triple product gives:

$$x^2 - xy + (1+a)y^2 - xy \tag{23}$$

Rearranging terms:

$$x^2 - 2xy + y^2 + ay^2 = (x-y)^2 + ay^2 \geq 0. \tag{24}$$

Part [3b]. Let A be a (3x3) matrix such that:

$$A \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}. \tag{25}$$

What is the result:

$$A \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} ? \tag{26}$$

Solution. Notice that:

$$A \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = A \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix} - 2A \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}. \tag{27}$$

Part [3c]. Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \tag{28}$$

represent (3x3) matrices, and

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{29}$$

Explain how the matrix products $[M]\,[A]$ and $[M]^2\,[A]$ act on the rows of A.

Solution. Each application of matrix M scrolls the rows upwards by one place. So, for example, $[M]\,[A]$ gives:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix}. \tag{30}$$

Applying $[M]$ for a second time gives:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix} = \begin{bmatrix} g & h & i \\ a & b & c \\ d & e & f \end{bmatrix}. \tag{31}$$

Applying $[M]$ for a third time brings the matrix rows in A back to their original configuration, indicating $[M]^3$ = $[I]$, the matrix identidy.

Part [3d]. Hence, write down the result of:

$$[M]^{2024}\,[A] \tag{32}$$

Solution. Given that $[M]^3$ = $[I]$, and that 2024 = 674 $\times$ 3 + 2, $[M]^{2024}$ is equivalent to $[M]^2$, which is equation 31.