Information Flow on Trees : A Survey

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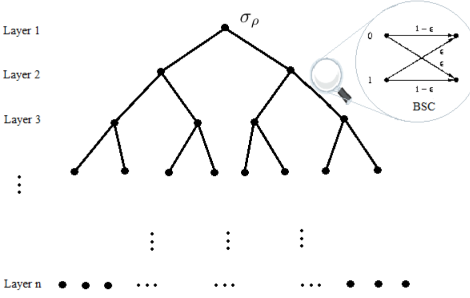
Introduction

Consider a tree $T = (V, E)$.

- Each edge behaves as an independent, identical channel M on alphabet $\mathcal{A}$.
- A symbol σ_ρ selected uniformly at the root from $\mathcal{A}$.
- For $(x, y) \in E$, x being the parent, $P(\sigma_y = j | \sigma_x = i) = M_{ij}$.



Question

Is it possible to reconstruct the root symbol σ_ρ by observing the symbols σ_{L_n} at any depth n.

Intuition

Obviously, the answer depends on T and M, but how?

Main Results

Definition

The reconstruction problem for T and M is solvable if $\forall \quad i, j \in \mathcal{A}$,

$$\lim_{n\to\infty} D_V(P_n^i, P_n^j) > 0 \tag{1}$$

where $P_n^l \equiv P(\sigma_{L_n} | \sigma_\rho = l)$.

- Intuitively, if the conditional distributions are different, we can distinguish.
- Similar definition possible in terms of the mutual information $I(\sigma_\rho; \sigma_{L_n})$.

Theorem

Let M be the Binary Symmetric Channel (BSC(ϵ)). Consider the problem of reconstructing σ_ρ from σ_{L_n} of T,

1. *If $br(T)(1-2\epsilon)^2 > 1$, then $\inf_{n\to\infty} I(\sigma_\rho; \sigma_{L_n}) > 0$.*
2. *If $br(T)(1-2\epsilon)^2 < 1$, then $\inf_{n\to\infty} I(\sigma_\rho; \sigma_{L_n}) = 0$.*

where $br(T)$ is the branching number of the tree and is a fundamental graph property [W. Evans et al., 2000].

Discussions

1. Very popular in statistical physics. Taking M to be the BSC(ϵ) gives the Ising Model.
$$\mathcal{G}(\sigma) = Z(t)^{-1} \exp\left(\frac{\sum_{u\sim v} J\sigma_u\sigma_v}{t}\right) \tag{2}$$
t : temperature, $Z(t)$: a normalizing constant, $J > 0$: the interaction strength and is related to ϵ as $\frac{\epsilon}{1-\epsilon} = \exp(-\frac{2J}{t})$. lzu.jpg
2. When only the census at the n^{th} layer is available, the reconstruction problem is census solvable when $br(T)|\lambda_2(M)|^2 > 1$ and not census solvable if $br(T)|\lambda_2(M)|^2 < 1$.
**Applicable for all channels M.
3. **Important conclusion : Reconstruction by global majority is as good as maximum likelihood reconstruction.**

Extension to DAGs

This problem was recently extended to the scenario of Directed Acyclic Graphs.

- Each edge is BSC(ϵ). Each node now has indegree $d \geq 2$. For every vertex in L_k, uniformly chose d vertices from L_{k-1} and construct the directed edges.
- Each node acts as a Boolean logic gate performing a (possibly node specific) Boolean operation on the noisy inputs to produce a single Boolean output which is then broad-casted downwards.
- Two fundamental differences from the tree model:
  - Unlike trees, the layer sizes do not need to scale exponentially.
  - The in-degree is more than one so information processing is possible at nodes.
- It was shown that the threshold is:

$$\epsilon_{maj} = \frac{1}{2} - \frac{2^{d-2}}{\lceil \frac{d}{2} \rceil \binom{d}{\lceil \frac{d}{2} \rceil}} \tag{3}$$

i.e., for $\epsilon < \epsilon_{maj}$, the majority decision rule can asymptotically recover the root-bit where as for $\epsilon > \epsilon_{maj}$ **even knowing the exact structure of the graph** and using maximum likelihood decision rule, it is not possible to recover the root [Makur *et al.*, 2019].