Node Repair on Connected Graphs

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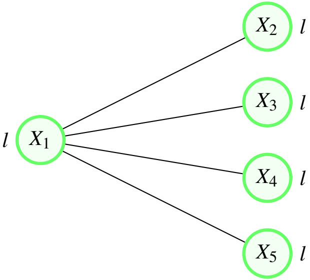
Node repair in distributed storage



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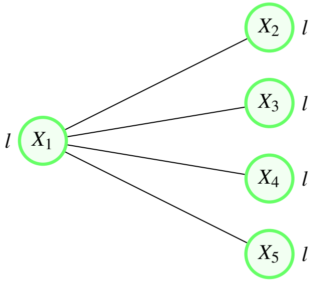

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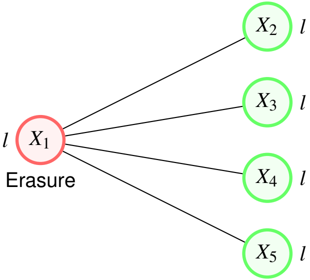

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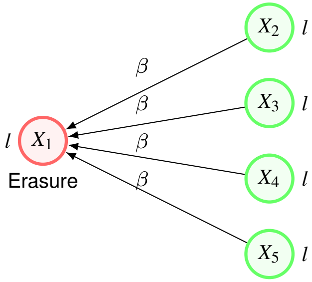

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- Total required transmission bounded by the Cut-set bound[1]

$$B \leqslant \sum_{i=0}^{k-1} \min\{l, (d-i)\beta\}$$

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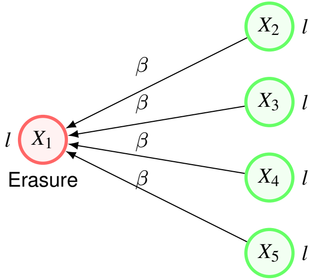

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- Different pairs of (l, β) satisfying the above with equality give rise to different points on the storage-bandwidth trade-off.

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Question 1 : Is it possible to process the data to reduce communication?

Question 2 : If so, then to what extent?

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Lemma

Let $v_f, f \in [n]$ be the failed node. For a subset of the helper nodes $E \subset D$ let R_E^f be a function of S_E^f such that

$$H(X_f|R_E^f, S_{D\setminus E}^f) = 0.$$

1) If $|E| \geqslant d-k+1$, then

$$H(R_E^f) \geqslant l.$$

2) If $|E| \leqslant d-k$, then

$$H(R_E^f) \geqslant \frac{|E|l}{d-k+1}.$$

Proof of Lemma

- Given $X_{D\setminus E}$ the information contained in R_E^f is sufficient to repair v_f, i.e.,

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- Take a set $A \subset E$ with $|A| = k - 1 - |D\setminus E|$. Now,

$$H(R_E^f, W_{D\setminus E}, W_A) = H(R_E^f, W_{D\setminus E}, W_f, W_A) \geqslant kl$$

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- The proof of Part (2) is similar.

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Proposition

Let R_j^f be the random variable denoting the information flow from the j-th layer to the $(j-1)$-th layer. Then

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Proof.

Take $E = \cup_{i=j}^t \Gamma_i(v_f)$. □

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Let $J_f = \{v \in V(T_f) \backslash \{v_f\} : |D^*(v)| \geqslant d-k+2\}$*. The total communication complexity* β_{total} *for the repair of node* v_f *on the repair tree* T_f *is bounded as*

$$\beta_{total} \geqslant \sum_{v \in J_f} l + \sum_{v \in V(T_f) \backslash (\{v_f\} \cup J_f)} \frac{|D^*(v)| l}{d-k+1}.$$

where $D^*(v)$: *set of descendants of* v *including itself.*

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- Every node $v \in J_f$ needs to transmit at least l symbols to its immediate parent.

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- The set A transmits the l-vector

$$\xi(f, A) = \sum_{h \in A} g^{(h)}(a_f) \begin{bmatrix} l_0^h + a_f^{k-1} l_{k-1}^h \\ l_1^h + a_f^{k-1} l_k^h \\ \vdots \\ l_{k-2}^h + a_f^{k-1} l_{2k-3}^h \end{bmatrix}.$$

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- It follows that the coefficients of the polynomial $g^{(f)}(z)$ is nothing but $\xi(f,D)$.

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- The same ideas of intermediate processing applies.

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- Threshold behavior: We say that t*-layer repair* of the failed node v *is possible* if

$$P(|N_t(v)| \geqslant d) \to 1 \text{ as } n \to \infty.$$

Application to random graphs: Repair threshold

Proposition

Let $d = \delta n, 0 < \delta < 1$ be a constant and let t be a fixed integer. Then t is the threshold depth for repair if

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- Using Intermediate Processing, the scaling of the bandwidth can be brought down from t to $t - s$.

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- The intermediate processing technique is applicable to all $\mathbb{F}$-linear MSR codes as well as interior point codes and achieves the minimum possible communication in some cases.
- For random graphs $\mathcal{G}_{n,p}$, in certain regimes, intermediate processing can give significant reductions in communication overhead.