More results on Regenerating Codes on Graphs

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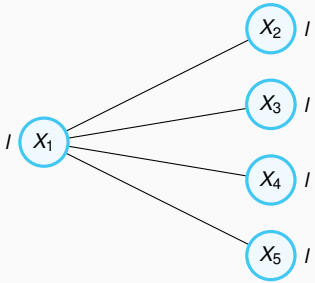
University of Maryland, College Park

Introduction

Node repair in distributed storage



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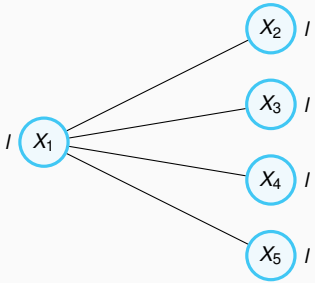

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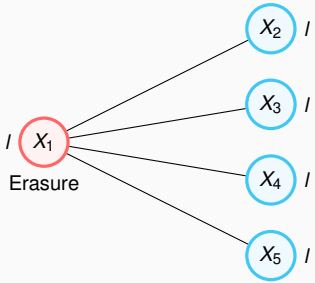

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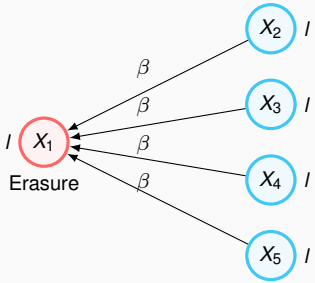

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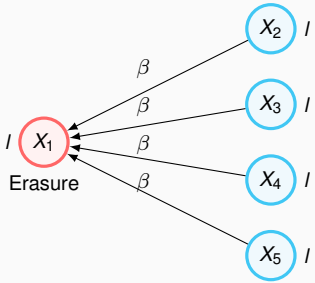

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- Different pairs of (l, β) satisfying the above with equality give rise to different points on the storage-bandwidth trade-off; important ones are the MSR and MBR points.

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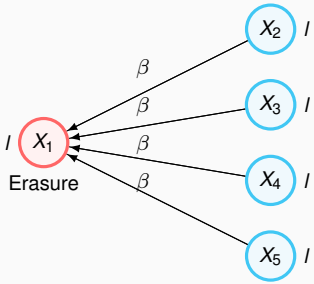

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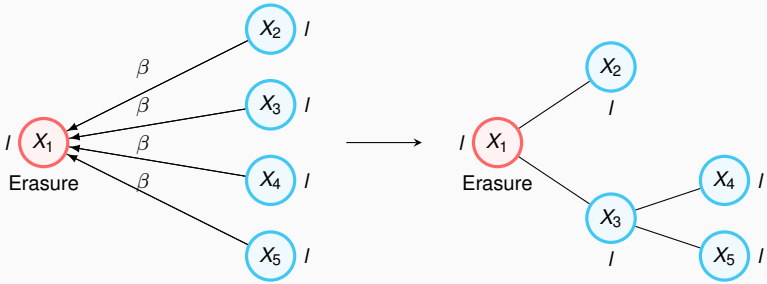

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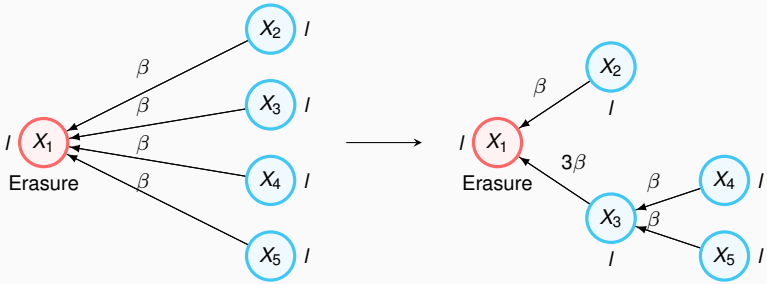

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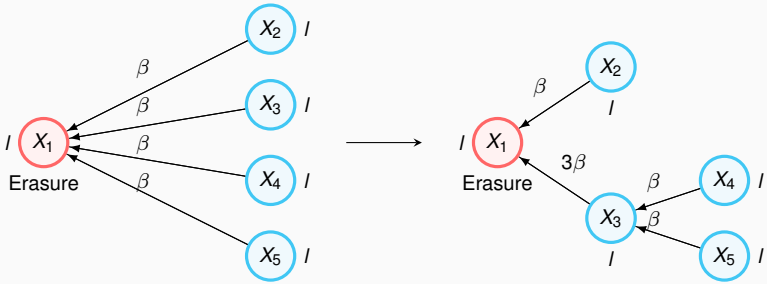

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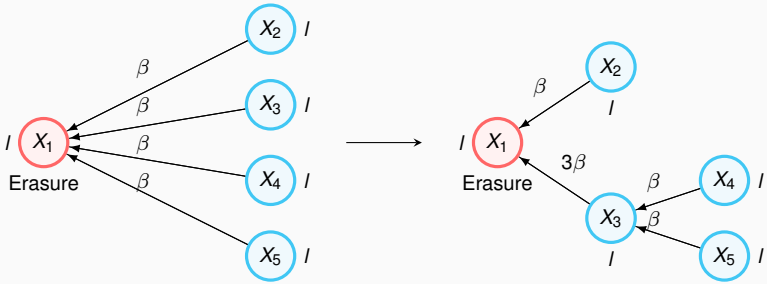

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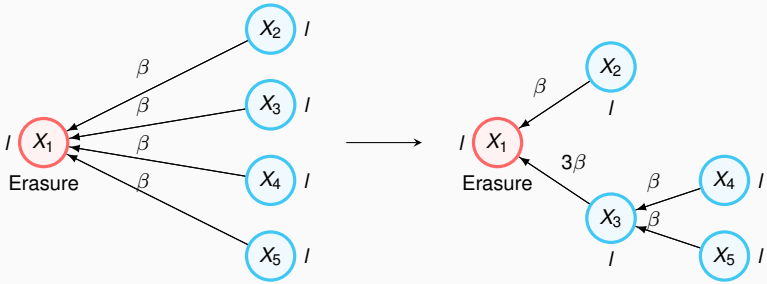

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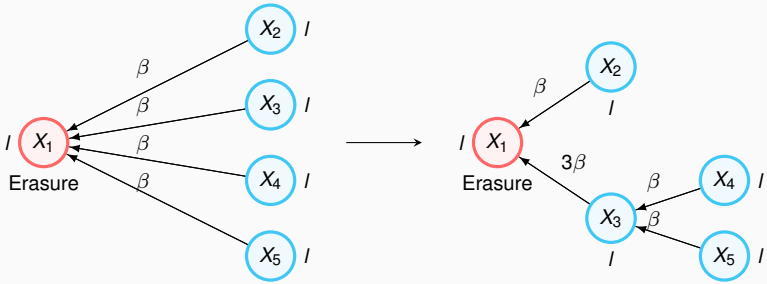

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Question : Is it possible to process the data to reduce communication?

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Generalised Version [Patra and Barg, 2022]

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Corollary

For MSR Codes $H(R_E^f) \geqslant I$.

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Are further savings possible in the above Graph Repair Setting?

Non-Uniform Contribution Model

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1. (RECONSTRUCTION) by accessing any k out of n nodes, the original file can be recovered;
2. (REPAIR) the contents of any node $f \in [n]$ can be recovered by contacting a set $D \subseteq [n] \setminus \{f\}, |D| = d$ of nodes and downloading β_i symbols from node $\tau^{-1}(i)$ for any bijective mapping $\tau : D \to [d]$.

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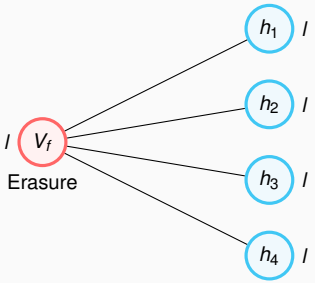

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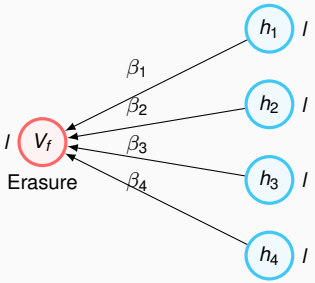

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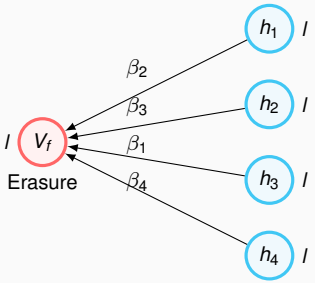

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Corollary

For MSR codes, we have

$$H(R_E^f) \geqslant l = \Delta_{d-k+1}(\mathcal{B}). \tag{2}$$

Stacking Construction

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1. *Take* $\mathcal{B} = \{\beta_j\}_{j=1}^{d}$ *non-decreasing,* $\mu_j = \mathbb{1}_{(\beta_j > \beta_{j-1})}$, $S := \{j : \mu_j = 1\}$.
2. *For each* $j \in S$ *take an MSR code* $\mathcal{C}_j$ *with parameters*

$$[n, k, d_j = d - j + 1,$$
$$l_j = (d - j - k + 2)(\beta_j - \beta_{j-1}),$$
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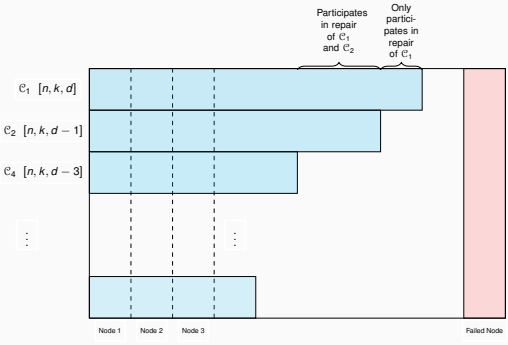

Figure 1: Stacking MSR codes

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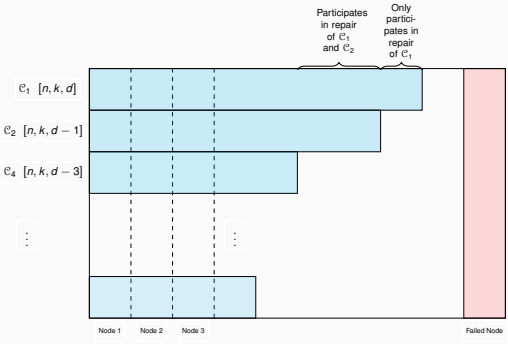

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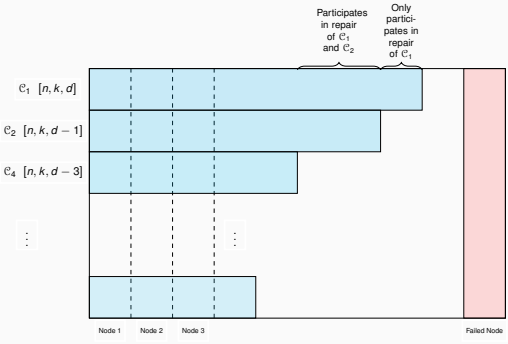

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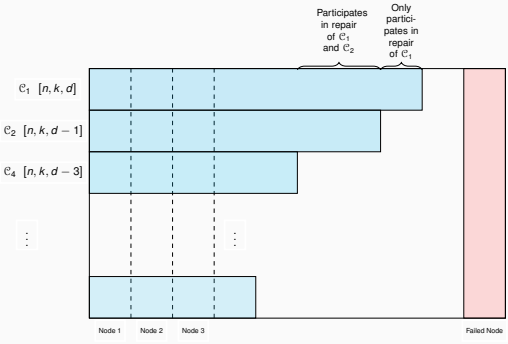

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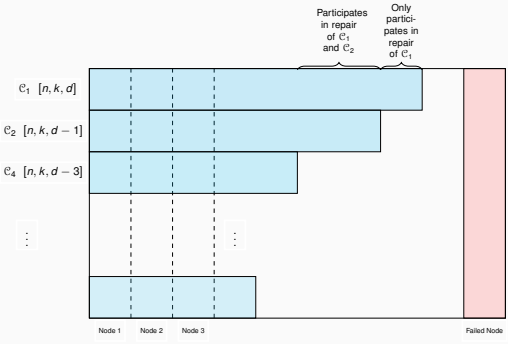

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Graph Repair

Non-Uniform Download from layers

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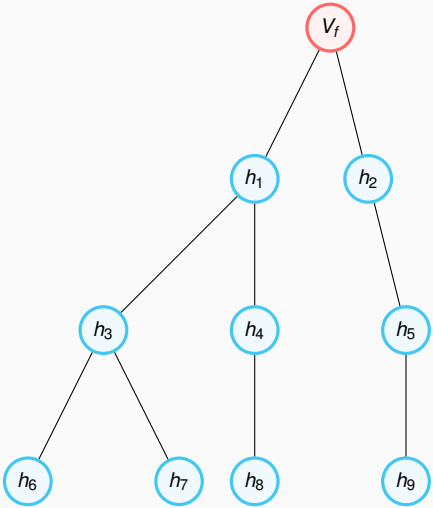
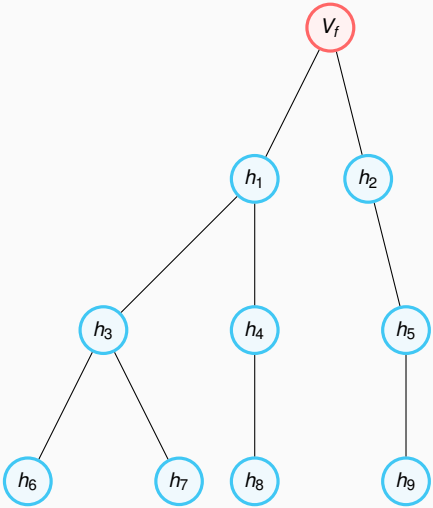


Non-Uniform Download from layers

- Helper nodes in layer i generate β_i symbols each for repair, $\beta_i \leqslant \beta_{i-1}$.



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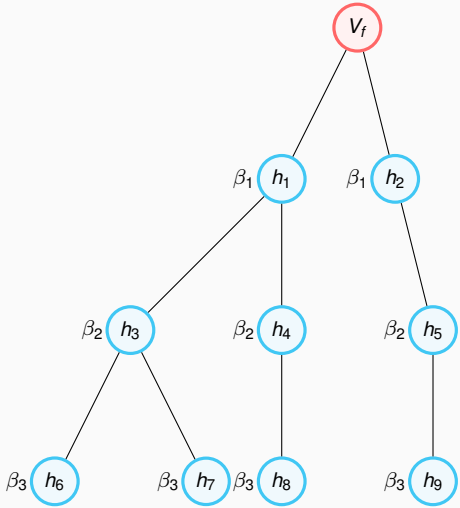

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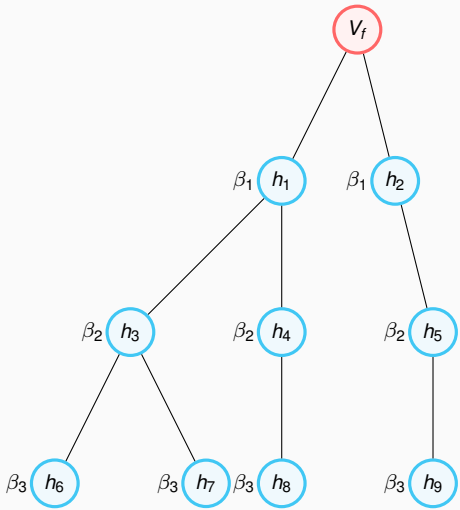

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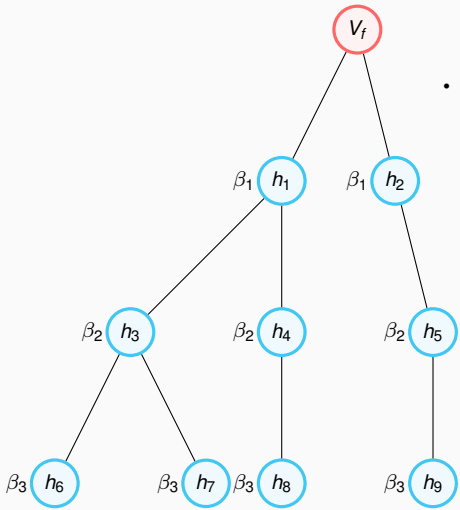

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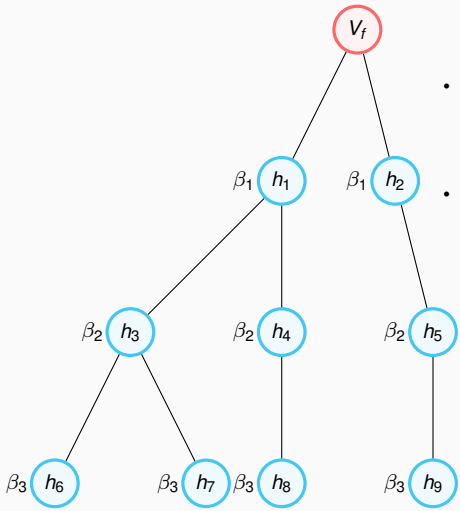

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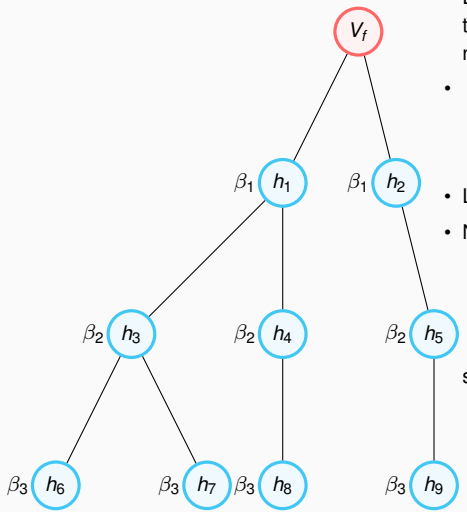

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- Non-Uniform Download is advantageous whenever:

$$\sum_{i=1}^{t} \sum_{j \in D_i \setminus J_i} \rho(j, \mathcal{P}(j)) \delta_i > 0$$

subject to

$$\sum_{i=t}^{t'+1} d_i \delta_i + \left(d - k + 1 - \sum_{i=t}^{t'+1} d_i\right) \delta_{t'} = 0.$$

where t' is the largest number such that $\sum_{i=t}^{t'} d_i \geqslant d - k + 1$.

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- **Theorem:** There exists an optimal solution with uniform downloads for some repair degree d.

Varying the Repair Degree: Example

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- Extreme case:

$$\begin{aligned}&\delta_t = \beta \implies \beta_t = 0 \\ \implies &\delta_{t-1} = -\frac{d_t\beta}{d-k+1-d_t} \\ \implies &\beta_{t-1} = \beta + \frac{d_t\beta}{d-k+1-d_t} = \frac{l}{d'-k+1}, \quad d' = d - d_t\end{aligned}$$

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- In both cases, **Intermediate Processing** of information further reduces the communication complexity.

Questions?