Node repair for Adversarial Graphical Networks

Adway Patra & Alexander Barg

(University of Maryland, College Park)

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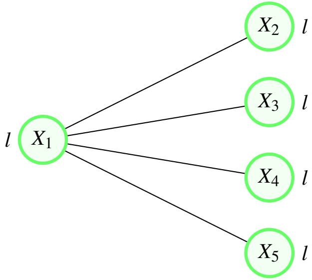
Node repair in distributed storage



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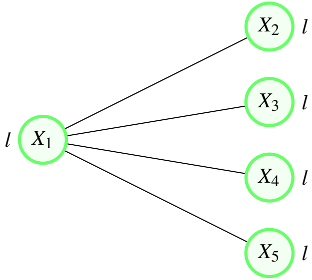

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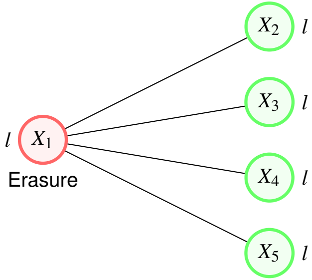

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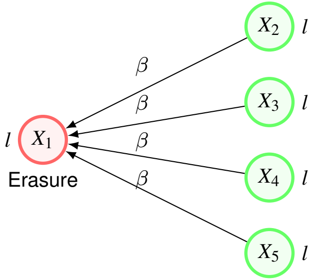

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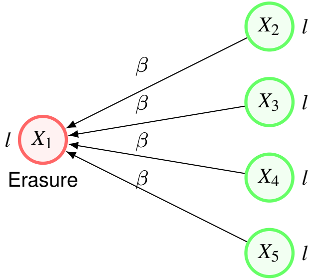

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- Different pairs of (l, β) satisfying the above with equality give rise to different points on the storage-bandwidth trade-off.

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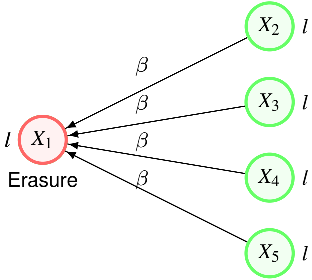

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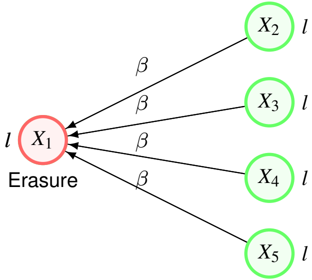

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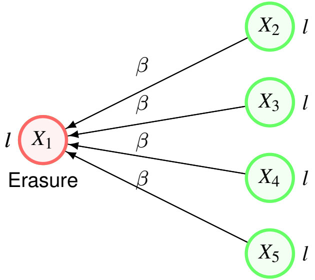

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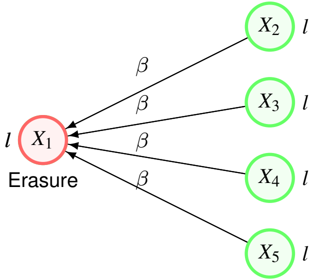

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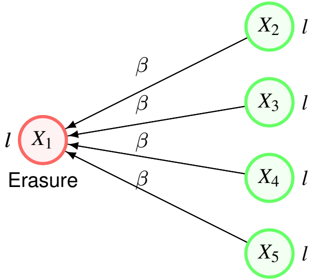

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- For interior points

$$d\beta > l > (d - k + 1)\beta.$$

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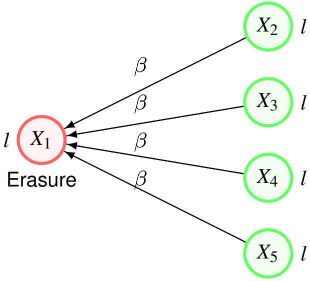

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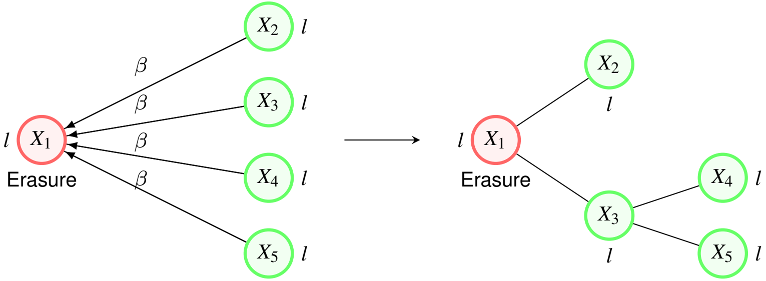

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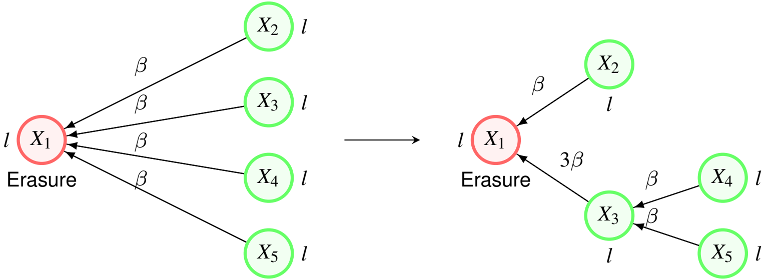

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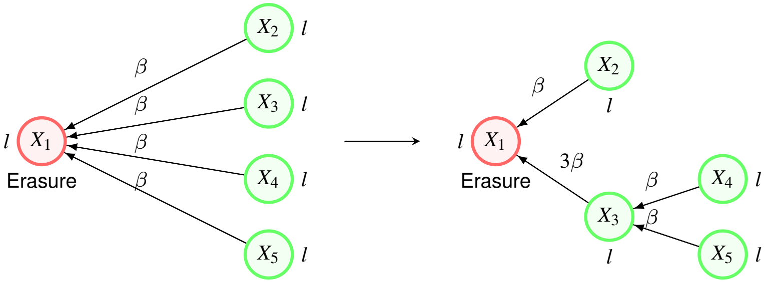

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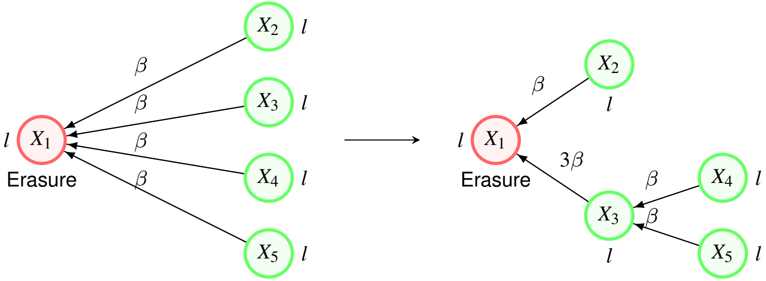

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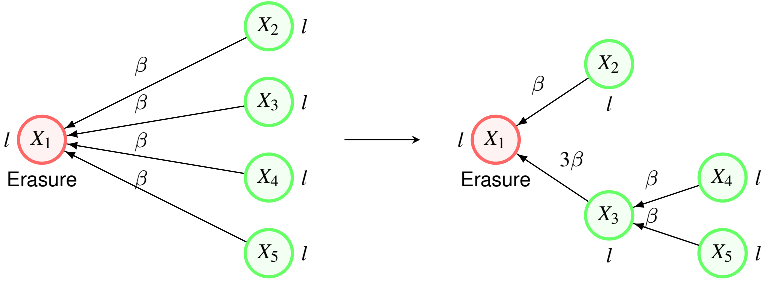

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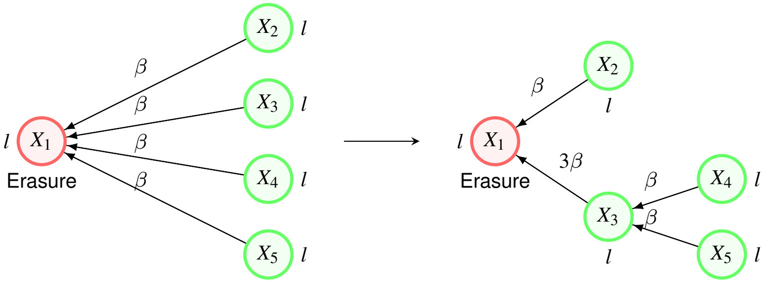

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Question : Is it possible to process the data to reduce communication?

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Generalised Version

For any Regenerating Code, if $|E| \geqslant d - k + 1$, then

$$H(R_E^f) \geqslant M - \sum_{i=1}^{k-1} \min\{l, (d-i+1)\beta\}.$$

Proof of Lemma

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$$\begin{aligned} H(R_E^f) + H(W_{D\setminus E}, W_A) &\geqslant H(R_E^f, W_{D\setminus E}, W_A) \\ H(R_E^f) &\geqslant M - H(W_{D\setminus E}, W_A) \\ &\geqslant M - \sum_{i=1}^{k-1} \min\{l, (d-i+1)\beta\} \end{aligned}$$

  using $H(W_i|W_X) \leqslant \min\{l, (d-|X|)\beta\}$ for any $i \notin X$.

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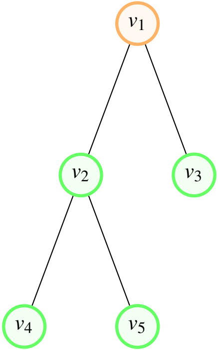
- For MSR case, $l = (d - i + 1)\beta$ and $M = kl$, hence

$$H(R_E^f) \geqslant l$$

Achieving the bounds: Using MSR Product Matrix codes[2]

Example

- Take an $[n = 5, k = 3, d = 4, l = 2, \beta = 1, M = 6]$ MSR Product Matrix code.



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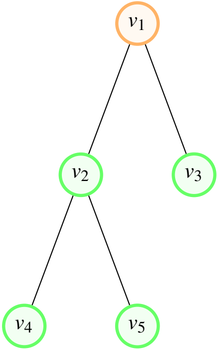

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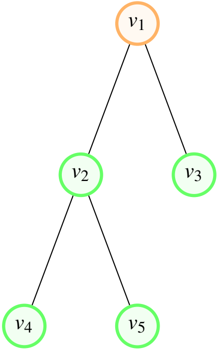

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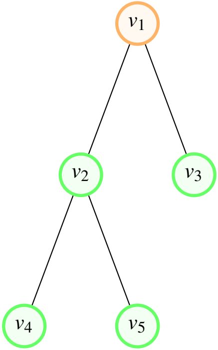

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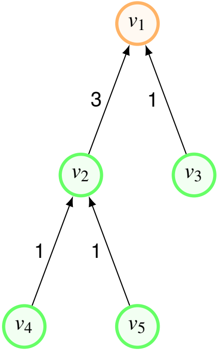

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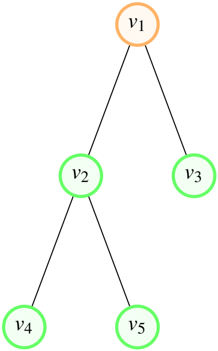
- Observe $C_1^T = \begin{bmatrix} I_2 & \lambda_1 I_2 \end{bmatrix} M\Phi_i^t = \mathcal{R}_1 \Psi_D^{-1} y_D = U^{1,D} y_D$



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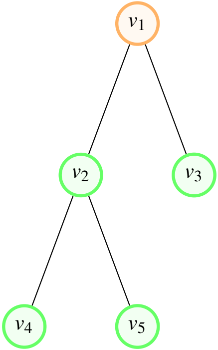

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- Observe $C_1^T = \begin{bmatrix} I_2 & \lambda_1 I_2 \end{bmatrix} M\Phi_i^t = \mathcal{R}_1 \Psi_D^{-1} y_D = U^{1,D} y_D$
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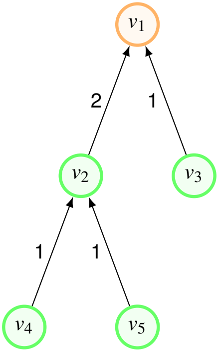

[2]Rashmi, Shah, Vijay Kumar, 2011

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Goal : Account for adversarial behavior without sacrificing the benefits of IP

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- Lower bounds and achievability?

Adversarial Setting: Lower Bounds

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Lemma

Suppose the data is encoded using an $[n, k, d, l, \beta, M]$ *MSR code on a graph. Let* v_f *be the failed node and* D *be the helper node set. Suppose that at most* t *nodes are controlled by a limited-power adversary and let* $E \subseteq D$ *be a subset of helper nodes containing them. If* $|E| \geqslant d - k + 1 + 2t$ *then*

$$c(E, v_f \cup D \setminus E) \geqslant l + 2t\beta.$$

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- The set E needs to convey the message R_E^f to the sink with $H(R_E^f) \geqslant l$ by the previous lemma.

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- Take an $[N, K, D]$ Gabidulin Code $\mathcal{C}_1$. Take an $[n, k, d, l = N, \beta, M]$ systematic MSR code $\mathcal{C}_2$. Encode each coordinate by $\mathcal{C}_1$ and then encode overall by $\mathcal{C}_2$.

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- Each systematic node now stores an N-length Gabidulin code-word.
- Hence if $D \geqslant 2t\beta + 1$, the failed node (or any faithful node along the way) receives a Gabidulin code-word with at-most $t\beta$ rank errors, it will be able to correct them.

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- If the total number of such nodes is limited, above construction still works with sufficiently large rank-metric distance.


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