Ray Splitting and Quantum Chaos

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Recent advances in the theory of the quasiclassical approximation for systems that are chaotic in the classical limit are extended to the case of ray splitting, in particular, to the splitting of an incident ray into a reflected and refracted component at a sharp interface. An instructive example is presented and novel results are found. These include evidence for ray split and periodic orbits in the spectral correlations and a new type of "scarred" eigenstate based on combining nonisolated periodic orbits whose quasiclassical contributions have a nontrivial phase from total internal reflection.

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Chaos is a well-defined concept usually applied to *deterministic nonlinear dynamical systems* which have *exponential sensitivity to initial conditions.* "Quantum chaos" is the field of study of quantum (or wave) systems whose classical limit is chaotic. The "classical limit" is $\lambda/a \to 0$, where λ is the wavelength, and a is the shortest relevant classical length. The *quasiclassical approximation* (QCA) is the most important tool of quantum chaos, as it treats the case $0 < \lambda/a \ll 1$.

Much interest exists in extending the QCA as widely as possible. There are important cases for which the classical limit does not exist or is physically irrelevant. If there is a characteristic length d for which $\lambda/d \gtrsim 1$, wave effects may persist. Simplification is achieved if $d/\lambda \to 0$ while for all other lengths $\lambda/a \ll 1$. The optical phenomena of *refraction* at well defined interfaces and *diffraction* [1] at edges or corners are no doubt the oldest and best understood effects of this type.

In this paper, we consider the case of refraction and reflection at an interface. A ray or classical trajectory *splits* into a reflected and refracted ray when it strikes the interface. This *ray splitting* is characteristic of the situation where there are just a few places that a length $d \ll \lambda$ exists. Another example is the surface of an elastic solid, where incident pressure waves split into reflected pressure and shear waves [2,3]. In general, if two or more distinct bulk waves can coexist at an interface, there will be a coupling between them describable as ray splitting. See also Refs. [2] and [4] for discussion of ray splitting and quantum chaos.

We have extended recent developments in the theory of the quasiclassical approximation in quantum chaos to the case of ray splitting at an interface. A typical model has been worked out in detail and there are a number of instructive new results. We concentrate here on the most important of these and we will give further details elsewhere [5]. In particular, we give here a generalization of Gutzwiller's trace formula, valid for ray splitting as well as nonisolated and stable periodic orbits. (The standard formula assumes hyperbolically unstable isolated periodic orbits.) It is expressed in terms of Bogomolny's transfer operator [6,7] which we generalized to ray splitting. We also use our ray-splitting model to numerically check predictions of the trace formula for peaks in the Fourier transform of the density of states. The transfer operator formalism allows several levels of the quasiclassical approximation which were checked numerically and found to be satisfactory [5]. Using the transfer operator approach we also found simple theories (one of which is discussed below) for some novel types of *scars* whose existence depends on refraction and reflection.

The model example we study is a *split circle billiard* (Fig. 1), a system which could be realized experimentally. It is described by the two-dimensional equation $[\nabla^2 + k_L^2 - u(x)V_0]\Psi(x,y) = 0$, with $\Psi = 0$ on the boundary of the unit circle $\partial C, x^2 + y^2 = r^2 = 1$, and $u(\cdot)$ is the unit step function. In the left semicircle $x < 0$, the potential energy vanishes, while on the right it is a constant, V_0 [8]. The above wave equation describes a quantum particle in units mass $m = \frac{1}{2}$, $\hbar = 1$, and energy $E = k_L^2$. Or if we consider a cylindrical microwave cavity with dielectric of index of refraction n_0 in $x < 0$ and vacuum in $x > 0$, Ψ could describe an electric field polarized in the

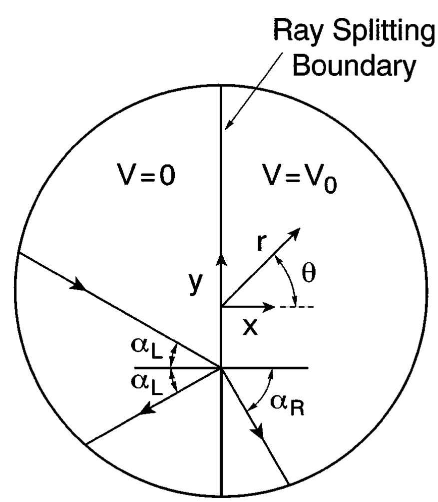

FIG. 1. Geometry of the ray-splitting billiard. A typical ray-splitting trajectory is also shown.

z direction, and $V_0 = k_L^2(1 - n_0^{-2})$. Or it could describe a drumhead with different mass membranes in left and right semicircles. We are interested in the short wavelength case $E, V_0 \gg 1$, $E/V_0 \sim 1$. The critical angle (from the normal) for internal reflection is given by $V_0/E = \cos^2 \alpha_c$, so that rays incident from the left at an angle α_L are reflected at angle α_L and refracted or transmitted at an angle α_R according to Snell's law, $\sin\alpha_L = \sin\alpha_c \sin\alpha_R$.

Bogomolny's operator $T(\theta, \theta')$ gives the semiclassical contribution of a ray emitted on ∂C at point θ' and arriving at point θ with no intermediate bounces from ∂C. T is a precise expression of Huygen's principle. Paths bouncing n times from ∂C are obtained by considering the nth power (iterate) of T. The right semicircle is described by $|\theta| < \pi/2$ (denoted $\theta \in C_R$) while $\theta \in C_L$ means $|\theta - \pi| < \pi/2$. Let $k_R \equiv \sqrt{E - V_0}$ be the wave number in C_R. If $E > V_0$ then $k_R = +i|k_R|$. The T operator can be written $T = T_d + T_r + T_t$. The *direct* orbit between θ and θ' that does not encounter the discontinuity gives a contribution

$$T_d(\theta, \theta') = -\sqrt{\frac{1}{2\pi i} \left| \frac{\partial^2 S_d(\theta, \theta')}{\partial\theta\partial\theta'} \right|} e^{iS_d(\theta,\theta')}, \quad (1)$$

where the action $S_d(\theta, \theta') = 2k_{R,L}|\sin\frac{1}{2}(\theta - \theta')|$ (θ and θ' both in C_L or both in C_R). This is the only kind of contribution for the circle billiard. The negative sign accounts for the Dirichlet condition on ∂C. For the *reflected* orbit we must generalize Bogomolny's formula Eq. (1) to include the *reflection coefficients* given by

$$r(\theta, \theta') = \pm(k_L \cos\alpha_L - k_R \cos\alpha_R)/(k_L \cos\alpha_L + k_R \cos\alpha_R) \quad (2)$$

encountered in the elementary problem of a plane wave incident on an interface. In Eq. (2) $\alpha_{L,R} = \frac{1}{2}(\theta - \theta')$, if $\theta, \theta' \in C_{L,R}$ and α_L is related to α_R by Snell's law. The positive sign is taken for $\theta \in C_L$. Then

$$T_r(\theta, \theta') = \frac{-r(\theta, \theta')}{\sqrt{2\pi i}} \sqrt{\left| \frac{\partial^2 S_r(\theta, \theta')}{\partial\theta\partial\theta'} \right|} e^{iS_r(\theta,\theta')}, \quad (3)$$

and $S_r = 2k_{L,R}|\cos\frac{1}{2}(\theta + \theta')|$.

The contribution to T of a *transmitted* or refracted orbit is

$$T_t(\theta, \theta') = \frac{-t(\theta, \theta')}{\sqrt{2\pi i}} \sqrt{\left| \frac{\partial^2 S_t(\theta, \theta')}{\partial\theta\partial\theta'} \right|} e^{iS_t(\theta,\theta')}. \quad (4)$$

In Eq. (5) (if $\theta' \in C_L$, $\theta \in C_R$) $S_t(\theta, \theta') = k_R L(\theta, \xi) + k_L L(\theta', \xi)$, where $L(\theta, \xi) = \sqrt{1 + \xi^2 - 2\xi \sin\theta}$ is the distance from a point θ on ∂C to a point on the interface $y = \xi$, $x = 0$ such that Snell's law is obeyed by the two rays. In fact, ξ minimizes S_t. The transmission coefficient is

$$t(\theta, \theta') = 2\sqrt{k_L k_R \cos\alpha_L \cos\alpha_R}/(k_L \cos\alpha_L + k_R \cos\alpha_R), \quad (5)$$

where $\tan\alpha_L = (\xi - \sin\theta_L)/\cos\theta_L$, θ_L being that argument of t which is in C_L. Note that $|r(\theta, \theta')|^2 + |t(\theta, \theta'')|^2 = 1$, where the ray reflected from θ to θ' is split and transmitted to θ''. This makes the operator T unitary in QCA. For angles $\alpha_L > \alpha_c$, k_R becomes positive imaginary, the reflection coefficient has unit magnitude but nontrivial phase, and the transmitted wave is evanescent rather than propagating.

These expressions for T may be written down almost by inspection. They can be checked by formulating the problem as an exact integral equation [5]. The Fredholm theory provides an expression for the Fredholm determinant $D(E) = \det(\mathbf{1} - \mathbf{T}(E))$ whose zeros give the spectrum [9]. The imaginary part of the logarithmic derivative of D is a generalization of the series known (in the hard chaos case) as the Gutzwiller trace formula. Using the relation between the determinant and the trace, we obtain the formal expression for the trace formula [7] for the density of states [10], $d(E) = \overline{d}(E) - \pi^{-1}\mathrm{Im}[d/dE \sum r^{-1}\mathrm{Tr}T(E)^r]$, where $\overline{d}(E)$ is the smoothed state density [11]. (See Ref. [2] for the Gutzwiller formula in the case of ray splitting with hyperbolically unstable isolated periodic orbits.)

If the integrals over ∂C involved in the trace are evaluated by the method of stationary phase, the result is expressed in terms of periodic orbits. We note that in the presence of ray splitting the term "periodic orbit" means a closed ray path traversed with periodically repeated particular choices of transmission or reflection at the ray splitting boundary. (See some examples in Fig. 2.) The trace of T^r gives the contribution of all periodic orbits with r bounces from ∂C. Such a contribution will have a phase $S_p = \oint P dQ$, the action along the orbit. The contribution of a given periodic orbit p is $a_p e^{iS_p}$. The weights a_p for nonisolated orbits can be evaluated as well as the ones for isolated orbits. In the electromagnetic version of the problem, where α_c is fixed and v_0 is proportional to k_L^2, $V_0 = \eta k_L^2$, each action $S_p = k_L s_p$ where $s_p(\eta)$ is independent of k_L. A Fourier transform with respect to k_L of our formal trace expression for $d(E)$ has peaks at the values of the transform variable $s = s_p$. (One does the transform for complex k_L thus avoiding the divergence problem.)

In Fig. 2 we show such a power spectrum [12] for our model (at $\eta = 0.5$) labeled by a schematic indicating the type of orbit giving the peak. [Several peaks appear at reduced actions half those of the orbits indicated. This is because the transform was made for those levels odd under reflection through the x axis. This symmetry can be geometrically represented by replacing the circle billiard by its upper semicircle, with Dirichlet conditions on the x

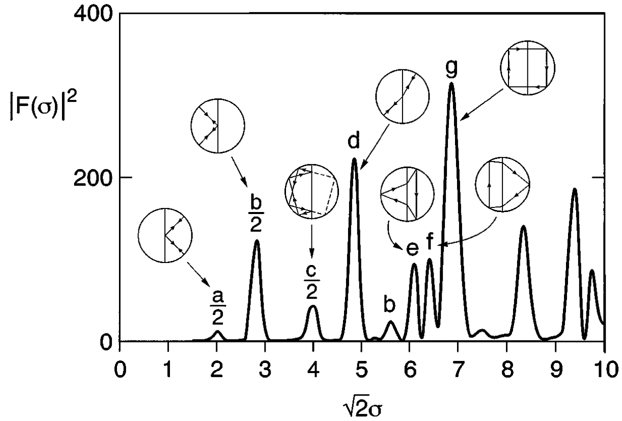

FIG. 2. Fourier transform $\int dk_L e^{-isk_L} d(k_L^2)$ (power spectrum) for $\eta = 0.5$. (The peak at a and the peak at c coincide for $\eta = 0.5$.)

axis. Neumann conditions give the even states. Retracing orbits thus desymmetrized give half the action, while orbits such as (d) are not so affected.] The first four peaks are associated with nonisolated families of periodic orbits, and the other labeled peaks are for isolated orbits [13]. We have also tracked these peaks as a function of η and found that they agree with the $s_p(\eta)$'s found from simple trigonometry. The position of the peaks is determined by simple geometry. By exploiting the T operator, we can find the amplitudes as well, even for nonisolated and stable orbits. We will report on these results elsewhere.

Figure 3 shows a few of many wave functions we have obtained numerically, together with their energy values. The notation 18+ indicates the 18th even state (which has energy 123.6). There are chaotic states, 22+, 32+, whispering gallery states 35+, 58−, internal reflection scars 18+, 31+, and states clearly based on periodic orbits refracting through the center of the circle, 19+, 63−. The regions of high probability density are dark.

We can also, in several cases, obtain semianalytic approximate solutions to the eigenvalue equation

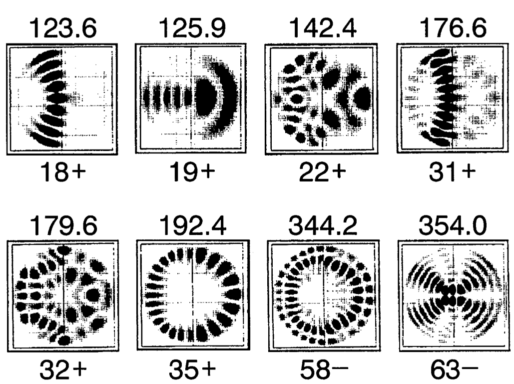

FIG. 3. Some eigenstates for $V_0 = 100$. Energies are given (above the pictured state), as well as the number of the state and its symmetry (+ for even and − for odd with respect to the x axis).

$$\int d\theta' \, T(\theta, \theta', E_a) \psi_a(\theta') = \psi_a(\theta) \,. \tag{6}$$

The most interesting example of this novel use of the transfer operator is probably that of the internal reflection scars [e.g., Fig. 3 (18+)] which are apparently based on classical periodic orbits totally internally reflected from the circle center. In the model, *any* orbit passing through the center is part of a periodic orbit. There is a continuous infinity of such periodic orbits, all with the same action, and so these periodic orbits *are not isolated.* They are marginally unstable. Prominent scars of nonisolated orbits are well known in other billiards, e.g., the Sinai billiard or the Bunimovich stadium billiard.

A remarkably simple theory of such scars features "second" WKB quantization. Since only the reflected orbit is important we approximate $T \simeq T_r$, Eq. (3). The state desired classically passes close to the center and so has low angular momentum. It thus varies comparatively slowly with θ. In fact, parametrize in WKB fashion, $\psi(\theta) = \sin(\sqrt{k_L}\, f(\theta))$, where we use the system symmetry to take the unknown function $f(\theta)$ either even or odd under $\theta \to 2\pi - \theta$. The integral Eq. (6) is done in the spirit of stationary phase. We encounter

$$F_{\pm}(\theta) = i \int d\theta' \, r(\theta, \theta') \sqrt{k_L \cos\left(\tfrac{1}{2}(\theta + \theta')\right)/2\pi} \times e^{2ik_L |\cos\left(\frac{1}{2}(\theta+\theta')\right)|} e^{\pm i\sqrt{k_L}\, f(\theta')} . \tag{7}$$

Assuming $f'(\theta) \sim 1$, the stationary point is at $\theta' = \pi - \theta$. This has the geometric meaning that the orbit reflects from the interface at the circle center. The prefactor is evaluated at the stationary point. This yields $\sqrt{k_L/2\pi}\, e^{iv(\theta)}$, where $r(\theta, -\theta) = e^{iv(\theta)}$ and $v(\theta) = -2\arccos(\cos\theta/\cos\alpha_c)$. We now expand the θ' dependence of the exponent in (7) about $-\theta$, i.e., $\cos\left(\frac{1}{2}(\theta + \theta')\right) \approx 1 - \frac{1}{8}(\theta + \theta')^2$, $f(\theta') \approx f(-\theta) + f'(-\theta)(\theta + \theta')$. Then $F_{\pm}(\theta) \approx i\exp i\{2k_L + [f'(\theta)]^2 + v(\theta)\}\exp\{\pm ik_L^{1/2} f(-\theta)\}$.

In order to have a solution of Eq. (6) the first exponential in F must be equal to $\mp i$, in the even (odd) case, respectively. This leads to the condition $[f'(\theta)]^2 + v(\theta) = \epsilon = 2\pi(p \mp 1/2) - 2k_L$. This is a WKB-like condition where f' plays the role of a momentum and $v(\theta)$ (the phase of the reflection coefficient) is an attractive potential. [Note $\epsilon \approx 1$, so it represents a small difference of two large numbers. In the language of energy levels, it is a quantum defect. This also means that the scale of variation of $\psi(\theta)$ is intermediate between k_L and unity, which is rather unusual.]

An appropriate boundary condition $\psi = 0$ at $|\theta - \pi| = \frac{1}{2}\pi$ is obtained by considering the grazing orbits. Then the direct orbit has nearly the same length as the reflected one, and contributes with the opposite sign, since the reflection coefficient is −1 at grazing incidence. This

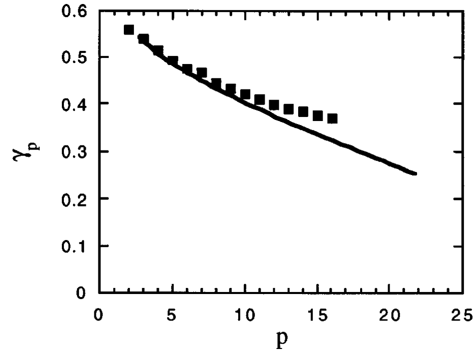

FIG. 4. The quantum defect in the semiclassical approximation of Eq. (8) (solid line) and the full quantum defects (squares).

yields the expression

$$\int_{\pi/2}^{\theta_\epsilon} d\theta \sqrt{\epsilon - v(\theta)} = \frac{\pi}{\sqrt{k_L}} (l + \delta). \tag{8}$$

Since $v(\theta)$ is an attractive "potential," there can be a "turning point" $\theta_\epsilon < \pi$ where $\epsilon = v(\theta_\epsilon)$ and in this case $\delta = 1/4$. If there is no turning point then in effect $\theta_\epsilon = \pi$ and $\delta = 0$ for odd modes and $\delta = \frac{1}{2}$ for even modes. The energies are labeled by two quantum numbers, p and l. The former counts the radial nodes and the latter the angular. The state shown has $l = 1$, $p = 3$. (Higher values of l in the range of energies we studied generally involve incident angles less than the critical, so the approximation keeping only T_r fails.) We carried out this WKB calculation and found approximate energies for the sequence of internal reflection scars. These energies are parametrized by the "quantum defect" γ_p such that $E_p = \pi^2(p + \gamma_p)^2$ and the results are compared with the numerical levels in Fig. 4.

To recapitulate, we have generalized the modern theory of the quasiclassical approximation to certain practically important cases for which the classical limit does not strictly exist, namely to the case of sharp interfaces where classical rays may split with certain quantum probabilities. Our main tool has been the generalization of the transfer operator to include probability amplitudes, including both magnitude and phase. We have found numerical confirmation that the resulting theory, an approximation at the quasiclassical level, is quite a good approximation. In particular we verified that split-ray periodic orbits give rise to distinctive energy level correlations. As an example of a result which is completely inaccessible to the standard type of trace formula, we developed a new method which explains a novel type of scarred eigenstate which depends explicitly on the phase shift of the total internal reflection. This method should be useful in other contexts.

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[9] Absolutely convergent series in terms of periodic orbits are known for this function.
[10] The sum over r diverges, and the result must be regarded as an analytic continuation of the series from unphysical parameters, e.g., from an energy with a large positive imaginary part to the physically significant real energy axis.
[11] $\overline{d}(E)$ with ray splitting is discussed by us elsewhere [R. E. Prange *et al.,* Phys. E **53**, 207 (1996)].
[12] For Fourier transformed spectral density without ray splitting, see, e.g., U. Eichmann, K. Richter, D. Wintgen, and W. Sandner, Phys. Rev. Lett. **61**, 2438 (1988).
[13] Evidence exists [5] for a periodic orbit involving a lateral ray. In this case, the lateral ray orbit is a sort of limiting case of the orbit indicated in Fig. 2(e), where the angle of incidence from the left is α_c and the part of the orbit parallel to the interface is exactly *at* the interface itself. The lateral ray is a higher order quasiclassical effect with a distinctive signature. Lateral rays are discussed in L. M. Brekhovskikh, *Waves in Layered Media* (Academic Press, New York, 1960).