PHYSICAL REVIEW LETTERS

VOLUME 53 | 3 DECEMBER 1984 | NUMBER 23

Effect of Noise on Time-Dependent Quantum Chaos

E. Ott and T. M. Antonsen, Jr.
Laboratory for Plasma and Fusion Energy Studies, University of Maryland, College Park, Maryland 20742

and

J. D. Hanson
Institute for Fusion Studies, University of Texas, Austin, Texas 78712

(Received 12 June 1984)

The dynamics of a time-dependent quantum system can be qualitatively different from that of its classical counterpart when the latter is chaotic. It is shown that small noise can strongly alter this situation.

PACS numbers: 03.65.Bz, 05.40.+j

What is the nature of a quantum system whose classical counterpart exhibits chaotic dynamics? The subfield dealing with this question has been called *quantum chaos.* A striking result in quantum chaos has been obtained by Casati *et al.*[1] These authors considered a particular Hamiltonian and a potential representing periodic impulses kicking the system. If the strength of the kicks is large enough, then, in the classical description the motion is chaotic, and the momentum variable, p, behaves diffusively. That is, the average value of p^2 apparently increases linearly with time. Casati *et al.* considered numerically the quantum mechanical version of the same problem with $\hbar$ small. They found that for early times, the average value of p^2 increased linearly with time at roughly the classical diffusive rate, but that for long time this linear increase slowed and eventually appeared to cease. Thus, there was no numerically discernible diffusion in the quantum case.

The observed saturation of the growth of $\langle p^2 \rangle$ is understandable if the Schrödinger operator for this problem has an essentially discrete quasienergy level spectrum.[1-4] Recently, Fishman, Grempel, and Prange[4] have presented strong arguments supporting the idea that the quasienergy spectra for systems of the type studied by Casati *et al.* are essentially discrete. These arguments are based on an analogy with Anderson localization of an electron in a solid with a random lattice. Futhermore, it has been pointed out that these results have implications for other physical systems[5-7] and experiments have been proposed. For example, the ionization of an atom by high-frequency electromagnetic waves and the interaction of electrons on the surface of a superconductor with an oscillating electric field have both been suggested[5,6] as systems for which the consequences of quantum localization in a classically chaotic system could be experimentally observed. A question then arises as to how sensitive the localization is to real effects not included in the model, e.g., finite bandwidth of the ionizing radiation, finite temperature, etc. In this Letter we crudely model such effects as noise. That is, we introduce a small random component into the quantum rotator equations[8] (see also Shepelyanski[3]). (Since our subsequent arguments are apparently not model dependent, we believe that they should be relevant to real physical experiments.) We find that the quantum interference leading to localization of p^2 is a delicate effect that is strongly affected by small noise. It is the goal of this paper to investigate the mechanisms by which small noise leads to diffusion, as well as the regimes of dependence of the quantum momentum diffusion coefficient on the noise and kicking strength.

We consider a Hamiltonian

$$H=\frac{P^2}{2I}+[\bar{\epsilon}R\cos\theta+\bar{\nu}\phi(\theta,t)]\sum_{n=-\infty}^{+\infty}\delta(t-nT), \quad (1)$$

where θ has period 2π, P is the angular momentum, I is the moment of inertia, $\bar{\epsilon}$ is the strength of a periodically applied (period T) horizontal impulsive force, R is the radius at which the force is applied, and the term $\phi(\theta,t)$ is a random function of time representing a noise component in the kicking with $\bar{\nu}$ a parameter governing the noise strength.

The classical problem corresponding to the Hamiltonian (1) yields the well-known *standard mapping*[9] (including noise),

$$p_{n+1}=p_n+\epsilon\sin\theta_{n+1}-\nu\phi'_{n+1}(\theta_{n+1}),$$
$$\theta_{n+1}=\theta_n+p_n,$$

where $\phi_n(\theta)=\phi(\theta,nT)$, $\phi'_n=d\phi_n/d\theta$, (p_n,θ_n) denote the values of $(p(t),\theta(t))$ just after the nth kick (at $t=nT$), and $\epsilon=\bar{\epsilon}RT/I$, $p=PT/I$, and $\nu=\bar{\nu}T/I$. One possible choice for ϕ_n that we will use in all of our subsequent calculations is $\phi_n(\theta)=\sqrt{2}\Delta_n\cos(\theta+\alpha_n)$, where Δ_n is a Gaussian random variable $\langle\Delta_n\Delta_{n'}\rangle=\delta_{nn'}$, and α_n is random with a uniform distribution in $[0,2\pi]$. For the case where ϵ is large most initial conditions for the classical map generate orbits which are diffusive with a momentum diffusion coefficient given approximately by[9,10] $D_{\rm cl}\simeq\epsilon^2/4+\nu^2/2$. Thus, if $\nu^2\ll\epsilon^2$ (which applies to all of our subsequent considerations), the noise has little effect on the value of $D_{\rm cl}$.

Turning to the quantum problem, we impose periodic boundary conditions, $\psi(\theta,t)=\psi(\theta+2\pi,t)$. Thus momenta are quantized at $p=l\hbar$ (l is an integer). Integrating Schrödinger's equation with Hamiltonian (1) through one time period,[1,11] setting $\psi_n=\psi(\theta,nT+0^+)$, and normalizing $\hbar$ to I/T gives

$$\psi_{n+1}(\theta)=\exp[(i\nu/\hbar)\phi_{n+1}(\theta)]L[\psi_n(\theta)], \quad (2a)$$

$$L[\psi(\theta)]\equiv\sum_l\int_0^{2\pi}(d\theta'/2\pi)[-i\hbar l^2/2+il(\theta-\theta')+i\epsilon\cos\theta/\hbar]\psi(\theta'). \quad (2b)$$

In what follows we shall consider $\epsilon^2\gg\nu^2$ and discuss the parameter dependence of D_q on ν, ϵ, and $\hbar$. We distinguish three regimes in terms of which we can state our main results as follows: (a) $(\epsilon/\hbar)^2\ll1$ (large $\hbar$) for which we find $D_q\simeq\nu^2/2$; (b) $(\epsilon/\hbar)^2\gg1$ and $(\nu/\hbar)^2(\epsilon/\hbar)^2\ll1$ (moderate $\hbar$) for which we find $D_q\sim\nu^2(\epsilon/\hbar)^4$; and (c) $(\nu/\hbar)^2\times(\epsilon/\hbar)^2\gg1$ (small $\hbar$) for which we find $D_q\simeq D_{\rm cl}$.

Thus, from our result for regime (c), in the "classical limit" (i.e., $\hbar\to0$) $D_q\to D_{\rm cl}$ when $\nu>0$ (see also Ref. 3). This is not so for $\nu=0$, since then the quantum diffusion coefficient is apparently zero for any $\hbar>0$ (hence, with $\nu=0$, $\lim D_q=0$ as $\hbar\to0$). Thus we may say that noise, however small, restores the classical limit. Furthermore, we emphasize that $D_q\simeq D_{\rm cl}$ can apply *even for very small noise* [i.e., $(\nu/\hbar)^2\ll1$] provided that we are in the semiclassical regime.

Regimes (a) and (b) may be treated by random-phase-approximation perturbation theory considering the effect of finite noise ($\nu>0$) as the perturbation. For $\nu=0$, we assume that (2) has an essentially discrete quasienergy spectrum.[2,4] Thus $\psi_n(\theta)$ may be expanded as $\psi_n(\theta)=\sum A_m\exp(-i\omega_m n)\times u_m(\theta)$, where from Eq. (2) the $u_m(\theta)$ and $\exp(i\omega_m)$ are the eigenfunctions and eigenvalues of the unitary operator L, $L[u_m]=\exp(-i\omega_m)u_m$. Since $\nu/\hbar\ll1$ for both regimes (a) and (b), the factor $\exp[i\nu\phi'_n(\theta)/\hbar]\simeq1+i\nu\phi'_n(\theta)/\hbar$ in Eq. (2), and, with the assumption that perturbation theory is valid, the probability per kick of a transition from u_m to $u_{m'}$ is $\alpha_{mm'}=(\nu/\hbar)^2|\langle u_{m'}|\phi'_n|u_m\rangle|^2_{\rm ave}$ where the subscript "ave" indicates an average over the ensemble of random ϕ_n. With use of the transition probability $\alpha_{mm'}$, the diffusion coefficient is

$$D_q(m)=\tfrac{1}{2}(\nu/\hbar)^2\sum_{m'}|\langle u_{m'}|\phi'_n|u_m\rangle|^2_{\rm ave}(p_{m'}-p_m)^2, \quad (3)$$

where p_m is the momentum expectation value for the state u_m. Note that, whenever Eq. (3) applies, D_q is proportional to ν^2.

We now consider regime (a). In this case the term $\exp[i(\epsilon/\hbar)\cos\theta]$ in L may be neglected to lowest order; thus the $u_m(x)$ are as in the freely rotating (unkicked) rotator, $u_m(x)\simeq(2\pi)^{-1/2}\exp(im\theta)$. For

$\phi_n=\sqrt{2}\Delta_n\cos(\theta+\alpha_n)$, we obtain $\alpha_{mm'}=(\nu/\hbar)^2 \times(\delta_{m,m'+1}+\delta_{m,m'-1})/2$. Since, in this approximation, u_m is an eigenfunction of the momentum operator corresponding to a momentum $p=m\hbar$, we obtain from (3) $D_q\simeq\nu^2/2$. This result is the same as the diffusion that one would obtain for the classical map with noise if ϵ were set equal to zero.

We now consider regimes (b) and (c). In these cases, $(\epsilon/\hbar)^2 >> 1$, and the eigenvalue problem for $u_m(\theta)$ is not analytically solvable. Thus we shall only be able to obtain estimates for D_q. First, we note that, on the basis of Anderson localization, Fishman, Grempel, and Prange[4] have argued that, in the momentum representation, the eigenfunctions are exponentially localized about the "lattice points" $p=l\hbar$. The localization length in p (which we denote Δ) is large compared to $\hbar$. Furthermore, for $(\epsilon/\hbar)^2 >> 1$, the momentum eigenfunctions,

$$\hat{u}_m(l)=(2\pi)^{-1}\int_0^{2\pi}\exp(-il\theta)u_m(\theta)\,d\theta,$$

are not smoothly varying on the lattice. That is, although *on average* there is a slow exponential decrease of $|\hat{u}_m(l)|$ with l away from the center of localization of $\hat{u}_m(l)$, there are also typically $\sim 100\%$ variations of $\hat{u}_m(l)$ on the lattice-spacing scale [i.e., typically $|\hat{u}_m(l)-\hat{u}_m(l\pm 1)|\sim|\hat{u}_m(l)|$]. This results from the factor $\exp(-i\hbar l^2/2)$ in Eq. (2b) which for large l gives each $u_m(l)$ a nearly random phase.

We now obtain an estimate of Δ using the arguments of Chirikov, Izrailev, and Shepelyanski.[2] We observe numerically, for the case with no noise, that $\langle p^2\rangle$ increases with time initially at roughly the classical rate, but then turns over at some time $n\sim n_*$. This is interpreted as being due to the excitation of many Anderson-localized modes by the initial condition (which is localized near $p=0$). Furthermore, those modes most strongly excited are those which are localized around momenta within Δ of $p=0$. Hence the effective number of modes excited by an initial condition with $p=0$ is of the order of $\Delta/\hbar$. Each mode has an associated eigenvalue $\exp(-i\omega_m)$. Thus the ω_m may be taken to lie in $[0,2\pi]$. Since there are $\Delta/\hbar$ modes, the typical frequency spacing between modes is $\delta\omega\sim 2\pi/(\Delta/\hbar)$. For $n\lesssim 1/\delta\omega$, the system does not yet "know" that the quasienergy spectrum is discrete. Thus we expect that $\langle p^2\rangle$ increases with time until $1/\delta\omega$, at which time the turnover in $\langle p^2\rangle$ should occur. Thus $n_*\sim 1/\delta\omega\sim\Delta/\hbar$. In addition, at the turnover the characteristic spread in momentum will be the localization width of the modes, i.e., $\langle p^2\rangle\sim\Delta^2$. Let n_d denote the time to classically diffuse the distance Δ, $n_d\sim\Delta^2/D_{\text{cl}}\sim\Delta^2/\epsilon^2$. Since the initial increase of $\langle p^2\rangle$ is at the classical rate, we have $n_*\sim n_d$ or $\Delta/\hbar\sim\Delta^2/\epsilon^2$, which yields the result[2] $\Delta\sim\epsilon^2/\hbar$.

Before considering regime (b), we ask what is the limit of validity of perturbation theory, Eq. (3). Localization is dependent on the maintenance of phase coherence for the time it would take a wave packet to classically diffuse the distance Δ in p (e.g., see Thouless[12]). Thus, if noise destroys this phase coherence in the time n_d, then the localized modes will also be destroyed. With localization no longer operable we expect a return to the classical result $D_q\simeq D_{\text{cl}}$. To see how much noise is needed to do this, we recall that an eigenstate in the momentum representation has $\sim 100\%$ variations down to momentum separations of $\hbar$ (the lattice spacing). Thus, if the cumulative effect of the noise scatters p by an amount equal to $\hbar$, then the phases have been randomized. Noting that $\nu^2/2$ is the component of momentum diffusion due to the noise, the time n_c for the noise to scatter p by $\hbar$ is $n_c(\nu^2/2)\sim\hbar^2$ or $n_c\sim\hbar^2/\nu^2$. Thus, if $n_c<n_d$, or $(\nu/\hbar)^2(\epsilon/\hbar)^2>1$, then we expect that $D_q\simeq D_{\text{cl}}$. This defines the boundary between regimes (b) and (c).

To estimate D_q when $(\nu/\hbar)^2(\epsilon/\hbar)^2<1$ and $(\epsilon/\hbar)^2>1$ [i.e., regime (b)], we note that the phase coherence of the waves is maintained for a time n_c. Thus we expect transitions between localized modes on this time scale. Since transitions are appreciable only for modes within a localization length of each other, $D_q\sim\Delta^2/n_c$, or $D_q\sim\nu^2(\epsilon/\hbar)^4$.

The above arguments are similar to those of Thouless[12] who considered the effect of finite temperature on localization in a solid. Thus our numerical experiments testing the above arguments (described below) may also be viewed as a test of Thouless's heuristic treatment of the low-temperature conductivity of disordered solids. To our knowledge no other numerical experiments testing Thouless's arguements exist.

The estimate $D_q\sim\nu^2(\epsilon/\hbar)^4$ can also be obtained directly from (3) as follows:

$$u_m(\theta)=\sum\hat{u}_m(l)\exp(il\theta).$$

From the fact that the $\hat{u}_m$ are localized, there are effectively of the order of $\Delta/\hbar$ appreciable terms in the sum over l. Thus, with use of the $\hat{u}_m(l)$ representation, the quantity $\langle u_{m'}|\phi_n'|u_m\rangle$, with $\phi_n=\sqrt{2}\Delta_n\cos(\theta+\alpha_n)$, will involve a sum over roughly $\Delta/\hbar$ appreciable terms. Since $\langle u_m|u_m\rangle=1$, $|\hat{u}_m(l)|^2\sim(\Delta/\hbar)^{-1}$. Now assuming that the $\hat{u}_m(l)$ are pseudorandom in l, we see that the sum involved in the calculation of $\langle u_{m'}|\phi_n'|u_m\rangle$ will be of the order of $(\Delta/\hbar)^{-1/2}$. Thus (3) yields $D_q\sim(\nu/$

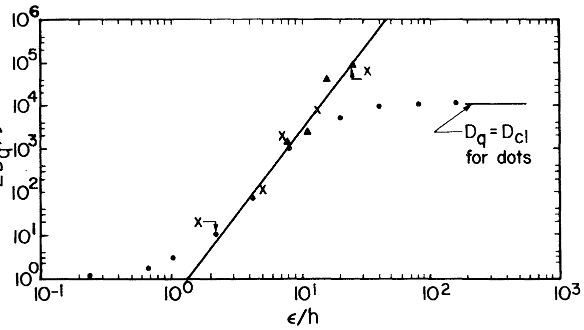

FIG. 1. $D_q/(\nu^2/2)$ vs $\epsilon/\hbar$ with $\nu = 0.0354$ in regime (b). Solid line corresponds to $D_q \propto (\epsilon/\hbar)^4$. Dots, $\epsilon = 5.0$, $\hbar$ varies; crosses, $\hbar = 5.0$, ϵ varies; triangles, $\epsilon = 55.26$, $\hbar$ varies. The iteration method is discussed by Hanson *et al.* (Ref. 11). For the dots, regime (a) corresponds to $\epsilon/\hbar \lesssim 0.8$, regime (b) to $2 \lesssim \epsilon/\hbar \lesssim 10$, and regime (c) to $\epsilon/\hbar \gtrsim 30$.

$\hbar)^2\Delta^2$ which again gives $D_q \sim \nu^2(\epsilon/\hbar)^4$.

As a test of these arguments, Fig. 1 shows numerical results obtained from long-time evolutions of Eq. (2). (Values of ϵ were chosen to avoid accelerator modes,[9] while values of $\hbar/4\pi$ are irrational to avoid quantum resonances.[13]) The dots show results for D_q versus $\epsilon/\hbar$ with $\epsilon = 5.0$, $\nu = 0.0354$, and $\hbar$ varying (horizontal axis). For $(\epsilon/\hbar)^2 << 1$ [regime (a)] there is good agreement with $D_q \simeq \nu^2/2$, and D_q apparently becomes asymptotic to D_{cl} for large $\epsilon/\hbar$ appropriate to regime (c). Figure 1 also shows other data (triangles and crosses) for regime (b). The triangles and dots have ν and ϵ fixed and $\hbar$ varying, while the crosses correspond to ν and $\hbar$ fixed and ϵ varying. The three sets of data fall close to each other and are consistent with an approximate proportionality of D_q to the fourth power of $\epsilon/\hbar$ in regime (b), as predicted theoretically (solid line in Fig. 1). In addition, we have obtained extensive data on the variation of D_q with ν (ϵ and $\hbar$ held fixed). Excellent agreement is found with the theoretically predicted proportionality to ν^2 in regimes (a) and (b) [cf. Eq. (3)].

In conclusion, the presence of small noise can greatly modify the behavior of a quantum mechanical system which is classically chaotic, particularly for systems in the semiclassical regime.

We thank S. Fishman, J. Ford, R. E. Prange, and R. Westervelt for fruitful discussions. This work was supported by the U.S. Department of Energy.

[1]G. Casati, B. V. Chirikov, F. M. Izrailev, and J. Ford, in *Stochastic Behavior in Classical and Quantum Hamiltonian Systems* (Springer, New York, 1979), p. 334.

[2]B. V. Chirikov, F. M. Izrailev, and D. L. Shepelyanski, Sov. Sci. Rev. C **2**, 209 (1981); T. Hogg and B. A. Huberman, Phys. Rev. Lett. **48**, 711 (1982).

[3]D. L. Shepelyanski, Physica (Utrecht) **8D**, 208 (1983).

[4]S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. Lett. **49**, 509 (1982). Actually the arguments of this reference do not imply a strictly discrete spectrum, and there *may be* continua present [G. Casati and I. Guaineri, Commun. Math. Phys. **95**, 121 (1984)]. That is why we have used the wording "essentially discrete." By this we mean that the evolution over a long but not too long time is as if the spectrum were discrete. See Prange, Grempel, and Fishman, in Proceedings of the Conference on Quantum Chaos, Como, Italy, June 1983 (to be published), for a heuristic discussion of why, for practical purposes, it may often be a good approximation to neglect the possible continuum aspects of the spectrum and approximate it as discrete. In our numerical experiments we see no evidence of continua, and, particularly for small $\hbar$, the presence of even tiny noise would seem to make the question moot [excluding low-order rational values of $\hbar$; cf. F. M. Izrailev and D. L. Shepelyanski, Theor. Math. Phys. **43**, 417 (1983)].

[5]R. V. Jensen, Phys. Rev. Lett. **49**, 1365 (1982); R. Blumel and U. Smilansky, Phys. Rev. Lett. **52**, 137 (1984).

[6]D. L. Shepelyanski, Institute of Nuclear Physics Report No. 83-61, 1983 (to be published).

[7]E. Ott, in *Long Time Prediction in Dynamics* (Wiley, New York, 1983), p. 281.

[8]Since we introduce randomness externally, our results do not address the question of whether, and to what extent, there is a quantum counterpart to *deterministic* chaos in classical systems. Indeed, for finite noise there is always diffusion [G. Casati and I. Guarneri, Phys. Rev. Lett. **50**, 640 (1983)].

[9]B. V. Chirikov, Phys. Rep. **52**, 265 (1979).

[10]A. B. Rechester and R. B. White, Phys. Rev. Lett. **44**, 1586 (1980).

[11]M. V. Berry, N. L. Balazs, M. Tabor, and A. Voros, Ann. Phys. **112**, 26 (1979); J. D. Hanson, E. Ott, and T. M. Antonsen, Phys. Rev. A **29**, 819 (1984).

[12]D. J. Thouless, Phys. Rev. Lett. **39**, 1167 (1977), and Solid State Commun. **34**, 683 (1980).

[13]Izrailev and Shepelyanski, Ref. 4. To avoid these resonances we use $\hbar = 4\pi r/(q + \gamma^{-1})$, where r and q are integers and γ is the golden mean.