Chaotic Fluid Convection and the Fractal Nature of Passive Scalar Gradients

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(Received 22 August 1988)

The chaotic convection of passive scalars by an incompressible fluid is considered. (The convection is said to be chaotic if nearby fluid elements typically diverge from each other exponentially in time.) It is shown that during the time evolution the square of the gradient of chaotically convected passive scalars typically concentrates on a fractal set. Considerations of the local stretching properties of the flow lead to a partition function which yields the dimension spectra of the resulting fractal measure. Fractal structure is a result of the nonuniform stretching of typical chaotic flows.

PACS numbers: 47.25.−c, 05.45.+b

The convection of passive scalars by fluid flows is a problem of great practical and fundamental interest. In the latter regard, it has recently been demonstrated experimentally[1] that the square of the gradient of passive scalars convected by fully developed large-Reynolds-number turbulent flows displays the same type of intermittent behavior and concentration on a multifractal measure as does the square of the vorticity of the fluid itself.[2] If $\phi(\mathbf{x},t)$ denotes the passive scalar (e.g., a contaminant, temperature perturbation, etc.), then the equation satisfied by ϕ is

$$d\phi/dt=[\partial/\partial t+\mathbf{v}(\mathbf{x},t)\cdot\partial/\partial\mathbf{x}]\phi=\xi\nabla^2\phi\,, \tag{1}$$

where $\mathbf{v}(\mathbf{x},t)$ is the fluid velocity, which we assume to be incompressible, $\nabla\cdot\mathbf{v}=0$, and ξ is a diffusion coefficient. Here we shall be interested in the measure μ giving the distribution of $\chi\equiv(\nabla\phi)^2$,

$$\mu(S,t)=\frac{\int_S|\nabla\phi|^2dV}{\int_{V_0}|\nabla\phi|^2dV}\,, \tag{2}$$

where we imagine that consideration is restricted to some region V_0, S denotes a subset of V_0, $\mu(S,t)$ is the measure of the region S with respect to the quantity $|\nabla\phi|^2$, and $\phi(\mathbf{x},t)$ is the solution of Eq. (1) with some smooth initial condition $\phi(\mathbf{x},0)$ (cf. Refs. 1 and 3 for measurements of $|\nabla\phi|^2$).

In this Letter we consider smooth velocity flows $\mathbf{v}(\mathbf{x},t)$ with a long spatial scale of variation. Thus we do not consider large-Reynolds-number turbulent flows for which there is a large range of spatial scales in $\mathbf{v}(\mathbf{x},t)$. We will show, however, that even when small scales in $\mathbf{v}(\mathbf{x},t)$ are absent, $\mu(S,t)$ can have fractal properties. In particular, we will demonstrate for typical *chaotic* flows $\mathbf{v}(\mathbf{x},t)$ and small ξ that, if we examine μ at some instant $t=t_1$, where t_1 is large but not too large (as explained below), then the measure μ is approximately a multifractal measure. In particular, down to some minimum spatial scale $r_{\min}$, the quantity $(\nabla\phi)^2$ concentrates on a fractal due to the typical nonuniform stretching action of the flow. In general, $r_{\min}$, the minimum spatial scale for variation of ϕ will be set by t_1 or by ξ, depending on how large t_1 is. Initially, the diffusive term, $\xi\nabla^2\phi$, in Eq. (1) is negligible for small enough ξ. As time proceeds, smaller scales are generated and the minimum scale $r_{\min}$ eventually reaches a value determined by the diffusion which is of order $(\xi/\omega_0)^{1/2}$, where ω_0 denotes a typical value of $\boldsymbol{\omega}=\nabla\times\mathbf{v}(\mathbf{x},t)$. In this Letter we consider the case where t_1 is large, but not too large, in the sense that $r_{\min}$ is very small but is larger than $(\xi/\omega_0)^{1/2}$. (This can occur if ξ is sufficiently small.) Thus, in this case, the diffusion can be neglected, and we henceforth set $\xi=0$.

In the above, by "chaotic flow" we mean that the trajectory of fluid elements, $d\mathbf{x}/dt=\mathbf{v}(\mathbf{x},t)$, has a sensitive dependence on initial conditions. Thus, taking an infinitesimal variation $\delta\mathbf{x}$ from the trajectory $\mathbf{x}(t)$, the equation for $\delta\mathbf{x}(t)$, $d\delta\mathbf{x}/dt=\delta\mathbf{x}\cdot\nabla\mathbf{v}$, typically has an exponentially increasing solution corresponding to a positive Lyapunov exponent,

$$h=\lim_{t\to\infty}\{t^{-1}\ln[|\delta\mathbf{x}(t)|/|\delta\mathbf{x}(0)|]\}>0\,.$$

For our considerations to follow, we assume that $\mathbf{v}(\mathbf{x},t)$ is a prescribed flow with smooth $\mathbf{x}$ dependence (determined by external conditions such as stirring, thermal convection, etc.). Then integration of $d\mathbf{x}/dt=\mathbf{v}(\mathbf{x},t)$ from time $t=mT$ to time $t=(m+1)T$ defines a map,

$$\mathbf{x}_{m+1}=\mathbf{M}(\mathbf{x}_m,m)\,, \tag{3}$$

where $\mathbf{x}_m$ denotes the position of a fluid element at time $t=mT$. [For the case of periodic flows the map is autonomous (i.e., independent of n) if T is chosen to be the period, $\mathbf{v}(\mathbf{x},t)=\mathbf{v}(\mathbf{x},t+T)$.]

We are interested in characterizing the fractal properties of the measure μ by the spectrum of fractal dimensions[4] D_q, where the index q is a continuous parameter. Roughly speaking, the fractal dimension D_q specifies the scaling of $\langle\mu_\epsilon^{(q-1)}\rangle$, where μ_ϵ is the measure in a small square of side ϵ and the average over squares, $\langle\cdots\rangle$, is with respect to the measure μ. The average then scales with ϵ as $\langle\mu_\epsilon^{(q-1)}\rangle\sim\epsilon^{(q-1)D_q}$ for small ϵ. In what follows we shall make use of the partition-function technique for obtaining the dimension spectrum associated with μ. Specifically, in Ref. 4 the following partition function is defined:

$$\Gamma(\tau,q,\epsilon,\{S_i\}) = \sum_i \mu_i^q/\epsilon_i^\tau\,, \tag{4}$$

where $\{S_i\}$ denotes a set of cubes of edge lengths $\epsilon_i \le \epsilon$ which cover the measure, and $\mu_i = \mu(S_i)$ is the measure in cube i. If $q<1$ (respectively, $q>1$), we choose the set of cubes $\{S_i\}$ so as to make Γ as large (small) as possible subject to the constraint $\epsilon_i \le \epsilon$; that is,

$$\Gamma(\tau,q,\epsilon) = \begin{cases} \text{min over } \{S_i\} \text{ of } \Gamma(\tau,q,\epsilon,\{S_i\}), \text{ if } q<1, \\ \text{max over } \{S_i\} \text{ of } \Gamma(\tau,q,\epsilon,\{S_i\}), \text{if } q>1. \end{cases} \tag{5}$$

We then take the limit $\epsilon \to 0$

$$\Gamma(\tau,q) = \lim_{\epsilon\to 0} \Gamma(\tau,q,\epsilon)\,. \tag{6}$$

It can be shown that $\Gamma(\tau,q)$ will be infinite if τ exceeds a critical value, and $\Gamma(\tau,q)$ will be zero if τ is less than this critical value,

$$\Gamma(\tau,q) = \begin{cases} 0 \text{ if } \tau<\tau_q, \\ \infty \text{ if } \tau>\tau_q, \end{cases} \tag{7}$$

where we have denoted the critical value τ_q. The dimension spectrum is then given by

$$D_q = \tau_q/(q-1)\,. \tag{8}$$

Equation (8) is a spectrum of dimensions since q is a continuous variable, and the higher-q D_q values characterize the more intense singularities of the multifractal measure.

We now wish to develop a partition function for the measure (2). In particular, we wish not to deal explicitly with the measure of small boxes ϵ_i. Rather it is our goal to estimate Γ using dynamical properties of the flow (in particular, the stretching properties inherent in the chaotic motion). We specialize to the case of two-dimensional flows[5] and employ a technique previously utilized for chaotic attractors and chaotic repellers of dissipative dynamical systems.[6] Suppose we set $\xi=0$ in Eq. (1) and start at $t=0$ with a uniform gradient field $\nabla\phi(\mathbf{x},0)$. Imagine that we divide the space by a square grid of unit grid size δ. We now iterate the map (3) n steps. If δ is small enough, the action of the map on a given square will be linear. Thus the map will take an initial square into a parallelogram. Let $\lambda_{1j}^n \equiv L_{1j}$ and $\lambda_{2j}^n \equiv L_{2j}$ be the magnitudes of the eigenvalues of the Jacobian matrix of the n-times-iterated map for initial conditions in the jth square, where $\lambda_{1j}^n > 1 > \lambda_{2j}^n$ (note that $\lambda_{1j}\lambda_{2j}=1$ since areas are preserved). Then the parallelogram will be long and thin, with long dimension $\lambda_{1j}^n\delta$ and short dimension $\lambda_{2j}^n\delta$, as shown in Fig. 1 (where the parallelogram has been drawn as a rectangle). We then cover the resulting parallelogram with smaller boxes of edge length $\epsilon_j = \lambda_{2j}^n\delta$. There are roughly

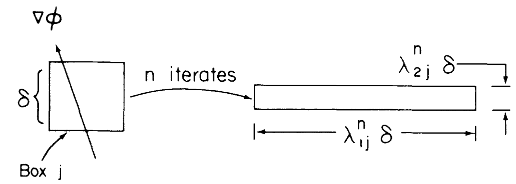

FIG. 1. Schematic illustration for the derivation of Eq. (10).

$\lambda_{1j}^n/\lambda_{2j}^n = \lambda_{1j}^{2n}$ such boxes. Let μ_j denote the measure initially in box j. Let $\hat\mu_j$ denote the measure in one of the small boxes covering the j parallelogram at iterate n. Since ϕ is convected by the flow, the gradient of ϕ can increase as nearby points in the fluid with different ϕ values approach each other. Since, by incompressibility $\lambda_{2j} = 1/\lambda_{1j}$, chaotic flows will generally lead to an exponential increase of $|\nabla\phi|$. This is illustrated in Fig. 1, where we see that $|\nabla\phi|$ increases by $\sim\lambda_{1j}^n$. Thus, since the area of one of the little boxes used to cover the parallelogram is smaller than the area of box j by a factor $\lambda_{2j}^{2n} = \lambda_{1j}^{-2n}$, we have from Eq. (2) $\hat\mu_j \cong \mu_j [\sum_j \mu_j \lambda_{1j}^{2n}]^{-1}$ where, in the sum in the denominator, we have taken into account the fact that the parallelogram is covered by λ_{1j}^{2n} small boxes. For smooth long-wavelength initial spatial variation of ϕ, all the μ_j are of the same order and we obtain $\hat\mu_j \sim (\sum_j L_{1j}^2)^{-1}$. From Eq. (4) we have $\Gamma \sim \sum_j L_{1j}^2 \hat\mu_j^q/\epsilon_j^\tau$. Inserting $\hat\mu_j$ and ϵ_j and setting $\tau=(q-1)D$ then yields

$$\Gamma \sim \left\{\sum_j L_{1j}^{[(q-1)(D-2)+2q]}\right\} \Big/ \left\{\sum_j L_{1j}^2\right\}^q\,. \tag{9}$$

(Interpret the symbol $\sim$ as "approximately proportional to.") We thus define the following new partition function Γ_λ based on the stretching properties of the flow,

$$\Gamma_\lambda(D,q,n) \equiv \langle L_1^{[(q-1)(D-2)+2q]}\rangle / \langle L_1^2\rangle^q\,, \tag{10}$$

where $L_1(\mathbf{x})$ is the largest Lyapunov number of the n-times-iterated map for the orbit originating at the point $\mathbf{x}$, and the angular brackets denote an average over space (i.e., over $\mathbf{x}$). Letting $n\to\infty$ in Eq. (10) is analogous to letting $\epsilon\to 0$ in Eq. (6) (recall that for Fig. 1, $\epsilon_j = \delta/\lambda_{1j}^n$). Thus we define a dimension $\tilde D_q$ as the value of D at which the quantity

$$\Gamma_\lambda(D,q) = \lim_{n\to\infty} \Gamma_\lambda(D,q,n)$$

goes from zero to infinity as $(q-1)D$ increases.[7] Since the optimization specified by Eq. (5) has not been carried out, $\tilde D_q \gtrsim D_q$. However, it appears to us that our covering is a rather natural one, and we conjecture that (at least for hyperbolic cases) $\tilde D_q = D_q$. [This can be confirmed with the generalized baker's map model (Ott and Antonsen[5]).]

Equation (10) implies a fractal dimension when the stretching is nonuniform. This follows by our inserting $D=2$ in Eq. (10) and noting that for any function $f(x)$ of a variable x where $d^2f/dx^2 \lesseqgtr 0$, we have $\bar{f}(x) \gtrless f(\bar{x})$, where the overbar denotes an average over x. Thus, letting $f(x)=x^q$ we find $\Gamma_\lambda(2,q,n) \le 1$ for $q<1$ and $\Gamma_\lambda(2,q,n) \ge 1$ for $q>1$. The inequalities become equalities only when the stretching is uniform. With nonuniform stretching as $n \to \infty$ we have $\Gamma_\lambda = \infty$ for $q>1$ and $\Gamma_\lambda = 0$ for $q<1$. Thus, by Eqs. (7) and (8), D_q must be less than 2, and hence the measure is fractal. The example to be discussed subsequently clearly shows this effect of nonuniform stretching.

It is of interest to investigate Eq. (10) for two limiting cases:

(i) For $q=0$, Eq. (10) yields $\Gamma_\lambda = \langle L_1^{(2-D)} \rangle$. Since $L_1 \to \infty$ as $n \to \infty$, we have $\tilde{D}_0 = 2$. The interpretation is that there is an area such that *any* subset S_0 of this area which itself has positive area has $\mu(S_0)>0$ in the limit of $t_1 \to \infty$, $\xi \to 0$ [cf. Eq. (4)].

(ii) For the limit $q \to 1$ (the information dimension), Eq. (10) gives to first order in $q-1$ the result

$$\Gamma_\lambda \cong 1-(q-1)\{\ln\langle L_1^2\rangle - \tfrac{1}{2} D \langle L_1^2 \ln L_1^2\rangle / \langle L_1^2 \rangle\}.$$

For $n \to \infty$ we have $\langle L_1^2 \ln L_1 \rangle / \langle L_1^2 \rangle \sim \langle \ln L_1 \rangle \sim n$. Thus the coefficient of the $q-1$ term in Γ_λ becomes large with n unless $D=\tilde{D}_1$ with

$$\tilde{D}_1 = 2 - 2\frac{\langle L_1^2 \ln L_1^2 \rangle - \langle L_1^2 \rangle \ln \langle L_1^2 \rangle}{\langle L_1^2 \ln L_1^2 \rangle}. \tag{11}$$

For the case of a smooth two-dimensional flow (rather than a discrete map), all the considerations above carry through with L_1 replaced by $\exp(h_1 t)$, where h_1 is the largest Lyapunov exponent calculated for the given initial condition in the time interval 0 to t, and t denotes the continuous time variable replacing the discrete time variable n.

As an illustrative example we now consider a particular time periodic, spatially smooth flow,

$$\mathbf{v}(\mathbf{x},t) = v'_{0x} y \mathbf{x}_0 + v_{0y} \sin x \, \delta_T(t) \mathbf{y}_0,$$

where v'_{0x} is a constant shear and $\delta_T(t) = T\sum_m \delta(t-mT)$. Integration of $d\mathbf{x}/dt = \mathbf{v}(\mathbf{x},t)$ through one period T then yields the standard map,[8] $x_{m+1} = x_m + w_m$, $w_{m+1} = w_m + K \sin x_{m+1}$, where $w = v'_{0x} T y$ and $K = v'_{0x} v_{0y} T^2$. We examine this flow in the case $K \gg 1$. A linearization of this map yields the following recursion relation for the solution with the largest Lyapunov exponent when $K \gg 1$, $\delta w_{m+1} = K \cos x_{m+1} \delta w_m$. Letting $\eta_{m+1} = \cos x_{m+1}$ we have for the total stretch after n iterates, $L_1 \simeq K^n |\eta_{n-1}\eta_{n-2} \cdots \eta_1 \eta_0|$. Following Chirikov's treatment,[8] we note that for large K the standard map produces an orbit which typically has $|x_{m+1}-x_m| \gg 2\pi$ so that the values of $x_{m+1} \bmod (2\pi)$ can be taken to be effectively random, independent, and uniformly distributed. Thus we see that this example involves nonuniform stretching; λ_1 on any given iterate is $K|\cos\theta|$ with θ random and uniformly distributed in $0 \le \theta \le 2\pi$. Hence by the argument given previously, the measure will be fractal. We believe that this example is typical in this regard (viz., uniform stretching would only occur in very special cases) and that fractal measures occur for chaotic flows encountered in practice. To proceed with our example, we may perform the averages appearing in Eq. (10) by neglecting correlations between η_i and η_{i+1},

$$\langle L_1^\gamma \rangle = K^{\gamma n} I^n(\gamma),$$

where

$$I(\gamma) = \int_0^{2\pi} \frac{d\theta}{2\pi} |\cos\theta|^\gamma = \pi^{-1/2} G(\tfrac{1}{2}(\gamma+1))/G(\tfrac{1}{2}(\gamma+2)),$$

and G is the gamma function. These averages may then be inserted in Eq. (10). Demanding that Γ_λ be finite as $n \to \infty$ produces the following transcendental equation for $\tilde{D}_q$:

$$\tilde{D}_q = 2 - \frac{q \ln 2 + \ln[I(2+(q-1)\tilde{D}_q)]}{(q-1)\ln K}. \tag{12}$$

If $\ln K$ is large, then $\tilde{D}_q$ approaches 2 from below as a result of the fact that the variation in the Lyapunov exponent ($\ln\lambda = \ln K + \ln|\cos\theta|$) for different initial conditions becomes relatively small (i.e., $\ln K \gg \langle \ln|\cos\theta| \rangle$). [Equation (12) implies $\tilde{D}_q > 2 - (\ln 2)/(\ln K)$.] Expanding for $q \simeq 1$ we obtain an explicit expression for the information dimension,

$$\tilde{D}_1 = 2\left[1 - \frac{3\ln 2 - 1}{2\ln(2K)-1}\right], \tag{13}$$

which is clearly less than 2, again demonstrating a fractal measure.

As a final comment we note that for many flows the time dependence of the vector field $\mathbf{v}(\mathbf{x},t)$ is itself chaotic. In such cases a possible model is to consider the map $\mathbf{M}(\mathbf{x},m)$ in Eq. (3) to be randomly selected according to some rule at each m (cf. Ott and Antonsen[5]). The problem then has similarities[5] with that of spin-glasses.[9]

In conclusion, we have shown that during the time evolution[10] the square of the gradient of passive scalars convected by chaotic flows in general results in a multifractal measure, and a theory giving the dimension spectrum of the measure has been formulated in terms of the dynamical stretching properties of the flow.

This work was supported by the U.S. Department of Energy and by the Office of Naval Research.

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[7]For the purpose of numerical computation we can obtain an explicit parametric relation for $\tilde{D}_q$ vs q. We set $\Gamma_\gamma = 1$ in Eq. (10) and take logarithms. This yields $q = \lim_{n\to\infty}(\ln\langle L_1^\eta\rangle/\ln\langle L_1^2\rangle)$, where $D_q = 2-(2q-\eta)/(q-1)$. Thus for each choice of η, the equation $q = \lim_{n\to\infty}(\ln\langle L_1^\eta\rangle/\ln\langle L_1^2\rangle)$ gives the corresponding q which when inserted in $D_q = 2-(2q-\eta)/(q-1)$ gives D_q. To numerically calculate the averages $\langle L_1^\eta\rangle$, sprinkle a large number of initial conditions uniformly in the ergodic region; iterate n times; calculate L_1 for each initial condition; and then perform the average over initial conditions. (As n gets larger more initial conditions will be needed to insure good statistics.)

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