Multifractal power spectra of passive scalars convected by chaotic fluid flows

Thomas M. Antonsen, Jr. and Edward Ott
Laboratory for Plasma Research, Department of Electrical Engineering, and Department of Physics, University of Maryland, College Park, Maryland 20742-3511
(Received 30 January 1990; revised manuscript received 1 August 1990)

The spatial power spectra of passively convected scalar quantities in fluid flows are considered for the case in which the flow has smooth large-sclae spatial dependence and Lagrangian chaos. Fundamentally different results apply for the small-diffusivity limit of the "initial-value problem" (in which an initial passive scalar density evolves in time with no passive scalar source present) and the "steady-state problem" (in which a statistically steady passive scalar source is present and one seeks time-asymptotic steady properties). Previous work has shown that the initial-value problem yields a situation where the gradient of the passive scalar tends to concentrate on a fractal. The purpose of this paper is to consider the implications of the prevously obtained fractal properties for the spatial power spectrum of passively convected scalars. The main result of this paper is that for the initial-value problem the spatial power spectrum is related to the fractal dimension spectrum and to the distribution of stretching rates (finite-time Lyapunov exponents) of the flow and is not necessarily a power law. In particular, for the initial-value problem in the case in which the flow has no Kolmogorov-Arnold-Moser (KAM) surfaces, the power spectrum is distinctly not a power law. However, if KAM surfaces are present, the power spectrum for the initial-value problem exhibits a k^{-1} power-law dependence in a range of k values. For the steady-state problem, it is shown that a k^{-1} power spectrum always applies. (This latter result has been previously derived for the steady-state problem and is known as "Batchelor's law.")

I. INTRODUCTION

The properties of scalar quantities (e.g., temperature or the concentration of an impurity) that are passively convected by an incompressible fluid flow have been of interest for many years [1–6]. The evolution of such a passive scalar is determined by the equation

$$\frac{d\phi}{dt}=\kappa\nabla^2\phi+S(\mathbf{x},t)\ . \tag{1}$$

In Eq. (1) the time derivative is taken following the motion of the fluid,

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{v}(\mathbf{x},t)\cdot\frac{\partial}{\partial \mathbf{x}}\ ,$$

where $\mathbf{v}(\mathbf{x},t)$ is the fluid velocity. Further, κ represents the diffusion coefficient for the scalar and $S(\mathbf{x},t)$ represents a source of the scalar. It is assumed that the fluid flow $\mathbf{v}(\mathbf{x},t)$ is incompressible ($\nabla\cdot\mathbf{v}=0$) and is determined by external dynamics (such as stirring), that the source $S(\mathbf{x},t)$ is prescribed, and that neither of these are affected by ϕ.

Equation (1) poses a fundamental problem in classical physics and naturally has attracted much attention. In particular, much effort has been devoted to the determination of the power spectrum of the correlation function,

$$C(\mathbf{r})=\langle\phi(\mathbf{x}+\mathbf{r})\phi(\mathbf{x})\rangle\ , \tag{2}$$

where the average in Eq. (2) can be taken to be over the domain in which Eq. (1) is solved. The power spectrum $F(k)$ is then defined as

$$F(k)=\int\frac{d^nk'}{(2\pi)^n}\delta(k-|\mathbf{k}'|)\overline{C}(\mathbf{k}')\ , \tag{3}$$

where

$$\overline{C}(\mathbf{k}')=\int d^nr\,C(\mathbf{r})e^{-i\mathbf{k}'\cdot\mathbf{r}} \tag{4}$$

is the Fourier transform of the correlation function and n is the dimensionality of the domain ($n=2$ or $n=3$ are the cases of interest). The integration over $\mathbf{k}'$ combined with the δ function in Eq. (3) represents an averaging over angles in $\mathbf{k}'$ space and leads to the integral relation

$$\int_0^\infty dk\,F(k)=C(0)=\langle\phi^2(\mathbf{x})\rangle\ . \tag{5}$$

Recent work by ourselves and others [2,3,6] has focused on another quantity of interest, namely, the fractal dimension of the measure of the gradient of the scalar. In particular, in Ref. [2] we defined a measure μ of a region of space s based on the distribution of $|\nabla\phi|^\gamma$,

$$\mu(s,t,\gamma)=\frac{\int_s|\nabla\phi|^\gamma d^nx}{\int_{V_0}|\nabla\phi|^\gamma d^nx}\ , \tag{6}$$

where V_0 is a reference domain in which (1) is satisfied and s is a subset of this domain. The situation was considered in which the flow was smooth on large scales, meaning that the smallest flow scale is much larger than the scales which develop in the variation of the passive scalar $\phi(\mathbf{x},t)$. Moreover, the flow was assumed to be chaotic and ergodic in V_0. Here we use the word chaotic with reference to the Lagrangian fluid trajectories rather than with reference to the time dependence of the Eulerian velocity field $\mathbf{v}(\mathbf{x},t)$. Thus by a chaotic fluid flow, in this paper we shall mean that the distance between infinitesimally nearby fluid elements typically diverges exponentially with time. [This definition of chaos includes time-dependent flows $\mathbf{v}(\mathbf{x},t)$ which are temporally nonperiodic. We expect such a nonperiodic time-dependent situation to occur, for example, in *low*- Reynolds-number turbulence.]

A distinction was found in Ref. [2] between the "initial-value" problem, in which Eq. (1) is solved with the source set equal to zero and an initially smooth ϕ evolves in time, and the "steady-state" problem, in which the source is considered to be statistically steady and the time-asymptotic properties of the measure are of interest.

For the initial-value problem it was found [2] that in the case of small diffusivity the measure μ exhibited fractal properties over some time interval determined by the diffusivity and the flow. To understand this, first consider the case where there is no diffusion $\kappa\equiv 0$. Then, as time increases, $\phi(\mathbf{x},t)$ develops finer and finer scale variations, and the regions of largest $|\nabla\phi|$ occupy a smaller and smaller fraction of space. This fraction approaches zero as $t\to\infty$, and in this limit, in an appropriate sense, the region of largest $|\nabla\phi|$ approaches a fractal set. At any large finite time, the region of largest $|\nabla\phi|$ will be approximately fractal in that, when viewed with finite spatial resolution [larger than some appropriate characteristic scale of the region of large $|\nabla\phi|$ (this length scale decreases with time)], the region looks fractal. If $\kappa\neq 0$, but is very small, then the preceding considerations apply up to some finite time t_κ at which the characteristic scale mentioned above becomes so small that diffusion cannot be neglected. After this time the measure was shown to no longer be fractal. Our interest is in the intermediate range of times where t is large enough that the measure is approximately fractal, but small enough that diffusion plays no role ($t<t_\kappa$). Such a range exists for sufficiently small κ (cf. Refs. [2]). In the steady-state problem the measure was shown [2] not to exhibit fractal properties in the time-asymptotic limit.

For the steady state problem, in the limit of vanishing diffusion coefficient, the following results have been obtained for $F(k)$:

$$F(k)\simeq k^{-5/3}\ , \tag{7a}$$

if the fluid flow is turbulent [5] and satisfies the Kolmogorov scaling hypothesis and k is in the so-called inertial range, and

$$F(k)\simeq k^{-1}\ , \tag{7b}$$

for either a high-Reynolds-number turbulent flow with k^{-1} less than the Kolmogorov viscous cutoff length [5], or if the flow is smooth with large spatial scales, but is chaotic [4] in the Lagrangian sense (cf. below). Equation (7b) is known as Batchelor's law.

The purpose of the present paper is to extend our previous work [2] on the fractal dimension spectrum of the measure (6) by exploring its relation to the power spectrum $F(k)$. The main results of the paper are as follows.

(1) We find that for cases in which the measure exhibits fractal properties (the initial-value problem) and the flow has no integrable regions [i.e., no Kolmogorov-Arnold-Moser (KAM) surfaces], the power spectrum does not obey a power law as in Eq. (7b). Instead, it exhibits a scaling dependence on k which is characteristic of a multifractal.

(2) It is possible to recover the spectrum of fractal dimensions for the measure μ from the power spectrum.

(3) If the initial-value problem is considered for a case in which the flow has KAM regions, then an approximate k^{-1} power spectrum results for a certain range of k values.

(4) For the steady-state problem our results are consistent with the known result for this case, Eq. (7b) (Batchelor's law).

II. INITIAL-VALUE PROBLEM

We consider solutions to Eq. (1) with the right-hand side equal to zero (no source and no diffusion) and with initial condition

$$\phi(\mathbf{x},t=0)=\exp(i\mathbf{k}_0\cdot\mathbf{x})\ ,$$

chosen for simplicity. [Due to the linearity of the problem (1) the solution for arbitrary initial conditions can be constructed by superposition.] Equation (1) states that $\phi(\mathbf{x},t)$ is constant along a fluid trajectory. Thus, let $\boldsymbol{\xi}(\mathbf{x},t;t')$ be the position of a fluid element at time t' which is located at the point $\mathbf{x}$ at time t. The vector $\boldsymbol{\xi}$ thus satisfies

$$\frac{d\boldsymbol{\xi}(\mathbf{x},t;t')}{dt'}=\mathbf{v}(\boldsymbol{\xi},t')\ , \tag{8}$$

with the initial condition

$$\boldsymbol{\xi}(\mathbf{x},t;t)=\mathbf{x}\ .$$

In terms of $\boldsymbol{\xi}$, the solution of Eq. (1) is

$$\phi(\mathbf{x},t)=\exp[i\mathbf{k}_0\cdot\boldsymbol{\xi}(\mathbf{x},t;0)]\ .$$

The appropriate expression for the correlation function, generalizing to our use of complex scalars, is

$$C(\mathbf{r},t)=\langle\phi(\mathbf{x}+\mathbf{r},t)\phi^*(\mathbf{x})\rangle$$

or

$$C(\mathbf{r},t)=\langle\exp\{i\mathbf{k}_0\cdot[\boldsymbol{\xi}(\mathbf{x}+\mathbf{r},t;0)-\boldsymbol{\xi}(\mathbf{x},t;0)]\}\rangle\ . \tag{9}$$

A. The relationship between the power spectrum and the distribution of finite-time Lyapunov exponents

We will be concerned with the dependence of the power spectrum, $F(k)$, for large values of $k\gg k_0$. This dependence is determined by the correlation function at small values of the separation, $\mathbf{r}$. This suggests that it is appropriate to linearize the trajectory $\boldsymbol{\xi}(\mathbf{x}+\mathbf{r},t;0)$,

$$\boldsymbol{\xi}(\mathbf{x}+\mathbf{r},t;0)\simeq\boldsymbol{\xi}(\mathbf{x},t;0)+\underline{M}(\mathbf{x},t;0)\cdot\mathbf{r}\ , \tag{10}$$

where

$$\underline{M}(\mathbf{x},t;0)=\partial\boldsymbol{\xi}(\mathbf{x},t;0)/\partial\mathbf{x}\ .$$

The quantity $\underline{M}\cdot\mathbf{r}$ is the initial separation of two fluid trajectories which at time t find themselves at $\mathbf{x}$ and $\mathbf{x}+\mathbf{r}$, respectively, when $\mathbf{r}$ is small.

We now substitute the linearized trajectory (10) into the expression for the correlation function (9), perform the Fourier transform (4), and the average (3). This results directly in the following expression for the power spectrum $F_1(k,t)$ for the initial-value problem:

$$F_1(k,t)=\langle \delta(k-|\mathbf{k}_0\cdot\underline{M}(\mathbf{x},t;0)|)\rangle \; . \tag{11}$$

We note that expression (11) clearly satisfies relation (5) for the integral of $F_1(k)$ for this problem. Here we have appended the subscript 1 to the power spectrum F to denote the initial-value problem; later on we shall use the symbol F_2 to denote the power spectrum for the steady-state problem.

We now discuss the validity of the linearization of the trajectories used to obtain (11). Clearly, with this linearization the correlation function has structure which is determined by trajectories which were initially separated by a distance $|\mathbf{k}_0|^{-1}$. So long as this distance is smaller than the characteristic scale for variation of the fluid velocity, l_v, the linearization is appropriate. We assume this to be the case and proceed. However, we expect our results to be roughly valid when $k_0 l_v \simeq 1$ as well.

We can now relate the power spectrum $F_1(k,t)$ to the distribution of Lyapunov exponents for the flow. In particular, the matrix $\underline{M}$ is constructed from linearized solutions of Eq. (8) which, for a chaotic flow, diverge exponentially in time. Since the flow is incompressible, some solutions converge exponentially as well. Thus, for large values of t,

$$|\mathbf{k}_0\cdot\underline{M}(\mathbf{x},t;0)|\simeq k_0 \exp ht \; , \tag{12}$$

where h is positive and is the magnitude of the most negative Lyapunov exponent for the linearized trajectories. The quantity

$$|\mathbf{r}||\mathbf{k}_0\cdot\underline{M}(x,t;0)|/k_0$$

is the maximum initial separation in the direction of $\mathbf{k}_0$ of two trajectories which at time t are separated by $|\mathbf{r}|$, and the maximum is taken with respect to the direction of $\mathbf{r}$. Since trajectories converge and diverge with exponential rates for large t, the direction of $\mathbf{r}$ producing the maximum initial separation will correspond to the linearized trajectory with the most negative Lyapunov exponent. For finite time, different trajectories will have different values of h,

$$h=h(\mathbf{x},t) \; .$$

However, in the limit of $t\to\infty$, if a trajectory ergodically visits the entire domain of interest, then, for almost every $\mathbf{x}$ in the domain, the exponent approaches the same value,

$$\lim_{t\to\infty} h(\mathbf{x},t)=\bar{h} \; .$$

The finite-time variations in the values of h can be characterized by a probability distribution [2,7] $P(h,t)$, where $P(h,t)dh$ is the probability that $h(\mathbf{x},t)$ is between h and $h+dh$, if $\mathbf{x}$ is chosen randomly with uniform probability in the relevant ergodic fluid region. In terms of $P(h,t)$, one obtains from (11) and (12) the power spectrum

$$F_1(k,t)=(kt)^{-1}P\left[t^{-1}\ln\frac{k}{k_0},t\right] \; . \tag{13}$$

Thus, the k dependence of the power spectrum of the correlation function is directly related to the distribution of finite-time Lyapunov exponents, and an experimental measurement of the former would determine the latter.

B. The distribution of finite-time Lyapunov exponents when KAM surfaces are absent

Equation (13) applies for both two- and three-dimensional incompressible flows. We now specialize to the case of two-dimensional incompressible flows. In this case a flow giving exponental divergence of nearby trajectories has two Lyapunov exponents which are equal in magnitude and opposite in sign. [Thus we can regard $h(\mathbf{x},t)$ as the positive exponent and $P(h,t)$ is its distribution.] One reasonable conjecture is that the distribution $P(h,t)$ is the same as that which results from the multiplication of many random (scalar) numbers. This results in a distribution of the form [2,7]

$$P(h,t)=\left[\frac{tG''(h)}{2\pi}\right]^{1/2}\exp[-tG(h)] \; , \tag{14}$$

where $G(\bar{h})=G'(\bar{h})=0$ (the prime denotes differentiation with respect to h). That is, P is peaked at $h=\bar{h}$ with deviations that scale as $t^{-1/2}$. This conjectured form has been verified numerically for cases where there are no KAM surfaces [8–10] [in particular, for the situation in which the time dependence of the flow is nonperiodic (e.g., the Eulerian velocity $\mathbf{v}(\mathbf{x},t)$ is itself temporally chaotic)]. Also, Eq. (14) is expected to apply for time-periodic flows, $\mathbf{v}(\mathbf{x},t)=\mathbf{v}(\mathbf{x},t+T)$, yielding maps for which KAM surfaces are essentially absent (e.g., the standard map at large nonlinearity parameter). [For hyperbolic dynamics for the stroboscopic map of a time-periodic flow, we also expect that Eq. (14) should be derivable rigorously, since in that case there is a splitting between stable and unstable directions.] In the presence of KAM surfaces bounding the relevant chaotic region, as typically arises for two-dimensional time-periodic flows, there are important modifications of Eq. (14). These modifications [8,9] are due to the "stickiness" of KAM surfaces, leading orbits near KAM surfaces to remain near them for long times. We discuss the effect of KAM surfaces on the power spectrum for the initial-value problem in Sec. II D. In the remainder of this section we treat the case where KAM surfaces are absent. We can then use Eq. (14) to determine the power spectrum from the function G. Defining $h_k=t^{-1}\ln k/k_0$, we have

$$F_1(k,t)=\frac{1}{k_0 t}\left[\frac{G''(h_k)t}{2\pi}\right]^{1/2}\exp\{-t[h_k+G(h_k)]\} \; . \tag{15}$$

The dependence of F_1 on k in this case cannot be approximated as a power law in k. To illustrate this, consider (15) for large t. We have

$$\frac{d\ln F_1}{d\ln(k/k_0)}\simeq-[1+G'(h_k)] \; . \tag{16}$$

Since $G'(h_k)$ is of order unity and a function of k, a simple power law does not apply. Further, to contrast the k dependence of $F_1(k)$ with Batchelor's law, we consider the product $kF_1(k,t)$ as a function of $\ln(k/k_0)$. (By (15), $kF_1 \simeq \exp[-tG(h_k)]$.) For typical $G(h)$ (Ref. [3]) the quantity $kF_1(k,t)$ plotted versus $\ln(k/k_0)$ has the shape of a pulse, as illustrated schematically in Fig. 1. Since $G'(\bar{h})=0$, the peak of the pulse propagates to larger values of $\ln k/k_0$ at the uniform velocity $\bar{h}$ while its width, w in $\ln k/k_0$, is given by $w \cong \sqrt{2t/G''(\bar{h})}$. Further, the area under the pulse is constant ($\int kF_1 d(\ln k) = \int F_1 dk$). Of course, this dependence of the power spectrum on k applies only for values of k less than the upper cutoff [5] $k_{\max} \simeq (\bar{h}/\kappa)^{1/2}$ determined by diffusion of the scalar. As the pulse of scalar variance approaches this upper cutoff, it is dissipated, and $F_1(k,t)$ for $k > k_{\max}$ quickly goes to zero.

In contrast, for the steady-state problem the power spectrum behaves as k^{-1} for all values of k greater than k_0 and less than the upper cutoff $k_{\max}$.

C. Relation of fractal dimensions to $F_1(k)$

In our previous work [2] we considered cases where Eq. (14) applies, and we related the distribution of Lyapunov exponents to the spectrum of fractal dimensions of the measure μ. We now show how this information can also be determined from $F_1(k)$.

The spectrum of fractal dimensions D_q (where q is a continuous parameter) characterizes the multifractal properties of the measure μ. Roughly speaking, D_q specifies the scaling of $\langle\langle \mu_\epsilon^{(q-1)} \rangle\rangle$ where μ_ϵ is the measure in a small box of side ϵ and the average $\langle\langle \ldots \rangle\rangle$ is over all boxes and taken with respect to the measure μ itself. The average defining D_q scales with ϵ as $\langle\langle \mu_\epsilon^{(q-1)} \rangle\rangle \sim \epsilon^{(q-1)D_q}$ for small ϵ (e.g., see Grassberger *et al.* [7] and references therein).

Specializing to the case of two dimensions, it was shown [2] that D_q can be determined by the time-asymptotic behavior of the "partition function" $\Gamma(D,q,t)$ defined as [11]

$$\Gamma(D,q,t) = \frac{\langle e^{\sigma h t} \rangle}{\langle e^{\gamma h t} \rangle^q}\ , \tag{17}$$

where $\sigma \equiv (q-1)(D-2)+\gamma q$. The average $\langle \ldots \rangle$ in (17) is over $\mathbf{x}$, or, equivalently, over h weighted by $P(h,t)$. As $t \to \infty$, Γ either diverges to infinity or tends to zero depending on σ (and hence on D). The dimension D_q was found [2] to be determined by the condition that $\lim_{t\to\infty}\Gamma$ goes from zero to ∞ as D passes through D_q. A similar partition function can now be defined from the power spectrum

$$\Xi(D,q,t) = \int_0^\infty dk\, k^\sigma F_1(R,t) \Big/ \left[\int_0^\infty dk\, k^\gamma F_1(k,t) \right]^q , \tag{18}$$

which, using (11) for $F_1(k)$ and performing the k integrations, can be seen to be equivalent to $\Gamma(D,q,t)$. Thus, the fractal properties of the measure μ can be determined from various k moments of the power spectrum of the correlation function.

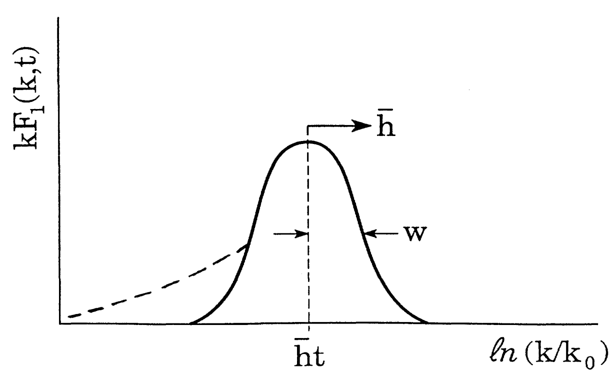

FIG. 1. Schematic of kF_1 vs $\ln(k/k_0)$.

D. The effect of KAM surfaces

We again consider two-dimensional flows and start with Eq. (13), which gives $F_1(k,t)$ with $P(h,t)$ being the distribution of positive Lyapunov exponents as before. We recall that the form of $P(h,t)$ given by Eq. (14) is equivalent to the distribution which results from the multiplication of many random, independent scalar numbers. This form was shown [8–10] to be appropriate when no KAM surfaces are present. When KAM surfaces are present, Eq. (14) must be modified due to the following effect. Chaotic trajectories can become "stuck" near KAM surfaces for long periods of time. While a trajectory is near the KAM surface, the divergence of nearby trajectories from it is greatly reduced. As shown in Ref. [9], this results in the following modification of the probability distribution function. For values of $h > \bar{h}$ the distribution function is determined by trajectories which have never become "stuck" near KAM surfaces (these are the ones with the largest Lyapunov exponent). In this case the form (14) applies (but with a different constant multiplying it).

For values of h sufficiently less than $\bar{h}$ the distribution function is determined by the orbits which were stuck near KAM surfaces. We denote the sticking time by τ. The linearized trajectory for these orbits will typically have exponentiated approximately at the rate $\bar{h}$ for a time $t-\tau$ (here t is the time interval over which the finite-time Lyapunov exponent h has been calculated). Thus, the effective Lyapunov exponent for these orbits is $h \simeq \bar{h}(1-\tau/t)$. Following the authors of Ref. [9], we let $W'(\tau)d\tau$ be the probability that an orbit has been stuck for a time in the range τ to $\tau+d\tau$. The resulting probability distribution for h is thus

$$P(h,t) = W'(t(1-h/\bar{h}))t/\bar{h} \tag{19}$$

for $0 < h \lesssim \bar{h}$.

The form of $W'(\tau)$ can be described as follows. For finite t some fraction of the orbits will never have been stuck. These contribute to $W'(\tau)$ as a δ function at $\tau=0$. These orbits will then cause $P(h,t)$ given by Eq. (19) to have a component which is also a δ function peaked at $h=\bar{h}$. [Actually, this peaked component is not truly a δ function but is broadened by the same statistical fluctuations in h that give rise to the distributions Eqs. (14) and (15)]. A second class of orbits, namely, those which have been stuck near KAM surfaces, also contribute to $W'(\tau)$. The authors of Ref. [9] note from numerical experiments that for this class of orbits $W'(\tau)\simeq\tau^{-\beta}$ with $2>\beta>1$ for the standard map. This power law only applies for τ much greater than the typical time scale of the flow, which can be estimated to be $\bar{h}^{-1}$.

The contribution of the "stuck" orbits to $F_1(k,t)$ for values of $h_k=t^{-1}\ln(k/k_0)$ for which the power law $W'(\tau)\simeq\tau^{-\beta}$ applies is obtained from Eqs. (13) and (19),

$$F_1(k,t)\simeq(k\bar{h})^{-1}[t(1-h_k/\bar{h})]^{-\beta} . \tag{20}$$

This contribution is sketched schematically with dashed lines in Fig. 1. The contribution to F_1 given by Eq. (20) is dominant only for values of $h_k=t^{-1}\ln k/k_0$ smaller than those contained in the broadened peak due to the orbits which have never "stuck"—that is, for $\ln k/k_0 \stackrel{\sim}{<} (t\bar{h}-w)$, where $w=\sqrt{2t/G''\bar{h}}$ is the width of the broadened peak. This corresponds to values of $\tau\sim w \gg \bar{h}^{-1}$, and thus the power law $W'\sim\tau^{-\beta}$ applies.

The result of these modifications of $P(h,t)$ on the local slope of the power spectrum can now be computed. For

$$h_k=t^{-1}\ln(k/k_0)\gtrsim\bar{h}-(w/t) ,$$

Eq. (16) applies as in the case of no KAM surfaces. For $h_k\lesssim\bar{h}-(w/t)$ we obtain from Eq. (20)

$$\frac{d\ln F_1}{d\ln(k/k_0)}=-\left[1-\frac{\beta}{t(\bar{h}-h_k)}\right] . \tag{21}$$

Thus, as $t\to\infty$, the power spectrum approaches a k^{-1} law for $k\lesssim k_0\exp(\bar{h}t-w)$. For $k\gtrsim k_0\exp(\bar{h}t-w)$, Eq. (16) applies, and thus the power spectrum k in this range appears to be a pulse (as in the case without KAM surfaces).

Finally, we point out that our previous discussion of the relationship between the power spectrum $F_1(k,t)$ and the spectrum of fractal dimensions implied by Eqs. (17) and (18) remains valid even in the presence of KAM surfaces since the particular form of $P(h,t)$ is not used in deriving the two partition functions Γ and Ξ.

III. THE STEADY-STATE PROBLEM

We now consider solutions to Eq. (1) with the source S not equal to zero. As in Secs. II B and II C, we consider the case where KAM surfaces are absent. However, we still consider the diffusion coefficient to be negligible. Taking the simplest possible source, namely, $S=\exp(i\mathbf{k}_0\cdot\mathbf{x})$, we find for $\phi(x,t)$,

$$\phi(\mathbf{x},t)=\int_0^t dt' \exp[i\mathbf{k}_0\cdot\boldsymbol{\xi}(\mathbf{x},t;t')] , \tag{22}$$

where we have assumed ϕ vanishes at $t=0$. Forming the correlation function, linearizing the trajectories as before, taking the Fourier transform, and averaging over angles in $\mathbf{k}'$ space results in the following expression for the power spectrum F_2 for the steady-state problem:

$$F_2(k,t)=\int_0^t dt'\int_0^t dt''\langle\delta(k-|\mathbf{k}_0\cdot\underline{M}(\mathbf{x},t;t')|)\exp\{i\mathbf{k}_0\cdot[\boldsymbol{\xi}(\mathbf{x},t;t')-\boldsymbol{\xi}(\mathbf{x},t;t'')]\}\rangle . \tag{23}$$

We will now argue that in the integrand of Eq. (23) the scale length for variation of the δ-function term is small compared to that for the exponential. This will have the consequence that the average of the product of the two terms can be replaced by the product of their averages.

We are concerned with large values of $k\gg k_0$. Hence, from the expression for $\underline{M}$ given by (12) and the δ function in (23), we are concerned with times

$$(t-t')\simeq h^{-1}\ln(k/k_0) .$$

Since nearby orbits separate exponentially with increasing $(t-t')$, the scale of variation of $\mathbf{k}_0\cdot\underline{M}$ with $\mathbf{x}$ is much smaller than the scale length for variation of the equilibrium velocity $\mathbf{v}(\mathbf{x},t),l_v$. In particular, the scale length in x can be estimated to be the initial separation required for two orbits to separate to the equilibrium scale length in the relevant time, $t-t'\simeq h^{-1}\ln(k/k_0)$. This gives for the scale length in x, of the δ-function term, $l_x\simeq l_v k_0/k \ll l_v$. The second term in the integrand of (23), the exponential term, involves an average taken at two different times on the same trajectory. This term will contribute when the times t' and t'' are close enough together, $|t'-t''|\simeq h^{-1}$, the characteristic time for variations of the flow velocity $\mathbf{v}$. Thus, the scale in x for this term will be roughly the scale length l_v. Since we are averaging over x and the two terms vary on such widely different scales, the two terms can be regarded as independent, yielding

$$F_2(k,t)=\int_0^t dt'\int_0^t dt'' C_s(|t'-t''|)\times\langle\delta(k-|\mathbf{k}_0\cdot\underline{M}(\mathbf{x},t;t')|)\rangle ,$$

where

$$C_s(|t'-t''|)=\langle\exp\{i\mathbf{k}_0\cdot[\boldsymbol{\xi}(\mathbf{x},t;t')-\boldsymbol{\xi}(\mathbf{x},t;t'')]\}\rangle$$

represents the correlation function of the source along a trajectory averaged over all trajectories. Letting $t\to\infty$ and noting that $C_s(|t'-t''|)$ decays to zero for time differences small compared with $h^{-1}\ln(k/k_0)$, we have

$$F_2(k,t\to\infty)=\tau\int_0^\infty dt' F_1(k,t') , \tag{24}$$

where

$$\tau=\int_{-\infty}^{\infty} dt\, C_s(|t|) .$$

Thus, the power spectrum for the steady-state problem is the time integral of the power spectrum for the initial-value problem.

Using the properties of $F_1(k,t)$, we can obtain the k dependence of $F_2(k,t\to\infty)$. Consider

$$\int_0^{t_*} dt \int_0^{k_*} dk\, F_1(k,t) \cong \int_0^{t_*} dt \int_0^{\infty} dk\, F_1(k,t) = t_* ,$$

where $t_*=\bar{h}^{-1}\ln(k_*/k_0)$; we have replaced k_* by ∞ in the second integral because $F_1(k,t)$ tends to zero rapidly for $k>k_0\exp(\bar{h}t)$, and we have used the integral relation (5) [$C(0)=1$ for our choice of initial condition]. We now reverse the order of integration and replace t^* by ∞ using the fact that $F_1(k,t)$ goes to zero rapidly at fixed k as $t\to\infty$ [cf. (15) and (20)]. We thus obtain

$$\int_0^{k_*} dk\, F_2(k,t\to\infty)=\tau t_* = \bar{h}^{-1}\tau \ln(k_*/k_0) .$$

Differentiating with respect to k_* yields (7b), Batchelor's law,

$$F_2(k,t\to\infty)=\tau(\bar{h}k)^{-1} . \tag{25}$$

Thus, for the steady-state problem in which we previously found the measure μ to have no fractal properties, the power spectrum is found to obey a power law, namely, Batchelor's law, Eq. (7b). Our present analysis allows for a simple interpretation of Batchelor's law. We have seen that in the initial-value problem the product $kF_1(k,t)$ is peaked at a value of $k=\bar{k}\simeq k_0\exp(ht)$ [cf. Eq. (14)] and has a fixed area under the curve $F_1(k,t)$ according to (5). This reflects the fact that a fixed amount of scalar variance $\langle \phi^2(x)\rangle$ is initially present at long wavelengths, $k\sim k_0$, and then "cascades" to shorter and shorter wavelengths with the typical value of k growing exponentially in time $d\bar{k}/dt=\bar{k}\bar{h}$. In the steady-state problem, scalar variance is continually injected at long wavelength, and then "cascades" to short wavelength. The time-asymptotic power spectrum $F_2(k,t\to\infty)$ then is a summation (integral) of the initial-value spectrum $F_1(k,t)$ at its various stages of evolution. Thus, $F_2(k,t\to\infty)$ is composed of contributions of scalar variance which were injected at different times in the past. This leads to Eq. (24) which states that $F_2(k,t\to\infty)$ is the time integral of $F_1(k,t)$. The inverse k dependence of F_2 follows from the rate at which the scalar variance "cascades" in k. Namely, F_2 at a particular value of k is inversely proportional to the rate at which the variance is cascading in k, namely, $dk/dt=k\bar{h}$. Hence, we obtain Batchelor's law, Eq. (25).

IV. CONCLUSION

In this paper we have studied the connection between the power spectrum for the correlation function and the fractal properties of the measure of the gradient of a passively convected scalar. A smooth chaotic flow was assumed, and the scalar was injected on a length scale smaller than the scale length of the flow. These assumptions allowed for a straightforward calculation of the power spectrum.

Important distinctions were found between the initial-value problem and the steady-state problem and between the case where the flow possesses KAM regions and the case where there are no KAM regions. In particular, in the initial-value problem, the measure has fractal properties, and the power spectrum of the correlation function exhibits a wave-number dependence which is not necessarily a power law. An expression was obtained for the power spectrum in terms of the distribution of the finite-time Lyapunov exponents for the flow, and a method of determining the fractal properties of the measure of the gradient from the power spectrum was demonstrated.

In the initial-value problem the shape of the power spectrum depends on whether the underlying flow contains KAM surfaces. In the absence of KAM surfaces the power spectrum has the shape of a pulse (see Fig. 1). When KAM surfaces are present, the power spectrum has two components. One component is a pulse, as in the case with no KAM surfaces. The other component, which is due to the orbits which have been stuck near KAM surfaces, has the character of a wake left by the pulse (see Fig. 1). For values of k which fall in the wake, the power spectrum has an approximate inverse k dependence. In the steady-state problem it was shown that the power spectrum always has an inverse k dependence (as predicted by Batchelor [5]).

At first it might seem that the steady-state problem is more appropriate to experiments in which the scalar is continually injected and the power spectrum is computed by taking a time average of the scalar measured at a limited number of spatial points, or by taking the time correlation function at a single point. However, if the scalar is injected in a localized region of space (e.g., at a boundary), and a mean flow away from the boundary is present, then the initial-value results may be relevant. In particular, in such a situation the time in the initial-value problem plays the role of the position downstream from the injection region (i.e., the mean time for a fluid element to travel from the injection point to the observation point). The validity of this consideration, of course, must be investigated on a case by case basis. Situations of this type could conceivably occur in recent experiments on fluid jets [6] and on thermal convection [12].

Finally, we wish to call attention to the similarity of the present problem to that of fluid turbulence. In particular, for the latter problem, it has been shown that the square of the vorticity $|\nabla\times\mathbf{v}|^2$ concentrates on a fractal in the limit of infinite Reynolds number and that the fractal nature of the vorticity is connected with the phenomenon of intermittency in the turbulence [13]. A coherent theory of the fractal distribution of vorticity in high-Reynolds-number fluid turbulence remains an important outstanding problem. In this paper we have considered the much simpler passive scalar problem and have found results that, in some respects, are roughly analogous to those in fluid turbulence.

ACKNOWLEDGMENTS

This work was supported by the U.S. Department of Energy and by the Office of Naval Research (Physics).

[1] Some recent works on the convection of passive scalars in flow fields are the following: H. Aref, J. Fluid Mech. **143**, 1 (1984); T. Dombre, U. Frisch, J. M. Greene, M. Henon, A. Mehr, and A. M. Soward, *ibid.* **167**, 353 (1986); J. Chaiken, C. K. Chu, M. Tabor, and Q. M. Tran, Phys. Fluids **30**, 687 (1987); D. V. Khakhar, H. Rising, and J. M. Ottino, J. Fluid Mech. **172**, 419 (1986); S. W. Jones and H. Aref, Phys. Fluids **31**, 485 (1988); T. H. Solomon and J. P. Gollub, *ibid.* **31**, 1372 (1988); Ya. G. Sinai and V. Yakhot, Phys. Rev. Lett. **63**, 1962 (1989); H. Chen, S. Chen, and R. H. Kraichnan, *ibid.* **63**, 2657 (1989).

[2] E. Ott and T. M. Antonsen, Jr., Phys. Rev. A **39**, 3660 (1989); Phys. Rev. Lett. **61**, 2839 (1988).

[3] In a recent experiment, R. Ramashankar and J. Gollub have observed and investigated the fractal nature of scalar gradients in a nonturbulent flow [Phys. Fluids A **3**, 1344 (1991)].

[4] A. Crisanti, M. Falcioni, G. Paladin, and A. Vulpiani, Physica A **166**, 305 (1990). At variance with our results in the present paper, Crisanti *et al.* assert a k^{-1} spectrum without distinguishing between cases with and without a source and with and without KAM surfaces.

[5] A. M. Obukov, Izv. Akad. Nauk. Geogr. Geofiz. **13**, 58 (1949); S. Corrsin, J. Appl. Phys. **22**, 469 (1951); G. K. Batchelor, J. Fluid Mech. **5**, 113 (1959); R. H. Kraichnan, Phys. Fluids **11**, 945 (1968); R. H. Kraichnan, J. Fluid Mech. **64**, 737 (1974).

[6] R. R. Prasad, C. Meneveau, and K. R. Sreenivasan, Phys. Rev. Lett. **61**, 74 (1988). See this paper for additional relevant references on experiments in turbulent fluids.

[7] P. Grassberger, R. Badii, and A. Politi, J. Stat. Phys. **51**, 135 (1988); H. Hata, T. Horita, H. Mori, T. Morita, and K. Tomita, Prog. Theor. Phys. **80**, 809 (1988); E. Ott, C. Grebogi, and J. A. Yorke, Phys. Lett. **135A**, 343 (1989).

[8] M. A. Sepulveda, R. Badii, and E. Pollak, Phys. Rev. Lett. **63**, 1226 (1989).

[9] T. Horita, H. Hata, R. Ishizaki, and H. Mori, Prog. Theor. Phys. B **3**, 1065 (1990).

[10] F. Varosi, T. M. Antonsen, Jr., and E. Ott, Phys. Fluids A **3**, 1017 (1991).

[11] Related partition functions also based on the finite-time stretching $h(\mathbf{x},t)$ have been used for the quite different problem of determining the fractal dimension spectrum of chaotic attractors of dissipative dynamical systems. R. Badii and A. Politi, Phys. Rev. A **35**, 1288 (1987); T. Morita, H. Hata, H. Mori, T. Horita, and K. Tomita, Prog. Theor. Phys. **78**, 511 (1987); E. Ott, T. Sauer, and J. A. Yorke, Phys. Rev. A **38**, 4212 (1989).

[12] F. Heslot, B. Castaing, and A. Libshaber, Phys. Rev. A **36**, 5870 (1987); B. Castaing, G. Gunaratne, F. Heslot, L. Kadanoff, A. Libshaber, S. Thomas, X. Z. Wu, S. Zaleski, and G. Zanetti, J. Fluid Mech. **204**, 1 (1989).

[13] B. B. Mandelbrot, J. Fluid Mech. **62**, 331 (1974); K. R. Sreenivasan and C. Meneveau, Phys. Rev. A **38**, 6287 (1988); R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, J. Phys. A **17**, 3521 (1984); U. Frisch and G. Parisi, in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics,* edited by M. Ghil, R. Benzi, and G. Parisi (North-Holland, New York, 1985), p. 84.