Single Fluid Ideal MHD

PHYS 761

In our previous discussions of fluids, we have focused on the assumptions of cold unmagnetized plasmas or cold magnetized plasmas with a straight, uniform field. We now want to consider a non-uniform plasma, including a strong magnetic field and pressure.

In these notes we will derive the ideal MHD equations from the two-fluid equations based on a set of assumptions. We will then explore some of the properties and consequences of ideal MHD.

References:

- J.P. Freidberg *Ideal MHD* - Chapters 2-3
- H. Goedbloed and S. Poedts *Principles of Magnetohydrodynamics* - Chapters 3-4

1. Derivation from two-fluid equations

We will begin with the two-fluid equations we obtained by taking moments of the kinetic equation including collisions. The continuity equation for species s is,

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \boldsymbol{u}_s) = 0, \tag{1}$$

where

$$n_s = \int d^3v \, f_{1s} \tag{2}$$

$$n_s \boldsymbol{u}_s = \int d^3v \, \boldsymbol{v} f_{1s}; \tag{3}$$

momentum balance is,

$$m_s n_s \frac{d\boldsymbol{u}_s}{dt} = q_s n_s \left(\boldsymbol{E} + \frac{\boldsymbol{u}_s \times \boldsymbol{B}}{c} \right) - \nabla p_s - \nabla \cdot \boldsymbol{\pi}_s + \boldsymbol{R}_s, \tag{4}$$

where

$$\delta\boldsymbol{v}_s = \boldsymbol{v} - \boldsymbol{u}_s \tag{5}$$

$$\boldsymbol{\pi}_s = m_s \int d^3v \left(\delta\boldsymbol{v}_s \delta\boldsymbol{v}_s - \frac{(\delta v_s)^2}{3}\mathbf{I} \right) f_{1s} \tag{6}$$

$$p_s = m_s \int d^3v \, \frac{(\delta v_s)^2}{3} f_{1s} \tag{7}$$

$$\boldsymbol{R}_s = m_s \int d^3v \, C_s(f_{1s}) \delta\boldsymbol{v}_s; \tag{8}$$

and the pressure equation is,

$$\frac{dp_s}{dt} + \gamma p_s \nabla \cdot \boldsymbol{u}_s + (\gamma - 1) \left(\boldsymbol{\pi}_s : \nabla \boldsymbol{u}_s + \nabla \cdot \boldsymbol{q}_s - Q_s \right) = 0, \tag{9}$$

where

$$\gamma = 5/3 \tag{10}$$

$$\boldsymbol{q}_s = \frac{m_s}{2} \int d^3v \, \delta\boldsymbol{v}_s (\delta v_s)^2 f_{1s} \tag{11}$$

$$Q_s = \frac{m_s}{2} \int d^3v \, (\delta v_s)^2 C(f_{1s}). \tag{12}$$

The fluid equations are coupled to Maxwell's equations,

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \tag{13}$$

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} + \cancel{\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}}. \tag{14}$$

Under the assumption that our system is not relativistic, the displacement current can be dropped.

We will assume a two-species plasma: ions with charge $q_i = e$ and electrons with charge $q_e = -e$. By applying certain simplifying assumptions, we will arrive at a set of single-fluid equations corresponding to ideal MHD equilibrium. We will consider a typical time scale $\tau^{-1} \sim \omega \sim \partial/\partial t$ and typical length scale $a^{-1} \sim \partial/\partial x$, with $v_{ti} \sim a/\tau$. The required basic physical assumptions are:

- Non-relativistic: $\omega/k \ll c$, $v_t \ll c$.
- Strongly magnetized and low frequency: $\omega \ll \Omega_i \ll \Omega_e$.
- $m_e/m_i \ll 1$: electron inertia is taken to be very small such that they respond very fast. This requires that $\omega \ll \Omega_e$ and $a \gg \rho_e$.
- Plasma is quasineutral: $a \gg \lambda_D$ and $\omega \ll \omega_p$. This will allow us to assume $(n_e - n_i) \ll (n_e + n_i)$, This implies that the electrons respond very quickly to any charge imbalance, maintaining local quasineutrality. This allows us to take $n_e = n_i \equiv n$.
- Temperatures equilibrate quickly: $\omega\tau_{eq} \sim \left(\frac{m_i}{m_e}\right)^{1/2} \frac{v_{ti}\tau_{ii}}{a} \ll 1$, where τ_{eq} is the temperature equilibration time. This allows us to take $T_e = T_i \equiv T$.

- Collisions are sufficiently strong that the distribution functions are nearly Maxwellian. For ions, the dominant collision frequency is between ions with timescale τ_{ii}. Thus we must have $\omega\tau_{ii} \ll 1$. For electrons, $\tau_{ee} \sim \tau_{ei}$. Thus the condition becomes $\omega\tau_{ee} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \omega\tau_{ii} \ll 1$. Additionally, the mean free path of both the electrons and ions should be much shorter than the length scales of interest. For ions, the condition is $v_{ti}\tau_{ii}/a \ll 1$, which is equivalent to the frequency assumption given above. For electrons the condition is $v_{te}\tau_{ee}/a \sim v_{ti}\tau_{ii}/a \ll 1$.
- Small resistivity: $(\rho_i/a)^2 \left(\frac{m_e}{m_i}\right)^{1/2} \frac{a}{v_{ti}\tau_{ii}} \ll 1$.

We now define one-fluid variables:

$$\text{mass density: } \rho = n_e m_e + n_i m_i \approx m_i n \tag{15}$$

$$\text{momentum density: } \rho \mathbf{v} = n_e m_e \boldsymbol{u}_e + n_i m_i \boldsymbol{u}_i \approx \rho \boldsymbol{u}_i \tag{16}$$

$$\text{current density: } \boldsymbol{J} = n_e q_e \boldsymbol{u}_e + q_i n_i \boldsymbol{u}_i = en(\boldsymbol{u}_i - \boldsymbol{u}_e) \tag{17}$$

$$\text{total pressure: } p = p_e + p_i = 2nT. \tag{18}$$

We can multiply the ion continuity equation by m_i to obtain the single fluid relation

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0}. \tag{19}$$

This is the continuity equation for the mass density of an ideal MHD plasma.

We can now multiply the electron and ion equations by e and subtract to obtain

$$\nabla \cdot \boldsymbol{J} = 0. \tag{20}$$

This is equivalent to the low frequency limit of Ampere's law,

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} \to \nabla \cdot \boldsymbol{J} = 0. \tag{21}$$

We will obtained the next single-fluid relation by considering the momentum equation. We can first note that $\boldsymbol{R}_e = -\boldsymbol{R}_i$, as the total momentum must be conserved due to collisions between electrons and ions.

We now add the electron and ion momentum equations together to obtain

$$\rho \frac{d\mathbf{v}}{dt} + \cancel{m_e n_e \frac{d\boldsymbol{u}_e}{dt}} = \frac{\boldsymbol{J} \times \boldsymbol{B}}{c} - \nabla p - \nabla \cdot \boldsymbol{\pi}_i - \nabla \cdot \boldsymbol{\pi}_e, \tag{22}$$

where we have ignored terms involving the electron inertia. There is no electric field term due to the quasineutrality assumption. Under the assumption of strong collisions, the anisotropic pressure terms can be shown to be small,

$$(\nabla \cdot \boldsymbol{\pi}_i)/\nabla p \sim v_{ti}\tau_{ii}/a \ll 1 \tag{23}$$

$$(\nabla \cdot \boldsymbol{\pi}_e)/\nabla p \sim v_{te}\tau_{ee}/a \ll 1. \tag{24}$$

The scaling of these moments can be done formally following a calculation of the Braginskii transport coefficients (see Friedberg Appendix B). These expressions are obtained by making the assumption that collisions are strong, $\omega\tau_{ee} \ll 1$ and $\omega\tau_{ii} \ll 1$, and strong magnetization, $\Omega_e\tau_{ee} \gg 1$ and $\Omega_i\tau_{ii} \gg 1$.

We now obtain the ideal momentum equation,

(25)
$$\boxed{\rho\frac{d\mathbf{v}}{dt} = \frac{\boldsymbol{J}\times\boldsymbol{B}}{c} - \nabla p}.$$

Let's now rewrite the electron momentum equation in terms of the single fluid variables,

(26)
$$\cancel{m_e n\frac{d\boldsymbol{u}_e}{dt}} = -en\boldsymbol{E} - en\frac{\boldsymbol{u}_e\times\boldsymbol{B}}{c} - \nabla p_e - \nabla\cdot\boldsymbol{\pi}_e + \boldsymbol{R}_e,$$

ignoring the electron inertia term. Noting that $\boldsymbol{u}_e = \mathbf{v} - \boldsymbol{J}/(en)$, we find

(27)
$$en\left(\boldsymbol{E} + \frac{\mathbf{v}\times\boldsymbol{B}}{c}\right) = \frac{\boldsymbol{J}\times\boldsymbol{B}}{c} - \nabla p_e - \nabla\cdot\boldsymbol{\pi}_e + \boldsymbol{R}_e.$$

From the momentum equation we can conclude that $(\boldsymbol{J}\times\boldsymbol{B})/c \sim \nabla p \sim \nabla p_e$. If we compare the $\mathbf{v}\times\boldsymbol{B}$ term with either of these,

(28)
$$\frac{\nabla p_e/(en)}{(\mathbf{v}\times\boldsymbol{B})/c} \sim \frac{cT}{ev_{ti}B} \sim \frac{cm_i v_{ti}}{eB} \sim \frac{\rho_i}{a},$$

where ρ_i is the ion gyroradius. We will now make the assumption of strong magnetic fields, or that $\rho_i/a \ll a$, such that $\boldsymbol{J}\times\boldsymbol{B}$ and ∇p can be neglected from the electron momentum equation. The collisional friction can be estimated as follows,

(29)
$$\boldsymbol{R}_e \approx \nu_{ei} m_e n_e \left(\boldsymbol{u}_i - \boldsymbol{u}_e\right) = \underbrace{\frac{\nu_{ei} m_e n}{e}}_{\eta}\boldsymbol{J},$$

where η is the resistivity. We can compare the size of this term with the other terms, noting that $\boldsymbol{J} \sim c\nabla p/B$

(30)
$$\frac{\eta\boldsymbol{J}}{en(\mathbf{v}\times\boldsymbol{B})/c} \sim \frac{m_e n}{e\tau_{ei}}\frac{cp}{aB}\frac{c}{env_{ti}B} \sim \left(\frac{m_e}{m_i}\right)^{1/2}\left(\frac{\rho_i}{a}\right)^2\frac{a}{v_{ti}\tau_{ii}} \ll 1.$$

Under these assumptions, we obtain the ideal Ohm's law,

(31)
$$\boxed{\frac{\mathbf{v}\times\boldsymbol{B}}{c} + \boldsymbol{E} = 0}.$$

This implies that the electric field moving in the frame of the plasma vanishes, considering the Lorentz transformation,

(32)
$$\boldsymbol{E}' = \gamma\left(\boldsymbol{E} + \frac{\mathbf{v}\times\boldsymbol{B}}{c}\right),$$

in the non-relativistic limit.

We now consider the energy equations. The ion energy equation in the single-fluid variables is,

(33)
$$\frac{dp_i}{dt} + \gamma p_i\nabla\cdot\mathbf{v} + (\gamma-1)\left(\boldsymbol{\pi}_i : \nabla\mathbf{v} + \nabla\cdot\boldsymbol{q}_i - Q_i\right) = 0,$$

and the electron energy equation yields,

(34)
$$\frac{\partial p_e}{\partial t} + \left(\mathbf{v} - \frac{\boldsymbol{J}}{en}\right) \cdot \nabla p_e + \gamma p_e \nabla \cdot \left(\mathbf{v} - \frac{\boldsymbol{J}}{en}\right) + (\gamma - 1)\left(\boldsymbol{\pi}_e : \nabla\left(\mathbf{v} - \frac{\boldsymbol{J}}{en}\right) + \nabla \cdot \boldsymbol{q}_e - Q_e\right) = 0.$$

We now consider the energy equations. Some of the terms are already small according to the orderings we have made,

$$\frac{\boldsymbol{\pi} : \nabla \boldsymbol{J}/(en)}{\partial p_e/\partial t} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \frac{\rho_i}{a} \frac{v_{ti}\tau_{ii}}{a} \ll 1 \tag{35}$$

$$\frac{\boldsymbol{J} \cdot \nabla p_e/(en)}{\partial p_e/\partial t} \sim \frac{\rho_i}{a} \ll 1 \tag{36}$$

$$\frac{\boldsymbol{\pi}_e : \nabla \mathbf{v}}{\partial p_e/\partial t} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \frac{v_{ti}\tau_{ii}}{a} \ll 1 \tag{37}$$

$$\frac{\boldsymbol{\pi}_i : \nabla \mathbf{v}}{\partial p_i/\partial t} \sim \frac{v_{ti}\tau_{ii}}{a} \ll 1. \tag{38}$$

If the approximation $\tau_{\mathrm{eq}}\omega \ll 1$, then the Q_e and Q_i terms are small. From the Braginskii calculation, we find

$$Q_i = \frac{3m_e n}{m_i \tau_e}(T_e - T_i) \tag{39}$$

$$Q_e = -\boldsymbol{R}_e \cdot \mathbf{v} - Q_i. \tag{40}$$

The contributions from the $\boldsymbol{q}_e$ and $\boldsymbol{q}_i$ terms can additionally be shown to be small.

In this case, the ideal energy equation reduces to,

$$\boxed{\frac{dp}{dt} + \gamma p \nabla \cdot \mathbf{v} = 0}, \tag{41}$$

by adding the electron and ion equations.

2. Consequences of ideal MHD

2.1. Conservation properties. We will consider the conservative form of each of the MHD equations. The continuity equation is already in conservative form,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{42}$$

The momentum equation can be written as,

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \left(\rho \mathbf{v}\mathbf{v} + \left(p - \frac{B^2}{8\pi}\right) \boldsymbol{bb} + \left(p + \frac{B^2}{8\pi}\right)(\mathbf{I} - \boldsymbol{bb})\right) = 0. \tag{43}$$

The first term corresponds with the Reynolds stress. The second corresponds with the total pressure parallel to the magnetic field, and the third perpendicular to the field. We can also obtain an energy equation in conservative form,

$$\frac{\partial}{\partial t}\underbrace{\left(\frac{\rho v^2}{2}+\frac{p}{\gamma-1}+\frac{B^2}{2}\right)}_{W}+\nabla\cdot\underbrace{\left(\left(\frac{\rho v^2}{2}+\frac{p}{\gamma-1}\right)\mathbf{v}+p\mathbf{v}+\frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B}\right)}_{\boldsymbol{S}}. \tag{44}$$

The total energy is W, including the kinetic, internal, and magnetic energy. The energy from the electric field must be small, as $E/B\sim v/c\ll 1$. The quantity $\boldsymbol{S}$ consists of the flux of kinetic and internal energy, the work done by compression, and the Poynting flux.

2.2. Fluid moves with field lines. We will find that the fluid moves with field lines in an ideal MHD plasma. To do so, we will need to consider how to define the velocity of a magnetic field line. We can generally express a divergence-free magnetic field in the following way,

$$\boldsymbol{B}=\nabla\alpha\times\nabla\beta, \tag{45}$$

for two scalar functions $\alpha(\boldsymbol{r})$ and $\beta(\boldsymbol{r})$.

Let's express the change in magnetic field in terms of the Clebsch representation,

$$\frac{\partial\boldsymbol{B}}{\partial t}=\nabla\left(\frac{\partial\alpha}{\partial t}\right)\times\nabla\beta+\nabla\alpha\times\nabla\left(\frac{\partial\beta}{\partial t}\right)=\nabla\times\left(\frac{\partial\alpha}{\partial t}\nabla\beta-\frac{\partial\beta}{\partial t}\nabla\alpha\right) \tag{46}$$

$$=-c\nabla\times\boldsymbol{E}. \tag{47}$$

This implies,

$$\boldsymbol{E}=-\frac{1}{c}\frac{\partial\alpha}{\partial t}\nabla\beta+\frac{1}{c}\frac{\partial\beta}{\partial t}\nabla\alpha-\nabla\phi \tag{48}$$

$$=-\frac{\mathbf{v}\times\boldsymbol{B}}{c}, \tag{49}$$

for some scalar ϕ.

We can see that $\boldsymbol{E}\cdot\boldsymbol{B}=0$; thus we can write

$$\boldsymbol{E}=-\left(\frac{1}{c}\frac{\partial\alpha}{\partial t}+\frac{\partial\Phi}{\partial\beta}\right)\nabla\beta+\left(\frac{1}{c}\frac{\partial\beta}{\partial t}-\frac{\partial\Phi}{\partial\alpha}\right)\nabla\alpha \tag{50}$$

$$=-\frac{\mathbf{v}\times\boldsymbol{B}}{c}=-\frac{1}{c}\mathbf{v}\times(\nabla\alpha\times\nabla\beta)=-\frac{1}{c}\left((\mathbf{v}\cdot\nabla\beta)\nabla\alpha-(\mathbf{v}\cdot\nabla\alpha)\nabla\beta\right). \tag{51}$$

We can therefore arrive at the following equations of motion

$$\frac{\partial\alpha}{\partial t}+\mathbf{v}\cdot\nabla\alpha=-c\frac{\partial\Phi}{\partial\beta} \tag{52}$$

$$\frac{\partial\beta}{\partial t}+\mathbf{v}\cdot\nabla\beta=c\frac{\partial\Phi}{\partial\alpha}. \tag{53}$$

If $\Phi=0$, we can see that $\mathbf{v}$ corresponds with the velocity of field lines. If $\Phi\neq 0$, we can write the above as,

$$\frac{\partial\alpha}{\partial t}+\left(\mathbf{v}+\frac{c\nabla\Phi\times\boldsymbol{B}}{B^2}\right)\cdot\nabla\alpha=0 \tag{54}$$

$$\frac{\partial\beta}{\partial t}+\left(\mathbf{v}+\frac{c\nabla\Phi\times\boldsymbol{B}}{B^2}\right)\cdot\nabla\beta=0. \tag{55}$$

Thus we find the velocity associated with a field line to be,

$$\boldsymbol{u} = \mathbf{v} + \frac{c\nabla\Phi \times \boldsymbol{B}}{B^2}. \tag{56}$$

A change of coordinates can be made such that that $\boldsymbol{u} = \mathbf{v}$. Let $\alpha = \alpha(\alpha_0, \beta_0, t)$ and $\beta = \beta(\alpha_0, \beta_0, t)$ where α_0 and β_0 are initial conditions. This can be inverted to find (α_0, β_0) as a function of (α, β). Thus we can find a coordinate transformation $(\alpha, \beta) \to (\alpha_0, \beta_0)$ such that

$$\frac{d\alpha_0}{dt} = \frac{\partial \alpha_0}{\partial t} + \mathbf{v} \cdot \nabla\alpha_0 = 0 \tag{57}$$

$$\frac{d\beta_0}{dt} = \frac{\partial \beta_0}{\partial t} + \mathbf{v} \cdot \nabla\beta_0 = 0. \tag{58}$$

This allows to see that, indeed, the field lines move with plasma, as the field line labels, α and β, are constant in a frame moving with the fluid.

2.3. Flux freezing. An important consequence of the ideal MHD equations is Alfvén's flux freezing theorem. This states that the magnetic flux through an open surface moving with an ideal MHD plasma does not change in time. As we will see, a consequence is that ideal MHD does not allow for changes in magnetic topology.

Consider the magnetic flux through a surface, S,

$$\Phi_S = \int_S \boldsymbol{B} \cdot \hat{\boldsymbol{n}}\, d^2x = \oint d\boldsymbol{l} \cdot \boldsymbol{A}. \tag{59}$$

We will compute the change in Φ_S as the surface moves with the plasma and the field evolves according to Maxwell's equations,

$$\frac{d\Phi_S}{dt} = \int_{S(t)} \frac{\partial \boldsymbol{B}}{\partial t} \cdot \hat{\boldsymbol{n}}\, d^2x + \frac{1}{\Delta t}\int_{\Delta S} \boldsymbol{B} \cdot \hat{\boldsymbol{n}}\, d^2x. \tag{60}$$

The change in the surface area due to velocity $\boldsymbol{u}$ is given by $(\boldsymbol{u} \times d\boldsymbol{l})\Delta t$,

$$\frac{d\Phi_S}{dt} = \int_{S(t)} \frac{\partial \boldsymbol{B}}{\partial t} \cdot \hat{\boldsymbol{n}}\, d^2x + \oint_{\partial S(t)} \boldsymbol{B} \times \boldsymbol{u} \cdot d\boldsymbol{l}. \tag{61}$$

The first term corresponds to the change in magnetic field at fixed position, while the second corresponds to the motion of the surface with velocity $\boldsymbol{u}$ (see Figure 1). For a more rigorous proof of the above, we reference to Appendix C in Freidberg and Chapter 4 in Goedbloed.

We found that the velocity of a field line can be written as,

$$\boldsymbol{u} = \mathbf{v} + \frac{\nabla\chi \times \boldsymbol{B}}{B^2}, \tag{62}$$

for single-valued function χ. In this case,

$$\oint_{\partial S(t)} \boldsymbol{B} \times \boldsymbol{u} \cdot d\boldsymbol{l} = \oint_{\partial S(t)} c\boldsymbol{E} \cdot d\boldsymbol{l} + \oint_{\partial S(t)} \nabla_\perp \chi \cdot d\boldsymbol{l}, \tag{63}$$

applying the ideal Ohm's law. The second term vanishes for single-valued χ. Applying Stokes' theorem we obtain

$$\frac{d\Phi_S}{dt} = \int_{S(t)} \hat{\boldsymbol{n}} \cdot \left(\frac{\partial \boldsymbol{B}}{\partial t} + c\nabla \times \boldsymbol{E}\right) d^2x. \tag{64}$$

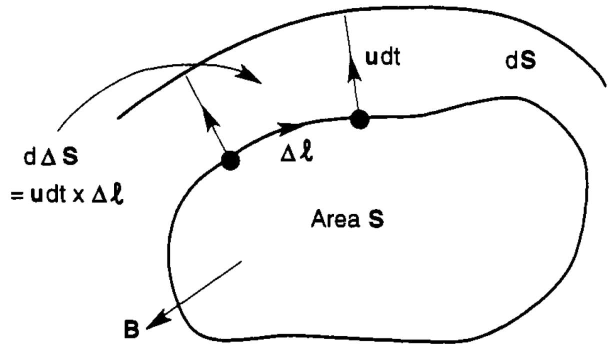


FIGURE 1. The area bounded by a closed contour changes due to the velocity of field lines. Figure reproduced from Goldston & Rutherford.

Thus from Faraday's law we obtain $d\Phi_S/dt = 0$: the magnetic flux is conserved in the frame moving with the plasma.

Consider a flux tube, a volume such that magnetic field lines lie tangent to its boundary surface. Consider two non-intersecting surfaces S_1 and S_2 cutting through the flux tube, as represented in Figure 2, and the volume V of the flux tube delimited by S_1 and S_2. We now demonstrate that the flux through a surface slicing through the flux tube, Φ_S, is independent of the choice of surface. Since $\nabla \cdot \boldsymbol{B} = 0$, then Gauss's law yields

$$0 = \int_V \nabla \cdot \boldsymbol{B}\, d^3x = \int_{S_1} \boldsymbol{B} \cdot \hat{\boldsymbol{n}}\, d^2x - \int_{S_2} \boldsymbol{B} \cdot \hat{\boldsymbol{n}}\, d^2x, \tag{65}$$

where the unit normals, $\hat{\boldsymbol{n}}$ are chosen to be oriented in the same direction for both surface integrals. There is no contribution from the sides of the flux tube, as the field lines are tangent to this surface. As it holds for any choice of bounding surfaces, then for a given flux tube we conclude that Φ_S across a surface S cutting through the flux tube is independent of the choice of surface.

Furthermore, the flux-freezing theorem implies that the flux through a given flux tube, Φ_S, does not change with time. This leads to the well-known result that ideal MHD plasmas do not allow changes in topology. A first example consists of two neighboring flux tubes with fluxes Φ_1 and Φ_2: under ideal MHD evolution, it is impossible for these two flux tubes to merge into one, as this would result in a single flux tube with flux $\Phi_1 + \Phi_2$ (see Figure 3a). A second example consists of two flux tubes that are initially linked to each other: as the flux through each tube must be preserved, the tubes cannot break and must remain linked (see Figure 3b). This can apply to the topology of a field line itself: we can imagine shrinking a flux tube down to a single field line, then the number of times this field line links another field line cannot change under ideal MHD evolution. It is in this sense that the topology of magnetic field lines is fixed under ideal MHD evolution.

2.4. Conservation of helicity. The flux-freezing condition can be seen as a local constraint on the magnetic topology. The ideal MHD equations also conserve the helicity, a global

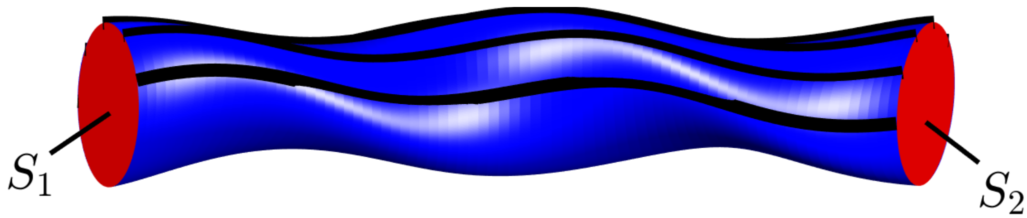
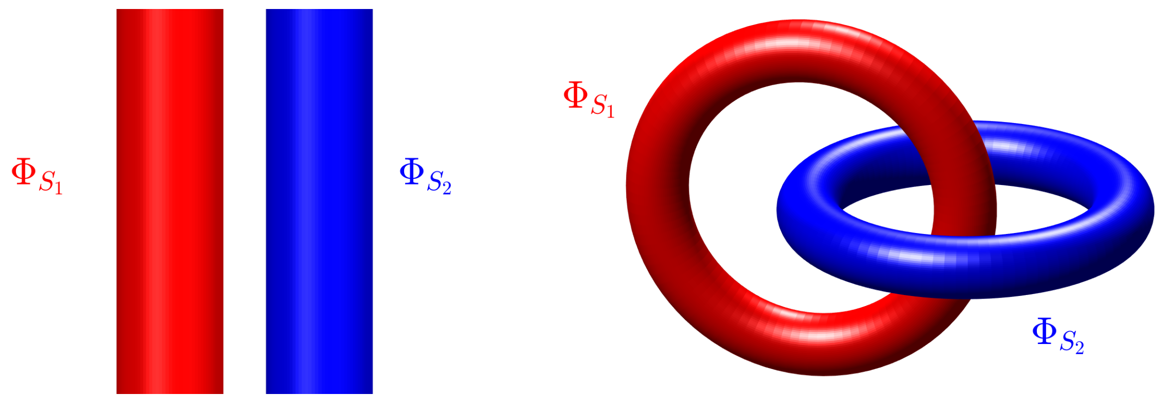


FIGURE 2. A flux tube is shown (blue) on which field lines lie (black). The magnetic flux through the flux tube can be computed with any any surface as shown in (65).



(A) Two flux tubes which are initially adjacent cannot merge under ideal MHD evolution, as their fluxes, Φ_{S_1} and Φ_{S_2}, must be preserved.

(B) Two flux tubes which are initially liked cannot unlike under ideal MHD evolution, as their fluxes, Φ_{S_1} and Φ_{S_2}, must be preserved.

constraint on topology,

$$K = \int_V d^3x\, \boldsymbol{A}\cdot\boldsymbol{B}, \tag{66}$$

where V is a flux tube volume. The change in the helicity can be computed

$$\frac{\partial K}{\partial t} = \int_V d^3x \left(\frac{\partial \boldsymbol{A}}{\partial t}\cdot\boldsymbol{B} + \boldsymbol{A}\cdot\frac{\partial \boldsymbol{B}}{\partial t}\right) + \int_S d^2x\, \mathbf{v}\cdot\hat{\boldsymbol{n}}\boldsymbol{A}\cdot\boldsymbol{B}. \tag{67}$$

The first term accounts for the Eulerian change to the fields, while the second accounts for the motion of the motion of the boundary. Applying Faraday's law and noting that $\boldsymbol{E} = -\nabla\Phi - 1/c\partial\boldsymbol{A}/\partial t$ we find

$$\frac{\partial K}{\partial t} = \int_V d^3x \left(\cancel{(c\nabla\Phi - \boldsymbol{E})\cdot\boldsymbol{B}} - c\underbrace{\boldsymbol{A}\cdot\nabla\times\boldsymbol{E}}_{\nabla\cdot(\boldsymbol{E}\times\boldsymbol{A})}\right) + \int_S d^2x\, \mathbf{v}\cdot\hat{\boldsymbol{n}}\boldsymbol{A}\cdot\boldsymbol{B} \tag{68}$$

$$= \int_S d^2x \left(\mathbf{v}\cdot\hat{n}\boldsymbol{A}\cdot\boldsymbol{B} - c\boldsymbol{E}\times\boldsymbol{A}\cdot\hat{\boldsymbol{n}}\right) \tag{69}$$

$$= \int_S d^2x \left(\mathbf{v}\cdot\hat{n}\boldsymbol{A}\cdot\boldsymbol{B} + (\mathbf{v}\times\boldsymbol{B})\times\boldsymbol{A}\cdot\hat{\boldsymbol{n}}\right) = 0, \tag{70}$$

where we have used $\boldsymbol{B}\cdot\hat{\boldsymbol{n}}=0$ on the boundary of the flux tube.

We can furthermore show that helicity is related to the topology of flux tubes. Consider two linked flux tubes, V_1 and V_2. We will compute the helicity through the volume of V_1,

$$K_1=\int_{V_1} d^3x\,\boldsymbol{A}\cdot\boldsymbol{B}=\oint dl\int_{S_1(l)} d^2x\,\boldsymbol{A}\cdot\boldsymbol{b}B, \tag{71}$$

where we have integrated along a field line on the surface of the flux tube, which is normal to surface $S(l)$. This can be rewritten in the following way,

$$K_1=\underbrace{\left(\oint d\boldsymbol{l}\cdot\boldsymbol{A}\right)}_{\Phi_2}\underbrace{\left(\int_S d^2x\,\boldsymbol{B}\cdot\hat{\boldsymbol{n}}\right)}_{\Phi_1}=\Phi_1\Phi_2. \tag{72}$$

noting that the flux is independent of the choice of surface. For a system of many linked flux tubes, we can by a similar argument, conclude that

$$K_i=\Phi_i\Phi_j N_{ij}, \tag{73}$$

where N_{ij} is the number of times flux tube j passes through the hole in flux tube i. Thus the total helicity of such as system is given by

$$K=\sum_{ij}\Phi_i\Phi_j N_{ij}. \tag{74}$$

Thus helicity conservation can be applied to constrain the evolution of both the local and global topology of an ideal MHD system.

At first glance it may seem that K is non-unique, as it is defined in terms of the vector potential. Suppose we perform a gauge transformation $\boldsymbol{A}\to\boldsymbol{A}+\nabla\chi$. In this case the new helicity is computed to be

$$K'=\int_V d^3x\,\boldsymbol{A}'\cdot\boldsymbol{B}=K+\underbrace{\int_V d^3x\,\boldsymbol{B}\cdot\nabla\chi}_{-\int_V d^3x\,\chi\nabla\cdot\boldsymbol{B}}=K. \tag{75}$$

Thus the helicity is independent of the choice of gauge.