Electromagnetic Waves in a Cold Plasma

Assume $\underset{\sim}{E} = \mathrm{Re}\{\hat{E}\, e^{-i(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{x})}\}$

$$\underset{\sim}{u}_{e,i} = \mathrm{Re}\{\hat{u}_{e,i}\, e^{-i(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{x})}\}$$

Take $\quad \underset{\sim}{B} = B_0\, \underset{\sim}{a}_z \qquad B_0 = \text{const}$

Solve for $\quad \underset{\sim}{J} = \sum_{e,i} q_{e,i}\, n_{e,i}\, \underset{\sim}{u}$

Linearize fluid equations

$$\frac{\partial}{\partial t}\underset{\sim}{u} + \underbrace{\underset{\sim}{u}\cdot\nabla\underset{\sim}{u}}_{\text{non linear}} = +\frac{q}{m}\left(\underset{\sim}{E} + \frac{\underset{\sim}{u}\times B_0 a_z}{c}\right)$$

$$-i\omega\hat{\underset{\sim}{u}} = \frac{q}{m}\hat{\underset{\sim}{E}} + \underset{\sim}{u}\times \underset{\sim}{a}_z\,\Omega_c \qquad \Omega_c = \frac{qB}{mc}$$

solve for $\hat{\underset{\sim}{u}} = \hat{u}_\perp + u_\parallel\, \underset{\sim}{a}_z$

$$\cancel{-i\omega(\hat{\underset{\sim}{u}}\times\underset{\sim}{a}_z) = \frac{q}{m}\hat{\underset{\sim}{E}}\times a_z + \Omega_c(\underset{\sim}{u}\times a_z)\times a_z}$$

$$\boxed{-i\omega u_\parallel = \frac{q}{m}\, a_z\cdot\hat{E}}$$ — motion parallel to magnetic field unaffected by $\underset{\sim}{B}$

$$-i\omega \hat{\underset{\sim}{u}}_\perp = \frac{q}{m}\underset{\sim}{E}_\perp + \hat{\underset{\sim}{u}}_\perp \times \underset{\sim}{a}_z \Omega_c$$

$$\hat{\underset{\sim}{u}}_\perp = \frac{1}{(-i\omega)}\left[\frac{q}{m}\underset{\sim}{E}_\perp + \Omega_c \hat{\underset{\sim}{u}}_\perp \times \underset{\sim}{a}_z\right]$$

$$\Omega_c(\underset{\sim}{u}_\perp \times \underset{\sim}{a}_z) = \frac{\Omega_c}{-i\omega}\left[\frac{q}{m}\underset{\sim}{E}_\perp \times \underset{\sim}{a}_z - \Omega_c \hat{\underset{\sim}{u}}_\perp\right]$$

$$-i\omega \underset{\sim}{u}_\perp = \frac{q}{m}\underset{\sim}{E}_\perp - \frac{\Omega_c}{i\omega}\left[\frac{q}{m}\underset{\sim}{E}_\perp \times \underset{\sim}{a}_z - \Omega_c \underset{\sim}{u}_\perp\right]$$

$$(\omega^2 - \Omega_c^2)\underset{\sim}{u}_\perp = i\omega \frac{q}{m}\underset{\sim}{E}_\perp - \Omega_c \frac{q}{m}\underset{\sim}{E}_\perp \times \underset{\sim}{a}_z$$

$$\boxed{\hat{\underset{\sim}{u}}_\perp = \frac{1}{\omega^2 - \Omega_c^2}\left[i\omega \frac{q}{m}\hat{\underset{\sim}{E}}_\perp - \Omega_c \frac{q}{m}\hat{\underset{\sim}{E}}_\perp \times \underset{\sim}{a}_z\right]}$$

Check against known limits

#1) $\Omega_c \to 0$ (unmagnetized)

$$\underset{\sim}{u}_\perp = \frac{i}{\omega}\frac{q}{m}\underset{\sim}{E}_\perp$$ OK (just like $u_\parallel$)

#2) $\Omega_c \to \infty$ (strong B)

$$\underset{\sim}{u}_\perp = \frac{1}{\Omega_c}\frac{q}{m}\underset{\sim}{E}_\perp \times \underset{\sim}{a}_z = \frac{c}{B}\underset{\sim}{E}_\perp \times \underset{\sim}{a}_z$$

"E×B" drift

#3) next term in ω/Ω

$$\underset{\sim}{u}_\perp = -\frac{i\omega}{\Omega_c^2}\frac{q}{m}\underset{\sim}{E}_\perp + \frac{1}{\Omega_c}\frac{q}{m}\underset{\sim}{E}_\perp \times \underset{-}{a}_z$$

"ExB"

Polarization drift

Solution looks good

Current density

$$\hat{\underset{\sim}{J}} = \sum_q q n \hat{\underset{\sim}{u}}_q$$

IN AMPERE'S LAB

$$\nabla \times \underset{-}{B} = \frac{4\pi}{c}\underset{\sim}{J} + \frac{1}{c}\frac{\partial}{\partial t}\underset{\sim}{E}$$

$$i\underset{\sim}{k} \times \hat{\underset{\sim}{B}} = \frac{4\pi}{c}\hat{\underset{\sim}{J}} - \frac{i\omega}{c}\hat{\underset{\sim}{E}}$$

$$4\pi\hat{\underset{\sim}{J}} = \sum_q \frac{4\pi q^2 n_q}{m}\left[\frac{i\omega}{\omega^2-\Omega_c^2}\underset{\sim}{E}_\perp - \frac{\Omega_c}{\omega^2-\Omega_c^2}\underset{\sim}{E}_\perp \times \underset{\sim}{a}_z - \frac{\underset{-}{a}_z}{i\omega}\underset{-}{a}_z\cdot E\right]$$

$$i\underset{\sim}{k}\times\underset{\sim}{\hat{B}} = -\frac{i\omega}{c}\Bigg[\underset{\sim}{\hat{E}} - \sum_q \frac{\omega_{pq}^2}{\omega^2-\Omega_c^2}\underset{\sim}{\hat{E}}_\perp - \frac{i\Omega_c}{\omega}\frac{\omega_{pq}^2}{\omega^2-\Omega_c^2}\underset{\sim}{\hat{E}}_\perp\times\underset{\sim}{\hat{a}}_z$$

$$- \sum_q \frac{\omega_{pq}^2}{\omega^2}\underset{\sim}{a}_z\underset{\sim}{a}_z\cdot\underset{\sim}{\hat{E}}\Bigg] = -\frac{i\omega}{c}\underset{=}{\epsilon}\cdot\underset{\sim}{E}$$

THIS CAN BE cast in matrix form
dielectric tensor

$$\underset{=}{\epsilon} = \begin{bmatrix} \epsilon_\perp & -i\epsilon_x & 0 \\ +i\epsilon_x & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{bmatrix}$$

$$\boxed{\epsilon_\perp = 1-\sum_q \frac{\omega_{pq}^2}{\omega^2-\Omega_c^2} \qquad \epsilon_x = \sum_q \frac{\Omega_c}{\omega}\frac{\omega_{pq}^2}{\omega^2-\Omega_c^2} \qquad \epsilon_\parallel = 1-\sum_q \frac{\omega_{pq}^2}{\omega^2}}$$

note $(\epsilon^t)^* = \epsilon$ transpose & conjugate

matrix is <u>Hermitian</u>.

Wave Propagation

Ampere's Law

$$i\underset{\sim}{k} \times \hat{B} = -i\frac{\omega}{c}\, \underline{\underline{\epsilon}} \cdot \underset{\sim}{\hat{E}}$$

Faraday's Law

$$i\frac{\omega}{c}\hat{B} = i\underline{k} \times \hat{E}$$

(Do not need to solve Poisson Equation)

$$\boxed{\underset{\sim}{k} \cdot \underline{\underline{\epsilon}} \cdot \hat{\underline{E}} = 0}$$

$$\underset{\sim}{k} \times (\underline{k} \times \underset{\sim}{E}) = -\frac{\omega^2}{c^2}\, \underline{\underline{\epsilon}} \cdot \underset{\sim}{E}$$

$$\frac{c^2}{\omega^2}\, \underset{\sim}{k} \times (\underline{k} \times \underline{E}) = -\underline{\underline{\epsilon}} \cdot \underset{\sim}{E}$$

wave no

$$\underset{\sim}{n} = \frac{c\underline{k}}{\omega}$$

$\cancel{(n^2 - \underset{\sim}{n}\underset{\sim}{n})}$

$$\underset{\sim}{k} \times (\underline{k} \times \underline{E}) = \underline{k}\,\underline{k} \cdot \underline{E} - k^2 \underset{\sim}{E}$$

$$\left[\frac{c^2 k^2}{\omega^2} \underset{\sim}{E} - \frac{c^2}{\omega^2} \underset{\sim}{k}\,\underline{k} \cdot \underline{E} - \underline{\underline{\epsilon}} \cdot \underset{\sim}{E} \right]$$

dispersion relation

$$\det\left[\underline{\underline{I}}\,\frac{k^2c^2}{\omega^2} - \frac{c^2\underline{k}\,\underline{k}}{\omega^2} - \underline{\underline{\epsilon}} \right] = 0$$

~~quadratic equation in~~

determines $\omega(\underset{\sim}{k})$

special cases $\underset{\sim}{k} \perp$ to $\underset{\sim}{B}$

$k \parallel$ to $\underset{\sim}{B}$

$\boxed{k_{\parallel} \text{ to } B}$

$$\det \begin{bmatrix} \frac{k_{\parallel}^2 c^2}{\omega^2} - \epsilon_{\perp} & +i\epsilon_x & 0 \\ -i\epsilon_x & \frac{k_{\parallel}^2 c^2}{\omega^2} - \epsilon_{\perp} & 0 \\ 0 & 0 & \epsilon_{\parallel} \end{bmatrix} = 0$$

$$\epsilon_{\parallel} \left[\left[\left(\frac{k_{\parallel}^2 c^2}{\omega^2} \right) - \epsilon_{\perp} \right]^2 - \epsilon_x^2 \right] = 0$$

THREE solutions

$\epsilon_{\parallel} = 0$ Plasma ~~Pole~~ Wave

$$\frac{k_{\parallel}^2 c^2}{\omega^2} = \epsilon_{\perp} \pm \epsilon_x$$ Light waves

Polarization

$$\begin{bmatrix} \frac{k_{\parallel}^2 c^2}{\omega^2} - \epsilon_{\perp} & +i\epsilon_x & 0 \\ -i\epsilon_x & \frac{k_{\parallel}^2 c^2}{\omega^2} - \epsilon_{\perp} & 0 \\ 0 & 0 & \epsilon_{\parallel} \end{bmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

) $\epsilon_{\parallel} = 0$ $\quad E_x, E_y = 0 \quad E_z \neq 0$

* THIS is an electrostatic wave $\underline{k}$ is parallel to $\underline{E}$ $\quad \underset{\sim}{E} = -i\underline{k}\phi$

* $1 - \sum_q \frac{\omega_p^2}{\omega^2} = 0$ unmagnetized plasma wave

motion along B unaffected by B_0

Second solution

$$\frac{k_{\parallel}^2 c^2}{\omega^2} - \epsilon_{\perp} = \pm \epsilon_x \qquad E_z = 0$$

$\underset{\sim}{E}$ is $\perp$ to $\underset{\sim}{k}$ (electromagnetic wave)

$$\left(\frac{k_{\parallel}^2 c^2}{\omega^2} - \epsilon_{\perp}\right) E_x + i\epsilon_x E_y = 0$$

$$\pm \epsilon_x E_x + i \epsilon_x E_y = 0$$

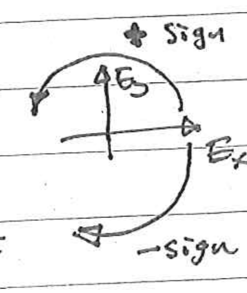
$$\boxed{\frac{E_y}{E_x} = \pm i}$$

circulary polarized with differening rotations

~~E is~~

$$\underset{\sim}{E} = \mathrm{Re}\left\{ \hat{E}_x (\hat{x} \pm i\hat{y}) e^{i(k_{\parallel} z - \omega t)} \right\}$$

work out $\qquad -i \quad i e^{-ic}$



$$E_x = \cancel{E_x}\cos(\omega t) \pm E_y \sin\omega t$$

Rotation of particles

$$\frac{dV_x}{dt} = \Omega_q V_y \qquad \Omega_q = \frac{qB}{mc}$$

$$V_x = \cos \Omega_q t$$ let's assume $B > 0$

$$V_y = -\sin \Omega_q t$$

if $\Omega_q > 0$ clockwise ions

if $\Omega_q < 0$ counter clockwise electrons

Thus (if $B>0$) $+$ sign rotates in electron sense

$-$ sign " in ion sense

$$\frac{k_\parallel^2 c^2}{\omega^2} = 1 - \sum_q \frac{\omega_{pq}^2}{\omega^2 - \Omega_{cq}^2} \pm \sum_q \frac{\Omega_c}{\omega} \frac{\omega_{pq}^2}{\omega^2 - \Omega_{cq}^2}$$

$$= 1 - \sum_q \frac{\omega_{pq}^2}{\omega(\omega \pm \Omega_{cq})}$$

$\Omega_e < 0$

$\Omega_i > 0$

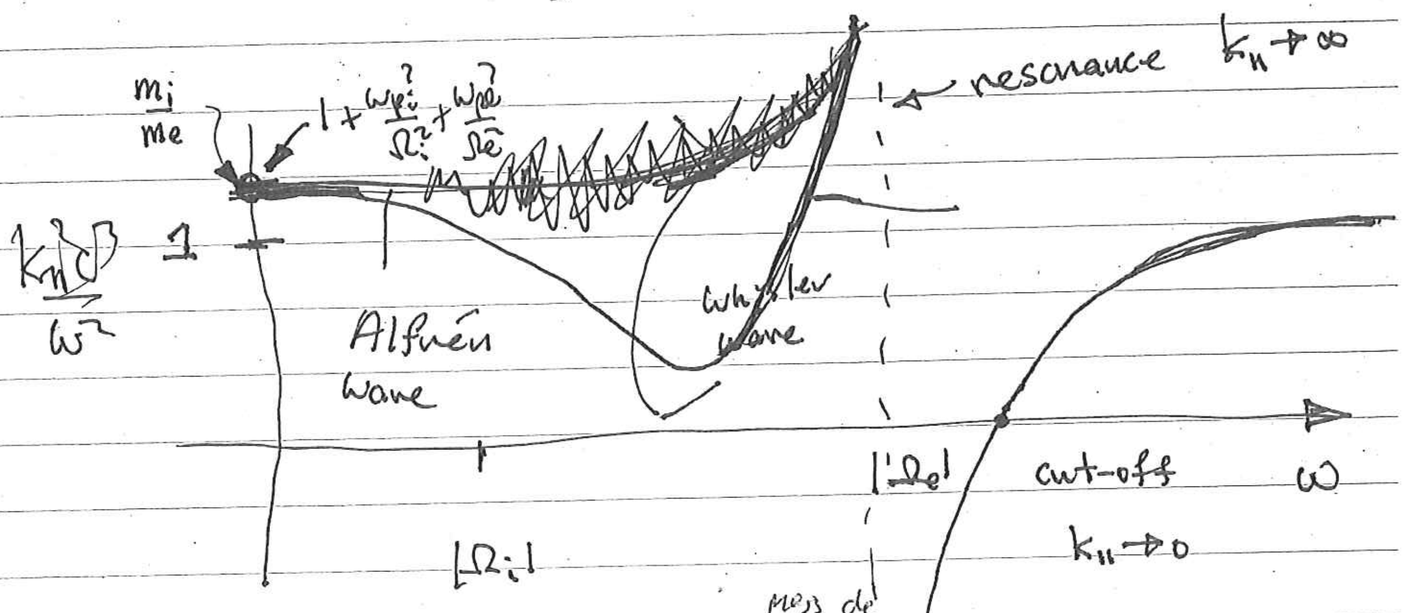
+ sign resonates with electrons

− sign resonates with ions

For + sign

$\omega_{pe}^2 = \frac{|\Omega_e|^2}{2}$

$$\frac{k_\parallel^2 c^2}{\omega^2} = 1 - \frac{\omega_{pi}^2}{\omega(\omega+|\Omega_i|)} - \frac{\omega_{pe}^2}{\omega(\omega-|\Omega_e|)}$$



mass de

$$\frac{\omega_{pi}^2}{\Omega_i^2} + \frac{\omega_{pe}^2}{\Omega_e^2} = \frac{4\pi q_i^2 n_i}{m_i\left(\frac{qB}{m_i c}\right)^2} + \underbrace{\frac{4\pi q_e^2 n_e}{m_e\left(\frac{qB}{m_e c}\right)^2}}_{\text{small}} \simeq \frac{4\pi n_i m_i}{B^2}$$

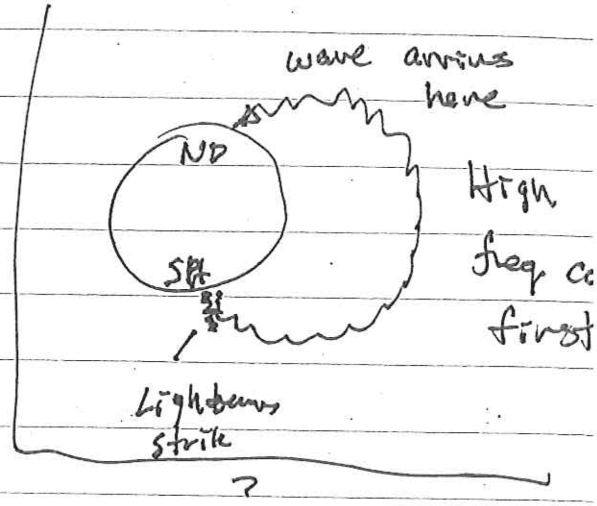
magnetic densn

for small $\omega << |\Omega_i|$

ExB cancel

$$\frac{k_\parallel^2 c^2}{\omega^2} = 1 - \frac{\omega_{pi}^2}{\omega|\Omega_i|}\left[1-\frac{\omega}{|\Omega_i|}\right] + \frac{\omega_{pe}^2}{\omega|\Omega_e|}\left[1+\frac{\omega}{|\Omega_e|}\right]$$

Whistlers Wave



$\Omega_e >> \omega >> \Omega_i, \omega_{pi}$

$$\frac{k_{\parallel}^2 c^2}{\omega^2} = 1 + \frac{\omega_p^2}{\omega|\Omega_e|} \simeq \frac{\omega_{pe}^2}{\omega|\Omega_e|}$$

$$k_{\parallel} = \frac{1}{c}\sqrt{\frac{\omega\,\omega_{pe}^2}{|\Omega_e|}}$$

$$\frac{\partial k_{\parallel}}{\partial \omega} = \frac{1}{V_g} = \frac{1}{2c}\sqrt{\frac{\omega_{pe}^2}{|\Omega_e|}}\frac{1}{\sqrt{\omega}}$$

$$V_g = 2c\frac{\sqrt{\omega}}{\sqrt{\frac{\omega_{pe}^2}{|\Omega_e|}}} = 2c\sqrt{\frac{\omega|\Omega_e|}{\omega_{pe}^2}} \sim \omega^{1/2}$$

Low freq Branch

$$\frac{k_{\parallel}^2 c^2}{\omega^2} = 1 - \underbrace{\frac{\omega_{pi}^2}{\omega|\Omega_i|} + \frac{\omega_{pe}^2}{\omega|\Omega_e|}}_{\text{cancel if } n_e = n_i} + \frac{\omega_{pi}^2}{\Omega_i^2} + \frac{\omega_{pe}^2}{\Omega_e^2}$$

$$\frac{\omega_{pi}^2}{\Omega_i^2} \gg \frac{\omega_{pe}^2}{\Omega_e^2} \qquad \frac{\omega_{pi}^2}{\omega_{pe}^2} = \frac{m_e}{m_i} \qquad \frac{|\Omega_i|}{|\Omega_e|} = \frac{m_e}{m_i}$$

$$\cancel{k_{\parallel}^2 = \frac{\omega^2}{c^2}\left(1 + \omega_{pi}^2\right.}$$

$$\omega^2 = k_{\parallel}^2 \frac{c^2}{\left(1 + \frac{\omega_{pi}^2}{\Omega_i^2}\right)} \cong k_{\parallel}^2 V_a^2$$

Alfven velociy

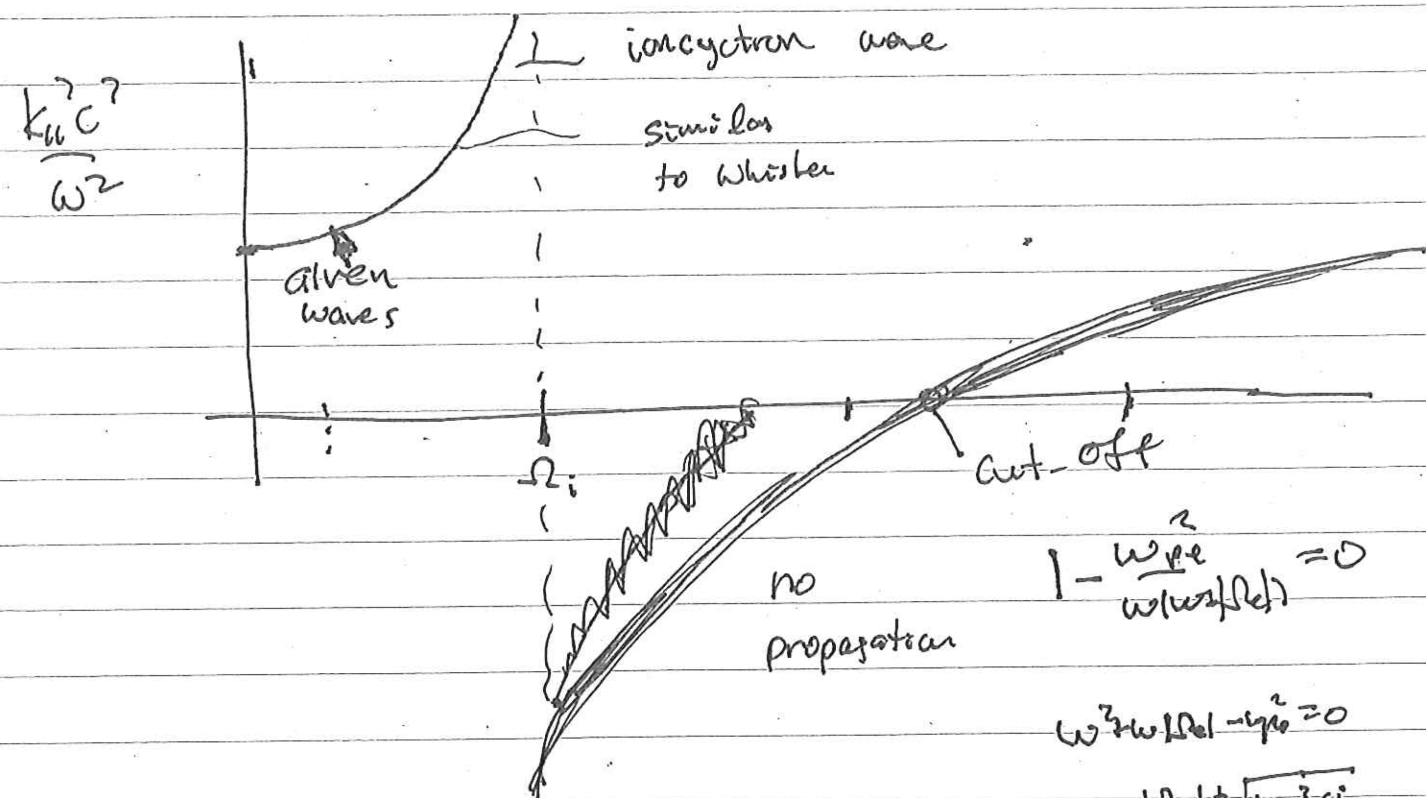
$$V_a = \sqrt{\frac{c^2 \Omega_i^2}{\omega_{pi}^2}} = \sqrt{\frac{B^2}{4\pi \underbrace{n_i M_i}_{\text{mass Density}}}}$$

Like a sound wave

but $P \rightarrow \frac{B^2}{4\pi}$ (magnetic pressure)

$$\frac{k_\parallel^2 c^2}{\omega^2} = 1 - \frac{\omega_{pi}^2}{\omega(\omega-\Omega_i)} - \frac{\omega_{pe}^2}{\omega(\omega+|\Omega_e|)}$$

For − sign



$$1 - \frac{\omega_{pe}^2}{\omega(\omega+|\Omega_e|)} = 0$$

$$\omega^2 + \omega|\Omega_e| - \omega_{pe}^2 = 0$$

$$\omega = \frac{-|\Omega_e| \pm \sqrt{|\Omega_e|^2 + 4\omega_{pe}^2}}{2}$$

$$0 = 1 - \frac{\omega_{pi}^2}{\omega(\omega-|\Omega_i|)} - \frac{\omega_{pe}^2}{\omega(\omega+|\Omega_e|)}$$

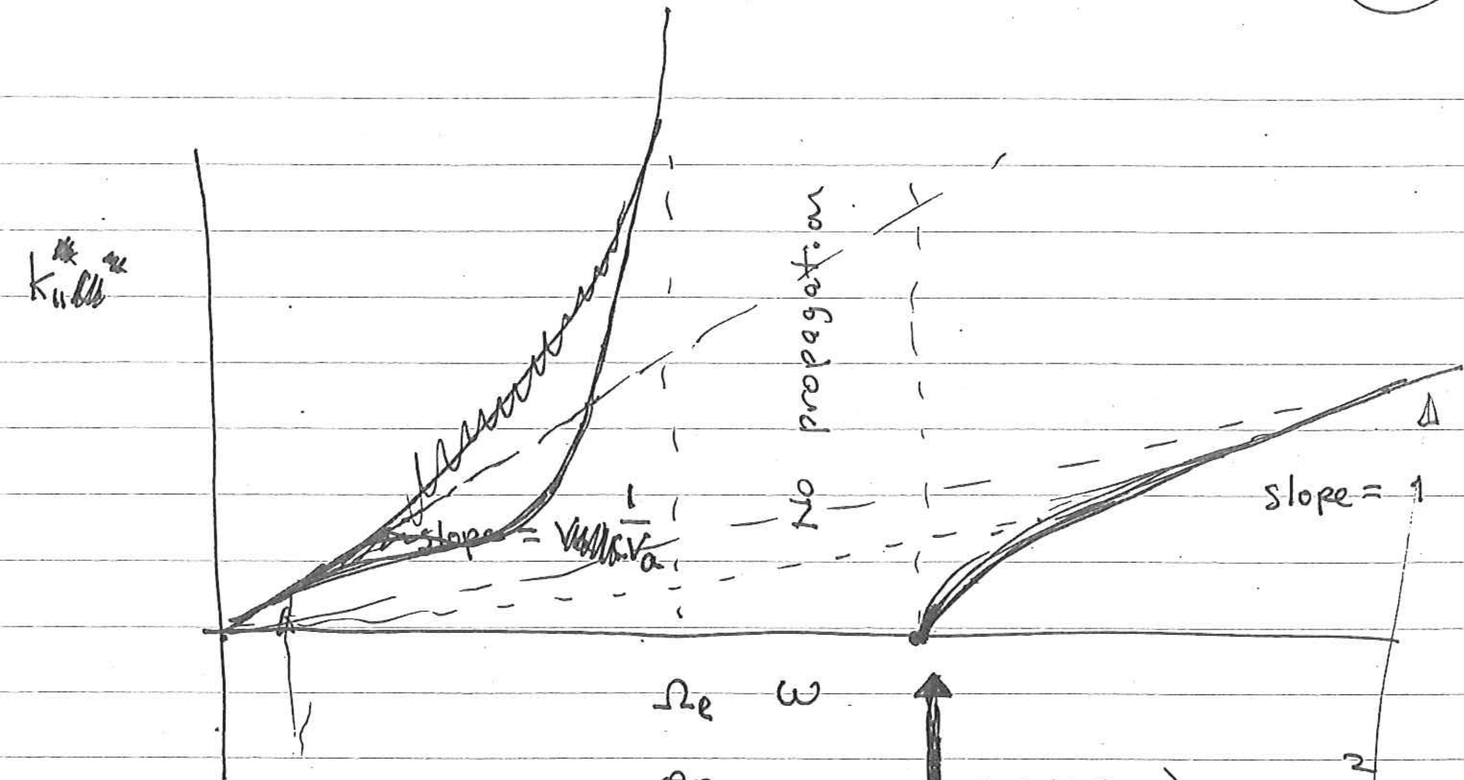
$$\omega(\omega-|\Omega_i|)(\omega+|\Omega_e|) = \omega_{pi}^2 \omega(\omega+|\Omega_e|) - \omega_{pe}^2 \omega(\omega-|\Omega_i|)$$

$$\cancel{\omega^2 - \omega(|\Omega_e|}$$

$$0 = 1 - \frac{\omega_{pi}^2}{\omega(\omega-|\Omega_i|)} - \frac{\omega_{pe}^2}{\omega(\omega+|\Omega_e|)}$$

$$\omega(\omega-|\Omega_i|)(\omega+|\Omega_e|) - \omega_{pe}^2\,\omega(\omega-|\Omega_i|) - \omega_{pi}^2\,\omega(\omega+|\Omega_e|) = 0$$

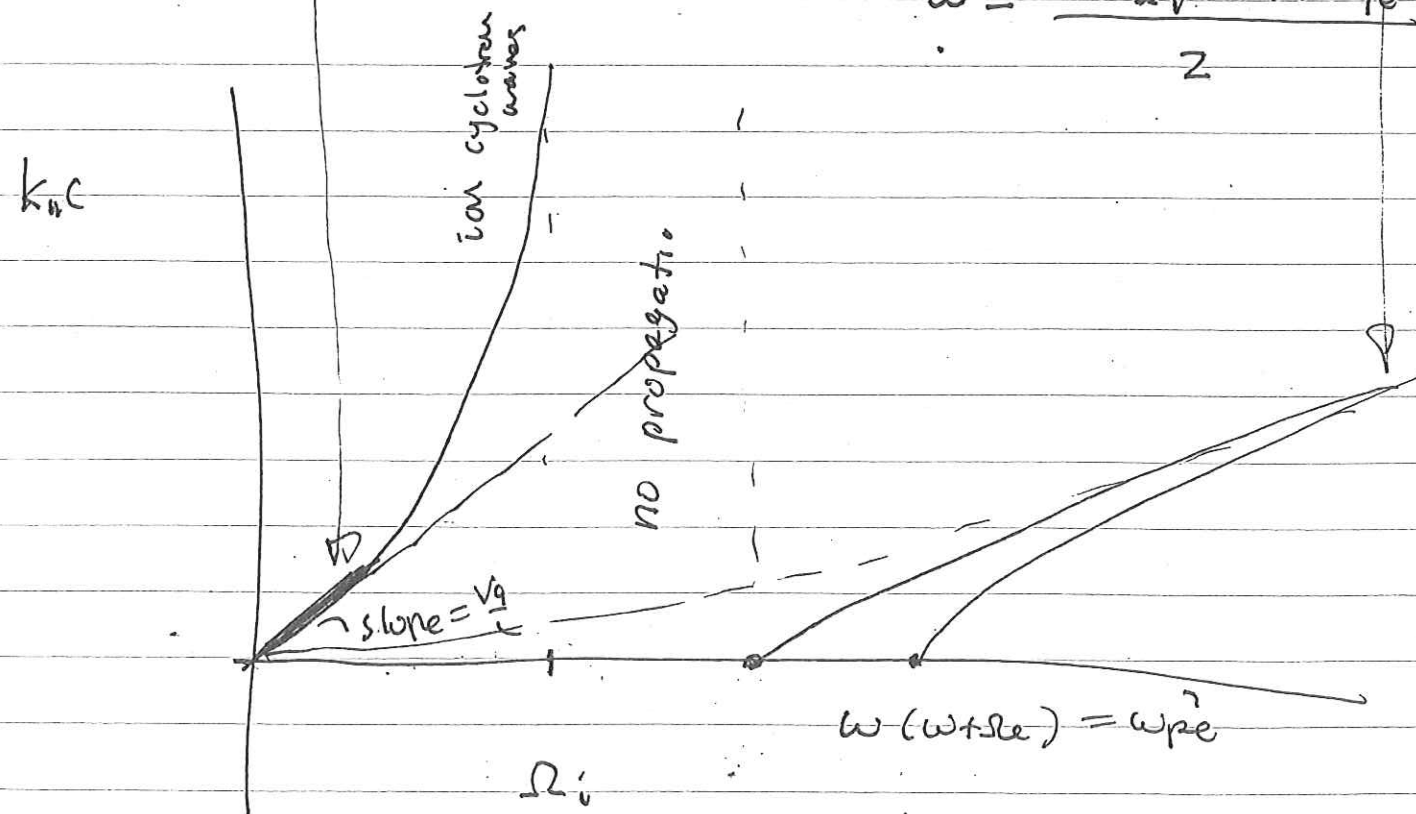
$$\omega^2 + \omega(|\Omega_e| - |\Omega_i|) - \Omega_e\Omega_i - \omega_{pe}^2(\omega - |\Omega_i|) - \omega_{pi}^2(\omega + |\Omega_e|) = 0$$



Res

$\cancel{\omega(\omega - |\Omega_e|) = \omega_{pe}^2}$

$$\omega = \frac{|\Omega_e| \pm \sqrt{|\Omega_e|^2 + 4\omega_{pe}^2}}{2}$$



L

$$\omega = \frac{\sqrt{|\Omega_e|^2 + 4\omega_{pe}^2} - |\Omega_e|}{2}$$

Propagation $\perp$ to B

Take

$$\underline{k} = k_x \underline{a}_x \qquad \underline{\underline{M}} = \underline{\underline{1}}\frac{k^2c^2}{\omega^2} - \frac{\underline{k}\,\underline{k}\,c^2}{\omega^2} - \underline{\underline{\epsilon}}$$

$$\det \begin{bmatrix} -\epsilon_\perp & +i\epsilon_x & 0 \\ -i\epsilon_x & \frac{k_x^2c^2}{\omega^2} - \epsilon_\perp & 0 \\ 0 & 0 & \frac{k_x^2c^2}{\omega^2} - \epsilon_{\parallel} \end{bmatrix}$$

$$\det = \left(\frac{k_x^2c^2}{\omega^2} - \epsilon_{\parallel}\right)\left(-\epsilon_\perp\left(\frac{k_x^2c^2}{\omega^2} - \epsilon_\perp\right) - \epsilon_x^2\right) = 0$$

solution

$$\frac{k_x^2c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2} \qquad \omega^2 = k_x^2c^2 + \omega_p^2$$

ORDINARY MODE
PROPAGATION IN
unmagnetized

$$\boxed{\begin{array}{c} E_z \neq 0 \\ E_x = E_y = 0 \end{array}}$$

$$\epsilon_\perp\left(\frac{k_x^2c^2}{\omega^2}-\epsilon_\perp\right) = -\epsilon_x^2$$

$E_z = 0$
$E_x, E_y \neq 0$

$$\frac{k_x^2c^2}{\omega^2} = \frac{(\epsilon_\perp-\epsilon_x)(\epsilon_\perp+\epsilon_x)}{\epsilon_\perp}$$

Extra-ordinary mode X-mode

cut-off's ($k_x \to 0$) $\epsilon_\perp = \pm\epsilon_x$

resonances ($k_x \to \infty$) $\epsilon_\perp = 0$

$$\epsilon_\perp = 1 - \frac{\omega_{pi}^2}{\omega^2-\Omega_i^2} - \frac{\omega_{pe}^2}{\omega^2-\Omega_e^2} = 0$$

multiply out

two solutions for ω^2

$$(\omega^2-\Omega_i^2)(\omega^2-\Omega_e^2) - \omega_{pi}^2(\omega^2-\Omega_e^2) - \omega_{pe}^2(\omega^2-\Omega_i^2) = 0$$

$$\omega^4 - \omega^2\left((\Omega_e^2+\Omega_i^2) + (\omega_{pe}^2+\omega_{pi}^2)\right) + \omega_{pi}^2\Omega_e^2 + \omega_{pe}^2\Omega_i^2 = 0$$

$$\omega^4 - \omega^2\left[(\Omega_e^2+\Omega_i^2) + (\omega_{pe}^2+\omega_{pi}^2)\right] + \omega_{pi}^2\Omega_e^2 + \omega_{pe}^2\Omega_i^2 = 0$$

$\omega \sim \Omega_e$

~~assume~~ $\cancel{\omega_{pi}^2/\Omega_e^2 \ll 1}$

assume $\omega_{pe}^2 \sim \Omega_e^2 \gg \omega_{pi}^2 \gg \Omega_i^2$

high freq solution

$$1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} = 0 \qquad \omega^2 = \Omega_e^2 + \omega_{pe}^2$$

(upper hybrid resonance)

Low frequency solution (Lower hybrid resonance)

$$1 + \frac{\omega_{pe}^2}{\Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2} = 0$$

$$\omega^2 = \Omega_i^2 + \frac{\omega_{pi}^2}{1 + \frac{\omega_{pe}^2}{\Omega_e^2}} \simeq \frac{\omega_{pi}^2}{\left(1 + \frac{\omega_{pe}^2}{\Omega_e^2}\right)}$$

if $\omega_{pi}^2 \gg \Omega_e^2$

$\Omega_e > \omega \sim \omega_{pi} \gg \Omega_i$ $\qquad$ $\omega^2 \simeq \Omega_e \Omega_i$

$\boxed{\text{involves both ions and electrons}}$

Polarizations ~~at resonance~~

~~at cut-offs~~

~~E_y~~ $\epsilon_\perp E_x + i\,\epsilon_x E_y = 0$

$$\frac{E_y}{E_x} = i\,\frac{\epsilon_\perp}{\epsilon_x}$$

at cut offs $\epsilon_\perp = \pm\epsilon_x$ circularly polarized

at resonance $\epsilon_x \to 0$ $E_y \to 0$ $E_x \neq 0$ $\underset{\sim}{k} \parallel \underset{\sim}{E}$

wave is electrostatic

$\boxed{\text{also } \dfrac{\epsilon_\perp}{\epsilon_x} = -1 \text{ circularly polarized}}$

?

at cyclotron resonance

$\omega - \Omega_e$

Check light waves.

SPECIAL case $B \to 0$. $\epsilon_\perp = 1 - \frac{\omega_p^2}{\omega^2}$

$\epsilon_x \to 0$

$\epsilon_\parallel = 0$ $E_x, E_y = 0$ $E_z \neq 0$

$\frac{k_\parallel^2 c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$ Two solutions

$E_z = 0$ $E_x, E_y \neq 0$

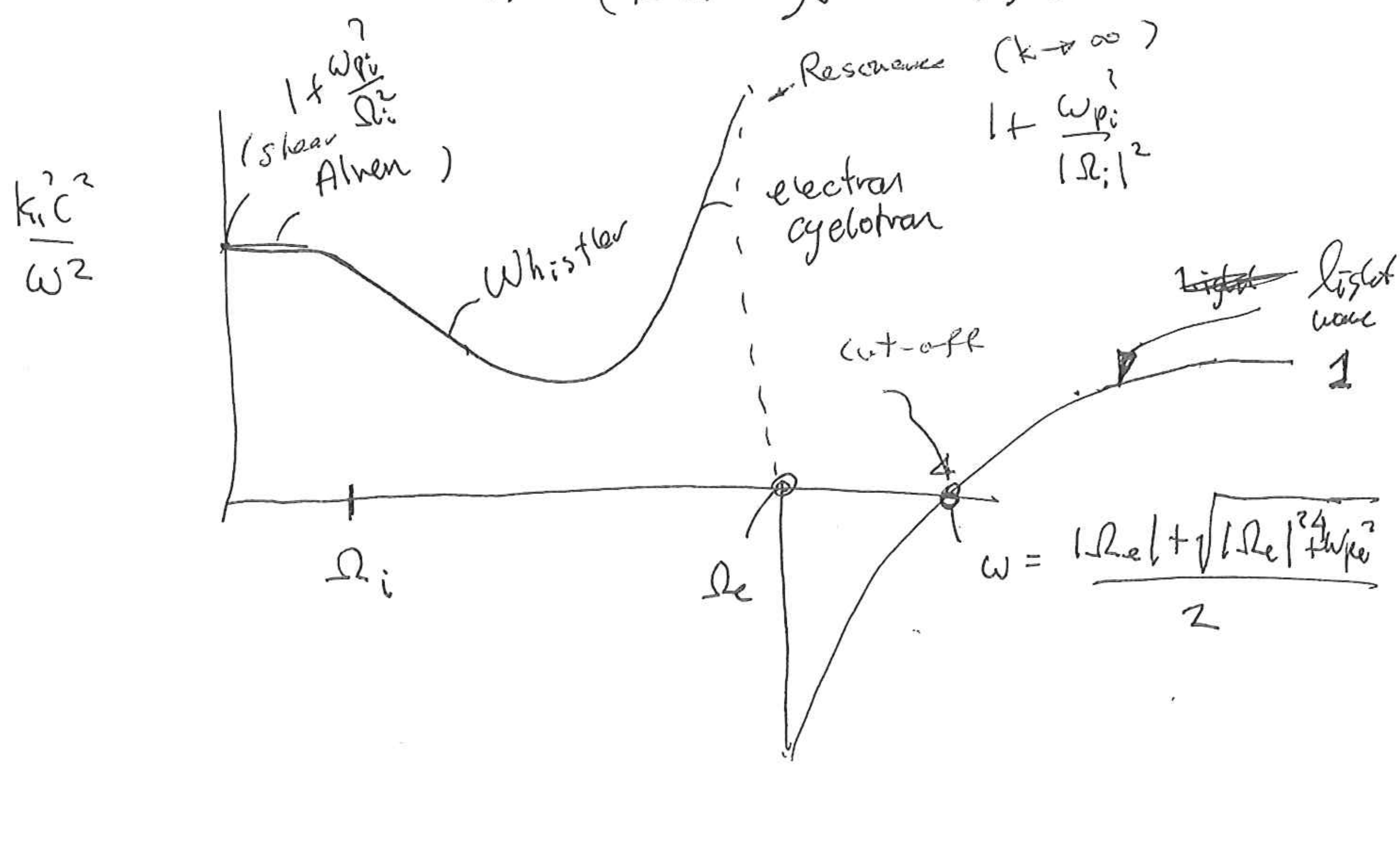
+ sign

$$\frac{k_\parallel^2 c^2}{\omega^2} = 1 - \frac{\omega_{pi}^2}{\omega(\omega+|\Omega_i|)} - \frac{\omega_{pe}^2}{\omega(\omega-|\Omega_e|)}$$

$$= 1 - \frac{\omega_{pi}^2}{|\Omega_i|}\left[\frac{|\Omega_i|}{\omega(\omega+|\Omega_i|)} + \frac{|\Omega_e|}{\omega(\omega-|\Omega_e|)}\right]$$

$$= 1 - \frac{\omega_{pi}^2}{|\Omega_i|}\frac{1}{\omega}\left[\frac{(\omega-|\Omega_e|)|\Omega_i| + (\omega+|\Omega_i|)|\Omega_e|}{(\omega+|\Omega_i|)(\omega-|\Omega_e|)}\right]$$

$$= 1 + \frac{\omega_{pi}^2}{|\Omega_i|}\left[\frac{|\Omega_e|-|\Omega_i|}{(|\Omega_e|-\omega)(\omega+|\Omega_i|)}\right]$$



$$\epsilon_\perp \pm \epsilon_x = 1 - \sum_q \frac{\omega_{pq}^2}{\omega^2 - \Omega_q^2} \pm \sum_q \frac{\Omega_q}{\omega} \frac{\omega_{pq}^2}{\omega^2 - \Omega_q^2}$$

$$\epsilon_\perp \pm \epsilon_x = 1 - \sum_q \frac{\omega_{pq}^2}{\omega^2 - \Omega_q^2}\left[1 \mp \frac{\Omega_q}{\omega}\right]$$

$$= 1 - \sum_q \frac{\omega_{pq}^2}{(\omega - \Omega_q)(\omega + \Omega_q)} \frac{\omega \mp \Omega_q}{\omega}$$

$$= 1 - \sum_q \frac{\omega_{pq}^2}{\omega(\omega \pm \Omega_q)}$$

$$\epsilon_\perp \hat{E}_x - i\epsilon_x \hat{E}_y = 0$$

$$\frac{\hat{E}_y}{\hat{E}_x} = -i\left(\frac{\epsilon_\perp}{\epsilon_x}\right) \qquad \text{if } \epsilon_\perp = \epsilon_x$$

+ sign	$\epsilon_\perp = -\epsilon_x$	$\frac{\hat{E}_y}{\hat{E}_x} = i$
− sign	$\epsilon_\perp = \epsilon_x$	$\frac{\hat{E}_y}{\hat{E}_x} = -i$

+ sign rotates in electron sense

− sign rotates in ion sense

$$\frac{k_x^2 c^2}{\omega^2} = 1 + \frac{\omega_{pe}^2}{\Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2} = \sum_q \frac{\Omega_q}{\omega} \omega_{pq}^2$$

$$\frac{\left(1 - \sum_q \frac{\omega_{pq}^2}{\omega(\omega \pm \Omega_q)}\right)\left(1 - \frac{\omega_{pq}^2}{\omega(\omega - \Omega_q)}\right)}{G_1}$$

$$\frac{\left(1 + \frac{\omega_{pi}^2}{\Omega_i^2}\right)^*}{1 + \cdot}$$

$$\boxed{\frac{k_x^2 c^2}{\omega^2} \simeq \frac{\omega_{pi}^2}{\Omega_i^2}}$$ compressional alfven

$$\frac{k_x^2 c^2}{\omega^2} = \frac{\omega_{pe}^2}{\Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2} - \sum_q \frac{\Omega_q}{\omega} \omega_{pq}^2$$

$$\frac{\left(1 - \sum_q \frac{\omega_{pq}^2}{\omega(\omega \pm \Omega_q)}\right)\left(1 - \frac{\omega_{pq}^2}{\omega(\omega - \Omega_q)}\right)}{G_1}$$

$$\frac{\left(1 + \frac{\omega_{pi}^2}{\Omega_i^2}\right)}{1 + }$$

$$\boxed{\frac{k_x^2 c^2}{\omega^2} \simeq \frac{\omega_{pi}^2}{\Omega_i^2}}$$ compressional alfven

Assume $\omega_{pe} \sim |\Omega_e|$

$$\omega_{pi} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \omega_{pe}$$

$$|\Omega_i| \sim \left(\frac{m_e}{m_i}\right) |\Omega_e|$$

if $\omega_{pe} \sim |\Omega_e|$

$$\omega_{pi} \gg |\Omega_i|$$

Low freq $\omega \ll |\Omega_e|, |\Omega_i|$

$$\frac{k_x^2 c^2}{\omega^2} \simeq \frac{(\epsilon_\perp - \epsilon_x)(\epsilon_\perp + \epsilon_x)}{\epsilon_\perp}$$

$$\epsilon_\perp \simeq 1 + \frac{\omega_{pi}^2}{\Omega_i^2} + \frac{\omega_{pe}^2}{\Omega_e^2}$$

$$\frac{k_x^2 c^2}{\omega^2} \simeq \epsilon_\perp - \frac{\epsilon_x^2}{\epsilon_\perp}$$

High freq

$$\boxed{\begin{aligned} &1 - \frac{\omega_{pe}^2}{\omega_{uc}(\omega_{uc} - |\Omega_e|)} = 0 \\ &1 - \frac{\omega_{pe}^2}{\omega_{Lc}(\omega_{Lc} + |\Omega_e|)} = 0 \end{aligned}}$$

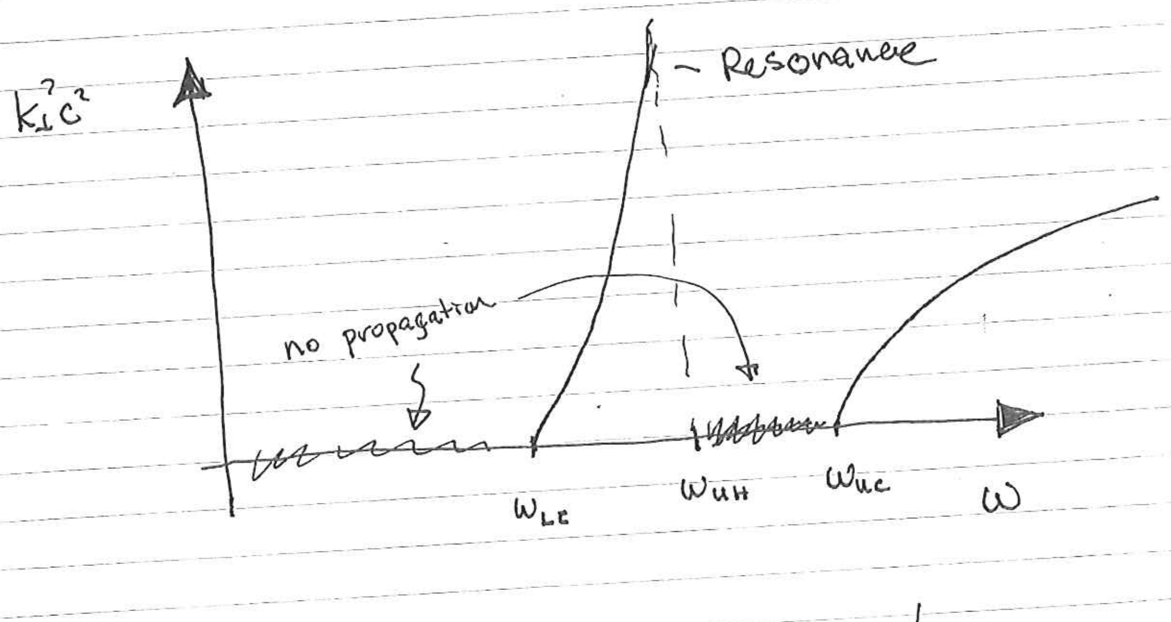
for $\omega \sim \Omega_e, \omega_{pe} \gg \Omega_i, \omega_{pi}$

$$k_\perp^2 c^2 = \frac{(\omega^2 + \omega\Omega_e - \omega_{pe}^2)(\omega^2 - \omega\Omega_e - \omega_{pe}^2)}{\omega^2 - (\omega_{pe}^2 + \Omega_e^2)}$$

upper hybrid resonance

cut-offs $\qquad \omega_{uc} = \dfrac{|\Omega_e| + \sqrt{|\Omega_e|^2 + 4\omega_{pe}^2}}{2} > \omega_{UH}$

$$\omega_{Lc} = \frac{\sqrt{|\Omega_e|^2 + 4\omega_{pe}^2} - |\Omega_e|}{2} < \omega_{UH}$$



Note no cyclotron resonance!

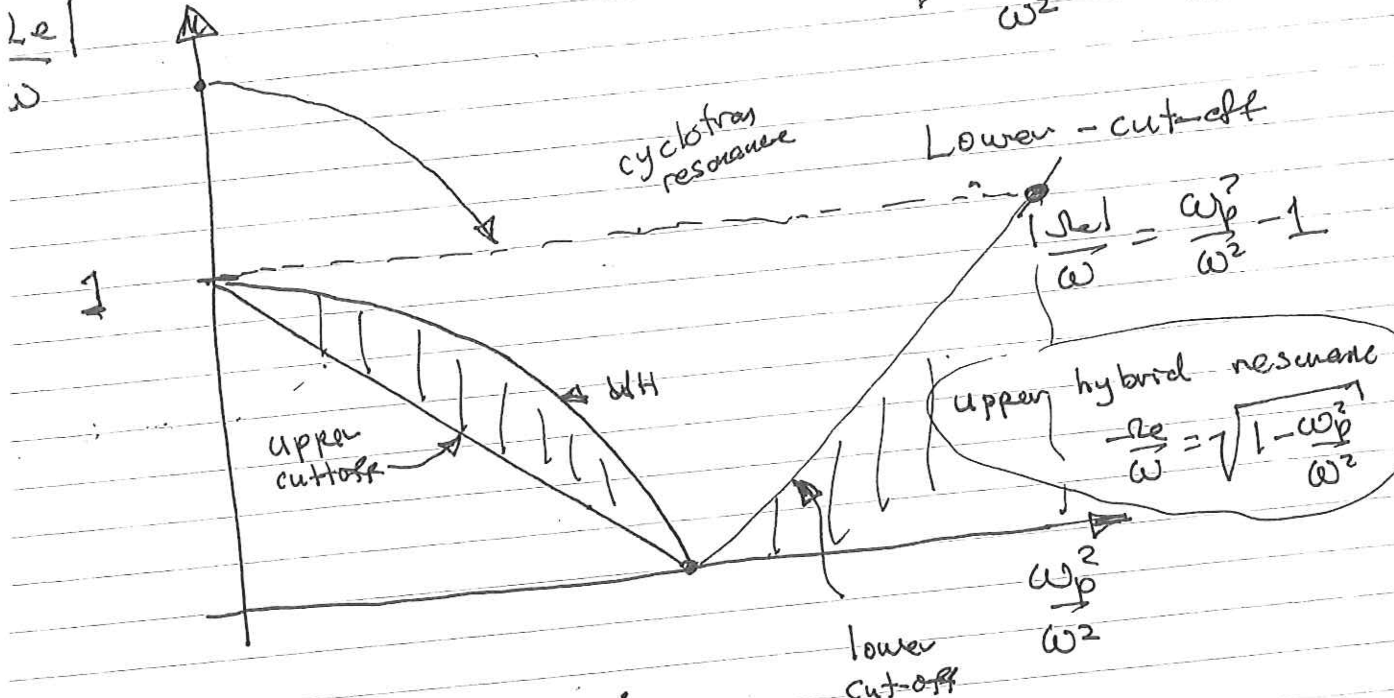
wave adopts polarization $\circlearrowright$

electrons $+\circlearrowleft$

upper cuttoff

$\cancel{\omega^2 = \omega_p^2}$

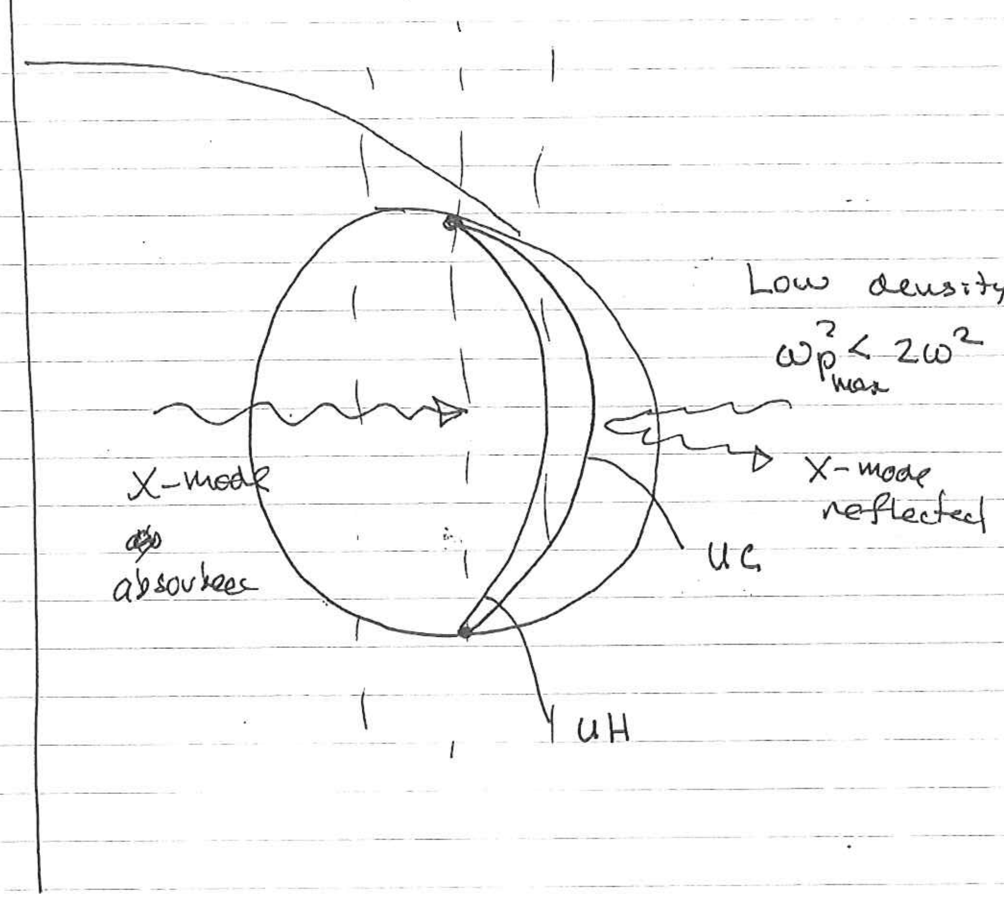
$$1 - \frac{\omega_p^2}{\omega^2} = \frac{|\Omega_e|}{\omega}$$



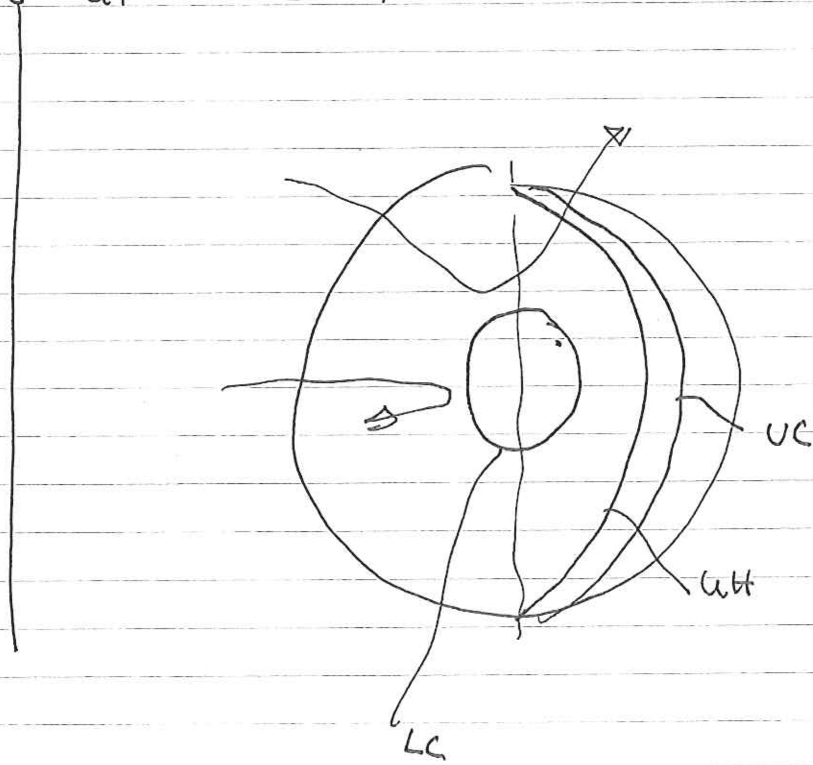
forbidden zones

density limit $\boxed{\frac{\omega_p^2}{\omega^2} < 2}$

Tokamak



Absorbtion occurs at $\omega = \Omega_{CR}$ due Thermal effects



Faraday Rotation

Consider case $\omega >> \Omega_e, \omega_{pe}$ (Ω_i, ω_{pi})

$$k_\parallel^2 c^2 = \omega^2 - \frac{\omega\,\omega_{pe}^2}{\omega - |\Omega_e|}$$

$+$ Rotates with electrons

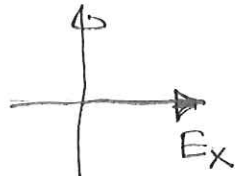
$$k_\parallel^2 c^2 = \omega^2 - \frac{\omega\,\omega_{pe}^2}{\omega + |\Omega_e|}$$

$-$ Rotates with ions

~~$k_{\parallel(+)} \neq k_{\parallel(-)}$~~

In limit ω_{pe}^2 or $\Omega_e \to 0$ $\quad k_{\parallel +} = k_{\parallel -}$

Right and left circularly polarized waves have the same wave vector. ~~Plane~~ ~~Polarized~~ Waves can also be represented as plane polarized.



$= \cancel{\frac{1}{2}(E_x}$

$$\hat{\underline{E}} = \frac{1}{2}(\underline{a}_x + i\underline{a}_y)E_+ + \frac{1}{2}(\underline{a}_x - i\underline{a}_y)E_-$$

if $E_+ = E_-$ wave is plane polarized in x-direction

if ~~E_{+}~~ $E_- = -E_+$ wave is plane polarized in y-direction

$$\underset{\sim}{E} = Re\left\{ \left((\underset{\sim}{a}_x + i\underset{\sim}{a}_y)\hat{E}_+ e^{ik_{\parallel +} z} + (\underset{\sim}{a}_x - i\underset{\sim}{a}_y)\hat{E}_- e^{ik_{\parallel -} z} \right) e^{-i\omega t} \right.$$

if $k_{\parallel +} = k_{\parallel -}$ and wave is plane polarized at $z = 0$ it will be plane polarized for all z in the same direction

if $k_{\parallel +} \neq k_{\parallel -}$ polarization will change with z

suppose wave is plane polarized at $z = 0$ $\hat{E}_+ = \hat{E}_-$

$$\overline{k_{\parallel}} = \frac{1}{2}k$$

$E =$ ~~$Re\{$~~

also assume

$$k_{\parallel +} = \bar{k}_{\parallel} + \frac{1}{2}\Delta k \qquad \hat{E}_+ = \hat{E}_-$$

$$k_{\parallel -} = \bar{k}_{\parallel} - \frac{1}{2}\Delta k$$

Then

$$\underset{\sim}{E} = Re\left\{ \underset{\sim}{a}_x \left(E_+ e^{ik_{\parallel +} z} + E_+ e^{ik_{\parallel -} z} \right) + i\, a_y \left(E_+ e^{ik_{\parallel +} z} + E_+ e^{ik_{\parallel -} z} \right) \right\}$$

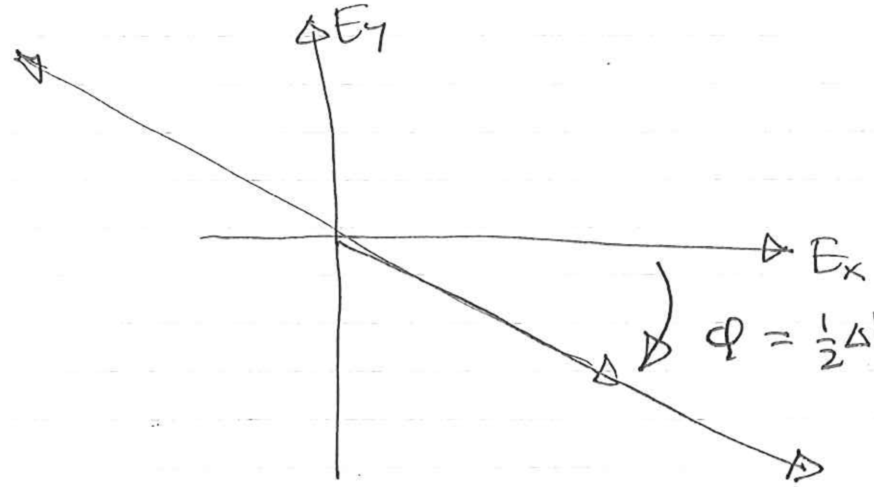
$$= Re\left\{ E_+ e^{i\bar{k}_{\parallel} z} \left[\underset{\sim}{a}_x \, 2\cos(\Delta k z/2) - \underset{\sim}{a}_y \, 2\sin(\Delta k z/2) \right] \right\}$$

$$E = \mathrm{Re}\left\{ 2\hat{E}_+ e^{i(\bar{k}_\parallel z - \omega t)} \right.$$

$$\left. \left[\underset{\sim}{a}_x \cos\left(\tfrac{1}{2}\Delta k_\parallel z\right) - \underset{\sim}{a}_y \sin\left(\tfrac{1}{2}\Delta k_\parallel z\right) \right] \right\}$$

Note ~~az~~ E_x & E_y are in phase

so this is a plane polarized wave.



Rotation is dependent on difference in k

for $\omega \gg \omega_{pe}$

$k_\parallel$

Now find $\bar{k}_{\parallel}$ and Δk

$$k_{\parallel +}^2 c^2 = 1 - \frac{\omega_p^2}{\omega(\omega - |\Omega_e|)}$$

$$k_{\parallel -}^2 c^2 = 1 - \frac{\omega_p^2}{\omega(\omega + |\Omega_e|)}$$

$$\frac{(k_{\parallel +}^2 + k_{\parallel -}^2)c^2}{\omega^2} = 2 - \frac{\omega_p^2}{\omega}\left(\frac{1}{(\omega - |\Omega_e|)} + \frac{1}{\omega + |\Omega_e|}\right)$$

$$2\bar{k}_{\parallel}^2 c^2 + \frac{1}{2}\Delta k^2 c^2 = 2\left(1 - \frac{\omega_p^2}{\omega^2 - |\Omega_e|^2}\right)$$

$$(\bar{k}_{\parallel} + \tfrac{1}{2}\Delta k)^2 = \bar{k}_{\parallel}^2 + \bar{k}_{\parallel}\Delta k + \frac{1}{4}\Delta k^2$$

$$\frac{\bar{k}_{\parallel} c}{\omega} \cong \sqrt{1 - \frac{\omega_p^2}{\omega^2 - |\Omega_e|^2}}$$

$$\cancel{2k}\ \frac{(k_{\parallel +}^2 - k_{\parallel -}^2)c^2}{\omega^2} = \frac{2\bar{k}_{\parallel}\Delta k\, c^2}{\omega^2} = -\frac{\omega_p^2}{\omega}\left(\frac{1}{\Omega - |\Omega_e|} - \frac{1}{\omega + |\Omega_e|}\right)$$

$$= -\frac{2\omega_{pe}^2|\Omega_e|}{\omega(\omega^2 - |\Omega_e|^2)}$$

THUS

$$\frac{\Delta k_0}{\omega} = -i \frac{\omega}{k_{\parallel} c} \frac{\omega_{pe}^2 |\Omega_e|}{\omega(\omega^2 - |\Omega_e|^2)} \propto n$$

measurement of line integrated density. (if B is known)