Rayleigh – Taylor instability

Let's add gravity to OUR SYSTEM OF EQN'S

$$\rho \frac{dV}{dt} = -\nabla p + \rho \underset{\sim}{g} + \frac{\underset{\sim}{J} \times \underset{\sim}{B}}{c}$$

$$\frac{dp}{dt} + \gamma p \nabla \cdot \underset{\sim}{V} = 0 \qquad \frac{4\pi}{c} \underset{\sim}{J} = \nabla \times \underset{\sim}{B}$$

$$\frac{d\rho}{dt} + \rho \nabla \cdot \underset{\sim}{V} = 0$$

$$\frac{\partial B}{\partial t} = \nabla \times (\underset{\sim}{V} \times \underset{\sim}{B})$$

$\underset{\sim}{g}$ = acceleration due to gravity

$\rho \underset{\sim}{g}$ = force density

- why add gravity

equilibrium

$$+\nabla\left(p + \frac{B^2}{8\pi}\right) = \frac{B^2}{4\pi} \underset{\sim}{K} + \rho \underset{\sim}{g}$$

gravity simulates the effect of curved field lines w/out having complications of geometry

<u>equibrium</u> (x, y, z)

$\underset{\sim}{B} = (0, 0, B_0)$

$p = p(x)$

$\underset{\sim}{J} = 0$

$\rho = \rho(\underset{\sim}{x})$ $\quad\quad \underset{\sim}{g} = (g, 0, 0)$

$$\frac{dp}{dx} = \rho g \quad \text{Equil}$$

let's assume incompressible and eliminate

p equation

$\nabla \cdot \underset{\sim}{\xi} = 0$ instead of $p_1 + \xi \cdot \nabla p_0 + \Gamma p \, \nabla \cdot \xi = 0$

$\underset{\sim}{\xi} = (\xi_x, \xi_y, 0)$

suppose $\Gamma \gg 1$

$\nabla \cdot \xi = \frac{p + \xi \cdot \nabla p}{\Gamma p}$

all quantities vary as $\exp[i k y - i\omega t]$

$$\underset{\sim}{B}_1 = \nabla \times \underset{\sim}{\xi} \times \underset{\sim}{B}_0$$

$$= \underset{\sim}{\xi}\, \nabla \cdot \underset{\sim}{B}_0 + \underset{\sim}{B}_0 \cdot \nabla \underset{\sim}{\xi} - \underset{\sim}{B}_0 \nabla \cdot \underset{\sim}{\xi} - \underset{\sim}{\xi} \cdot \nabla \underset{\sim}{B}_0$$

$= 0$ no perterbed magnetic field

$\underset{\sim}{J}_1 = 0$ $\nabla \times (\xi \times B)$

$\boxed{\text{Equation of momentum Balance}}$

$$-\omega^2 \rho \underset{\sim}{\xi} = -\nabla p_1 + \rho_1 \underset{\sim}{g}$$

$\hat{z} \cdot \nabla \times (\ )$ elm

to eliminate p_1 $\hat{z} \cdot \nabla \times \{\text{above}\}$

$$\hat{z} \cdot \nabla \times (-\omega^2 \rho \underset{\sim}{\xi}) = \frac{d}{dx}(-\omega^2 \rho \xi_y) + ik\omega^2 \rho \xi_x$$

but $ik\xi_y + \frac{d}{dx}\xi_x = 0$ $\xi_y = -\frac{1}{ik}\frac{d}{dx}\xi_x$

$$\frac{\omega^2}{ik}\left[\frac{d}{dx}\rho\frac{d}{dx}\xi_x - k^2 \rho \xi_x\right] = z \cdot \nabla \times (-\omega^2 \rho \underset{\sim}{\xi})$$

O.K.

$$z \cdot \nabla \times \left(- \cancel{\nabla p_1}^{\,0} + \rho_1 \underset{\sim}{g} \right)$$

$$= \overset{z}{z} \cdot \overset{y}{\nabla} \times \rho_1 \overset{x}{\underset{\sim}{g}} = - i k g \tilde{\rho}_1$$

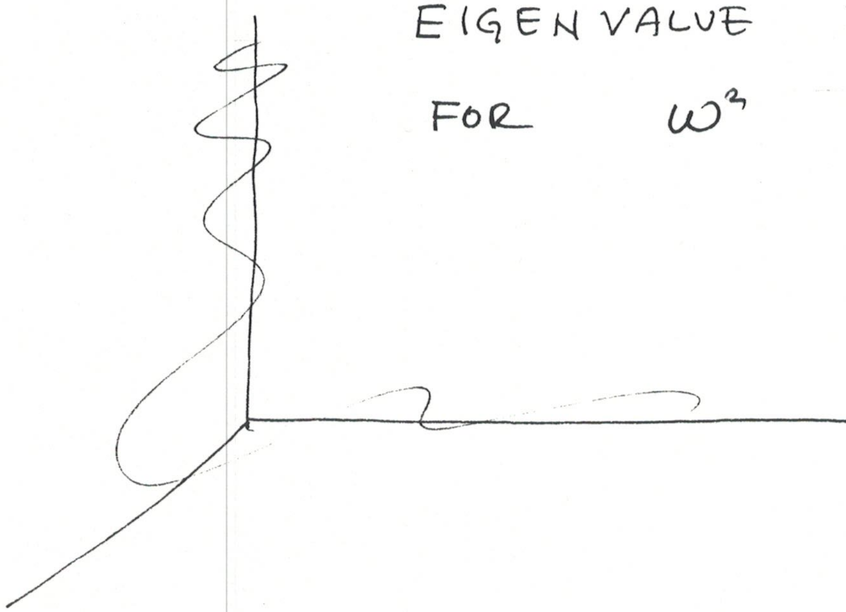
$$\boxed{\tilde{\rho}_1 + \xi_x \frac{\partial \rho_0}{\partial x} = 0}$$

FINALLY

$$\omega^2 \left[\frac{d}{dx} \rho_0 \frac{d}{dx} \xi_x - k^2 \rho_0 \xi_x \right] = - k^2 g \frac{\partial \rho_0}{\partial x} \xi_x$$

EIGENVALUE EQUATION

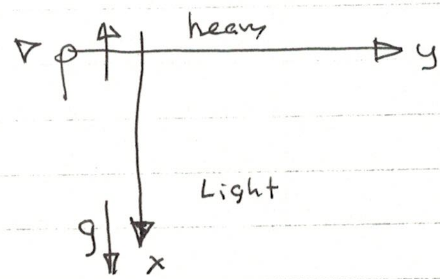
FOR ω^3



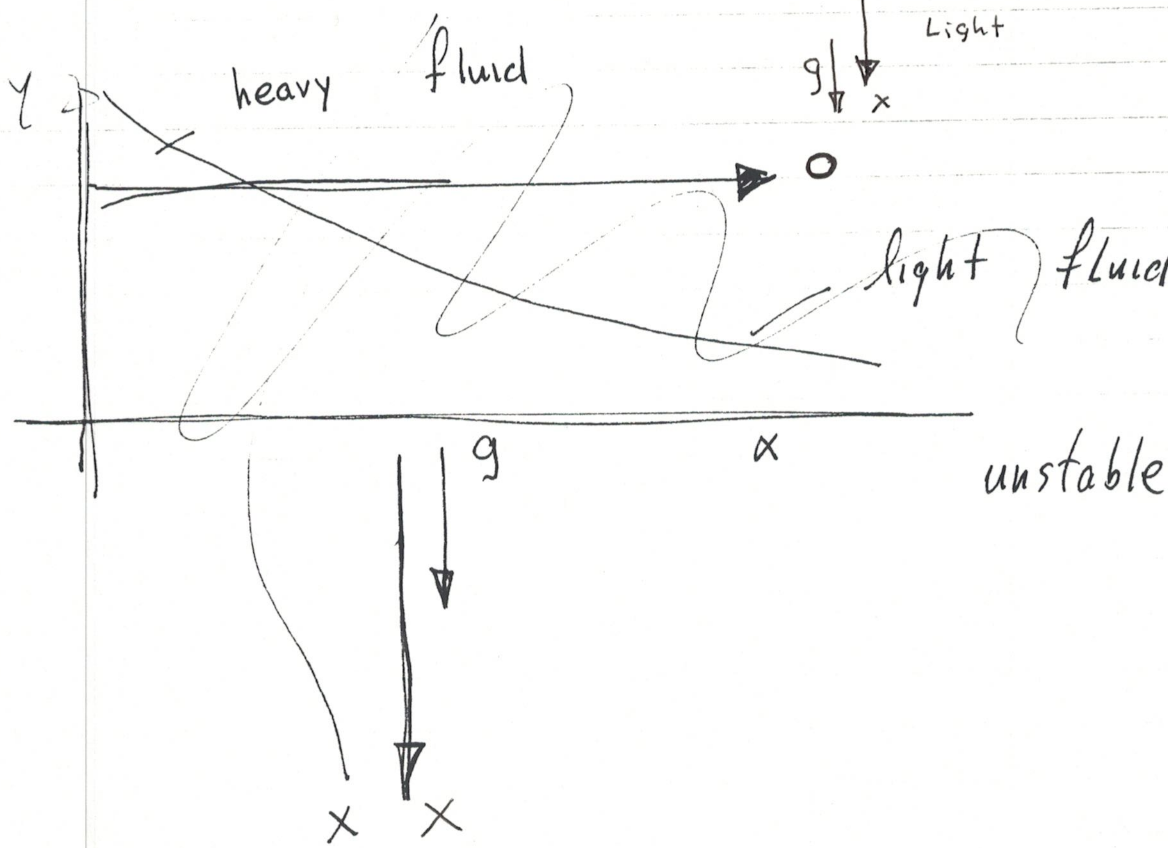
LETS NEGLECT THE x derivative and solve locally $k^{*} \gg \frac{d}{dx}^{*}$

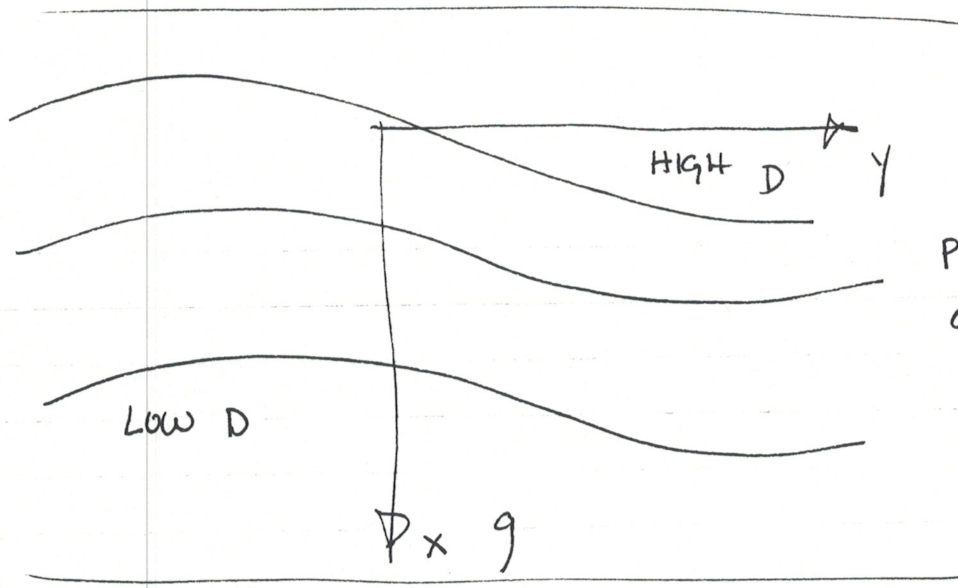
$$\omega^2 = \frac{g \frac{\partial \rho_o}{\partial x}}{\rho_o} = \frac{g}{r_n}$$

IF $\frac{g}{\rho_o} \frac{\partial \rho_o}{\partial x} < 0$ $\quad \omega^2 < 0$ heavy fluid supported by light fluid



unstable





perturbed density contours

system moves to a lower energy state

Quadratic form

$$\xi_x(x_1, x_2) = 0$$

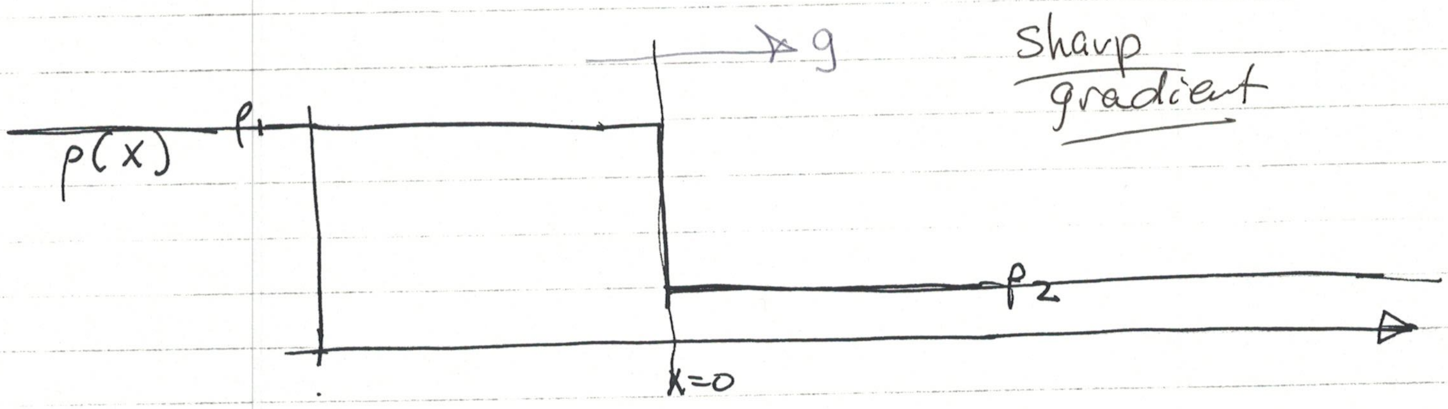
$$\int_{x_1}^{x_2} dx\, \xi_x\, \omega^2 \left[\frac{d}{dx} \rho_0 \frac{d}{dx} \xi_x - k^2 \rho_0 \xi_x \right] = - \int_{x_1}^{x_2} k^2 g \frac{d\rho_0}{dx} \xi_x^2\, dx$$

$$\omega^2 = \frac{\int_{x_1}^{x_2} dx\, k^2 g \frac{d\rho_0}{dX} \xi_x^2}{\int_{x_1}^{X} dX\, \rho_0 \left[\left(\frac{d\xi_x}{dX} \right)^2 + k^2 \xi_x \right]}$$

a variation WRT TO ξ_x reproduces

~~THE ω^2 are extreme values of~~

Two limits are interesting



for $x > 0$

$$\rho_2 \omega^2 \left(\frac{d^2}{dx^2} - k^2 \right) \xi_x = 0$$

$$\xi_x = \xi_x(0+) \exp(-kx)$$

for $x < 0$

$$\rho_1 \omega^2 \left(\frac{d^2}{dx^2} - k^2 \right) \xi_x = 0$$

$$\xi_x = \xi_x(0-) \exp(kx)$$

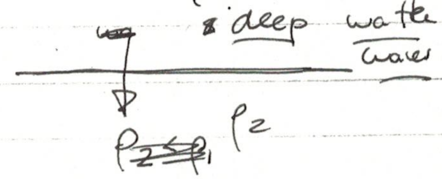
ξ is continuous $\quad \xi_x(0+) = \xi_x(0-)$

integrate
across discontinuity

$$\omega^2 \left[\rho_2 \frac{d\xi_x}{dX}\Big|_{0+} - \rho_1 \frac{d\xi_x}{dX}\Big|_{0-} \right] = -k^2 g \left(\rho_2-\rho_1\right) \xi_x(0)$$

$$-k\,\xi_x(0) \qquad -k\,\xi_x(0)$$

$$\omega^2 = \frac{+k\, g\, (\rho_2-\rho_1)}{(\rho_2+\rho_1)}$$



$\rho_2>\rho_1$

$\omega \sim \sqrt{k}$ Taylor

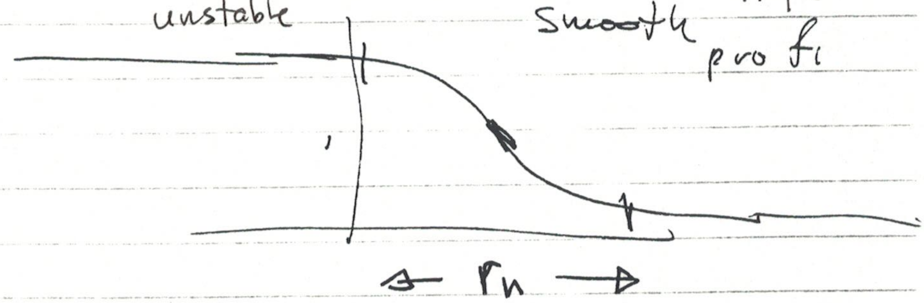
$\rho_1>\rho_2$

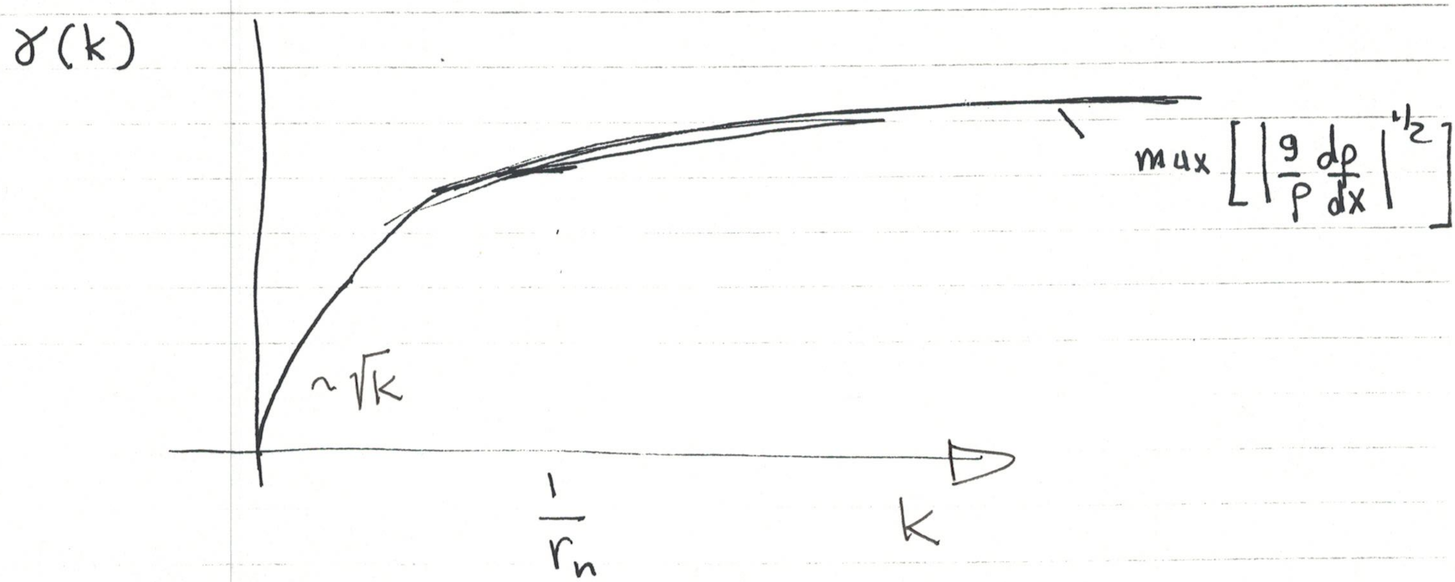
IF $\rho_2<\rho_1$

unstable

smooth profile

$$\gamma = \sqrt{\frac{g\,k\,\Delta\rho}{2\bar{\rho}}}$$





<u>notice</u> all this happened and

B_0 was ~~power~~ powerless to

stop the instability

magnetic field

Infact perturbation had ~~to~~ no

perturbed magnetic field ~~pur~~ "flute

instability"

$(\xi_x, \xi_y, 0)$

Lets include a $k_{\parallel} B_0$ $\qquad \nabla \cdot \xi = 0$

$$\underset{\sim}{B}_1 = \underset{\sim}{B}_0 \cdot \nabla \xi = i k_{\parallel} B_0 \underset{\sim}{\xi}$$

$$\underset{\sim}{J}_1 = \frac{c}{4\pi} \nabla \times \underset{\sim}{B}_1 = \qquad J_{1z} = \frac{c}{4\pi} \left[\frac{d}{dx} B_{1y} - i k B_{1x} \right]$$

$$\hat{z} \cdot \nabla \times \frac{\underset{\sim}{J}_1 \times \underset{\sim}{B}_0}{c} = \nabla \cdot \left(\frac{\underset{\sim}{J}_1 \times \underset{\sim}{B}_0}{c} \right) \times \hat{z} = \nabla \cdot (\cancel{J_1 B_0 z}$$

$$\frac{\underset{\sim}{B}_0 J_{1z}}{c} - \frac{B_0 \underset{\sim}{J}_1}{c}$$

$\nabla \cdot \underset{\sim}{J}_1 = 0$

$$= \frac{B_0 i k_{\parallel}}{4\pi} \left[\frac{d}{dx} \underbrace{i k_{\parallel} B_0 \xi_y}_{B_y} - i k \underbrace{i k_{\parallel} B_0 \xi_x}_{B_{1x}} \right]$$

$$i k \xi_y + \frac{d}{dx} \xi_x = 0 \qquad \xi_y = -\frac{1}{ik} \frac{d}{dx} \xi_x$$

$$\hat{z} \cdot \nabla \times \frac{\underset{\sim}{J}_1 \times \underset{\sim}{B}_0}{c} = - \frac{i k_{\parallel}^2 B_0^2}{4\pi k} \left[\frac{d^2}{dx^2} \xi_x - k^2 \xi_x \right]$$

k

$$\frac{d}{dx}\left(\omega^2\rho - \frac{k_{\parallel}^2 B_0^2}{4\pi}\right)\frac{d}{dx}\xi_x - k^2\left(\omega^2\rho - \frac{k_{\parallel}^2 B_0^2}{4\pi}\right)\xi_x = \underbrace{-k^2 g\frac{\partial\rho_0}{\partial x}}_{\text{instability}}\xi_x$$

notice shear alfven dispersion relations

Local limit $k^2 \gg \frac{d}{dx}$

$$\underbrace{\omega^2\rho - \frac{k_{\parallel}^2 B_0^2}{4\pi}}_{\text{shear Alfven wave}} = -k^2 g\frac{\partial\rho_0}{\partial x}$$

unstable if $\frac{k_{\parallel}^2 B_0^2}{4\pi} + k^2 g\frac{\partial\rho_0}{\partial x} < 0$

energy released by rearrang mass density

stabilizing term due to field line bending

Quadratic Form

ξ_x times above & integrate

$$+\ \omega^2 \int dx\, \rho \left[\left(\frac{d\xi_x}{dx}\right)^2 + k^2 \xi_x \right]$$

Kinetic energy of flow

Bending energy

$$= \int dx \left[\frac{k_\parallel^2 B_0^2}{4\pi} \left[\left(\frac{d\xi_x}{dx}\right)^2 + k^2 \xi_x \right] + k^2 g \frac{\partial \rho_0}{\partial x} \xi_x^2 \right]$$

energy released by gavity + displ

$$\boxed{\underline{\text{Energy}}\ \ \underline{\text{principle}}}$$