Non linear Waves

So far we have been solving for the properties of waves assuming the wave amplitudes are infinitessimally small.

$\hat{\underset{\sim}{E}}$ and $\hat{u}$ are small perturbations

neglect terms which are products of two small numbers.

Result of this approximation

$$\underset{\sim}{E} = Re\left\{ \sum_{\underset{\sim}{k},s} \hat{\underset{\sim}{E}}_k \, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega_s(\underset{\sim}{k})t)} \right\}$$

$\underset{\sim}{k}$ different possible $\underset{\sim}{k}$-vectors

$\omega_s(\underset{\sim}{k})$ different solution of

$\det|\underset{\sim}{G}| = 0$ for each $\underset{\sim}{k}$

B

Because $\underset{\sim}{E}_k$ satisfies linear equations ~~so~~ the terms in sum are independent and the amplitudes are arbitrary (as long as they are small)

* What happens when wave amplitudes are not small

* product term are important

Start with one wave

$$\underset{\sim}{E}_{k} = Re\left\{ \underset{\sim}{E}_0 \, e^{i(\underset{\sim}{k}_0 \cdot \underset{\sim}{x} - \omega_0 t)} \right\}$$

$$n =$$

PRO. $\underset{\sim}{u} =$

Products then produce 2^{nd} harmonics

~~And are~~

$$\underset{\sim}{E} = \sum_n Re\left\{ \underset{\sim}{E}_{0,n} \, e^{ni(\underset{\sim}{k}_0 \cdot \underset{\sim}{x} - \omega_0 t)} \right\}$$

frequencies $\omega_0, 2\omega_0, 3\omega_0$ ----

wave numbers $\underset{\sim}{k}_0, 2\underset{\sim}{k}_0, \ldots 3\underset{\sim}{k}_0$

amplitude of n^{th} harmonic scales a $(E_{0,1})^n$.

Example: electrostatic plasma waves

J.M. Dawson Phys. Rev. 113, 383 (1959)

Suppose there are two waves present

$$\underset{\sim}{E} = Re\left\{ \hat{\underset{\sim}{E}}_0 e^{i(\underset{\sim}{k}_0 \cdot \underset{\sim}{x} - \omega_0 t)} + \hat{\underset{\sim}{E}}_1 e^{i(\underset{\sim}{k}_1 \cdot \underset{\sim}{x} - \omega_1 t)} \right\}$$

product $e^{i(\underset{\sim}{k}_0 \pm \underset{\sim}{k}_1)\cdot \underset{\sim}{x} - i(\omega_0 \pm \omega_1)t}$

$e^{i(-\underset{\sim}{k}_0 \pm \underset{\sim}{k}_1)\cdot \underset{\sim}{x} - i(-\omega_0 \pm \omega_1)t}$

lets suppose $\omega_0 > \omega_1$

freq's

$\omega_0 + \omega_1$	$\underset{\sim}{k}_0 + \underset{\sim}{k}_1$
$\omega_0 - \omega_1$	$\underset{\sim}{k}_0 - \underset{\sim}{k}_1$

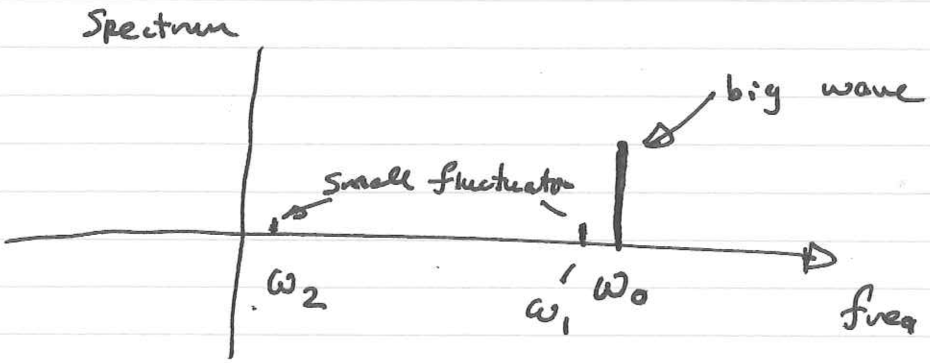
D

Intermodulation

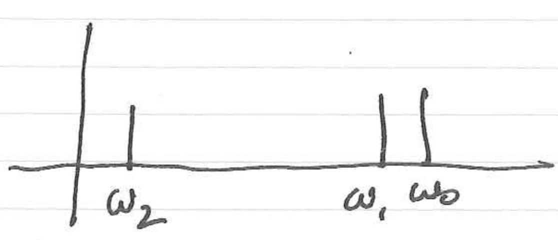
Things become interesting when there is another linear wave with frequency ω_2 and wave number $\underset{\sim}{k}_2$ such that

$$\left.\begin{aligned} \omega_2 &= \omega_0 - \omega_1 \\ \underset{\sim}{k}_2 &= \underset{\sim}{k}_0 - \underset{\sim}{k}_1 \end{aligned}\right\} \text{ frequency and wave number matching}$$

Then parametric decay occurs

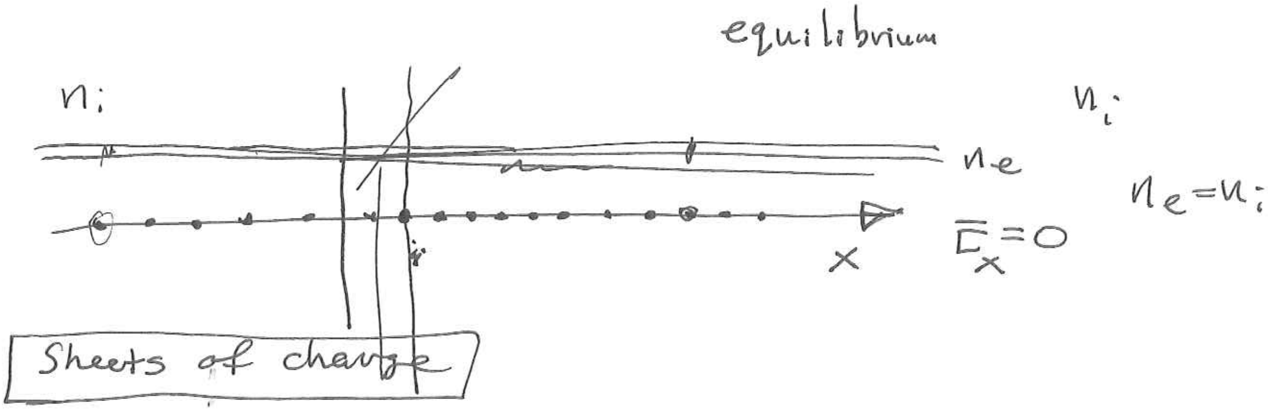


fluctuations grow exponentially and big wave decreases



Non linear Plasma Oscillations

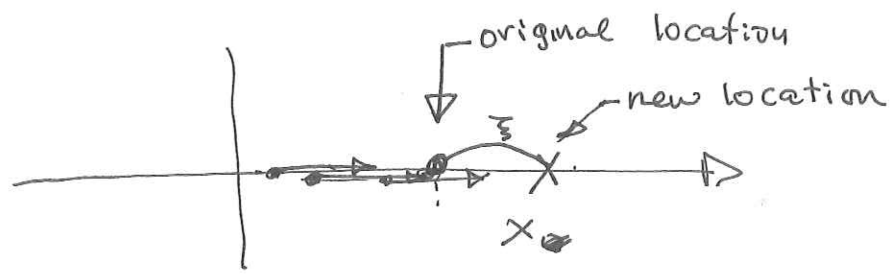
Ion density – uniform & imobile



now imagine electron at location x_0

is moved to $x_e = x_0 + \xi(x_0, t)$

displacement



Also suppose that ~~elect~~ electrons don't cross $d x_e(x_0)/dx_0 > 0$ $1 + \frac{\partial \xi}{\partial x_0} > 0$

Amount of new + charge on left

$K_s = \xi n_0 q_i$

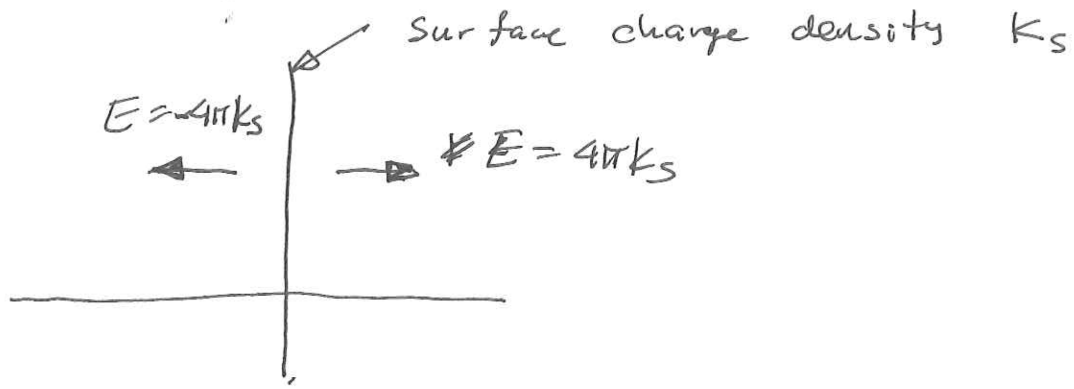
$E_x = 4\pi K_s$

$$m_e \frac{d^2 \xi}{dt^2} = q_e E_x$$

$$E_x = \underbrace{4\pi q_i n_{0i}}\, \xi = -\underbrace{4\pi q_e n_{0e}}\, \xi$$

charge balance in equilibrium

~~th.~~ 1D think about sheets of charge



$$m_e \frac{d^2}{dt^2} \xi = -4\pi q_e^2 n_{0e} \xi$$

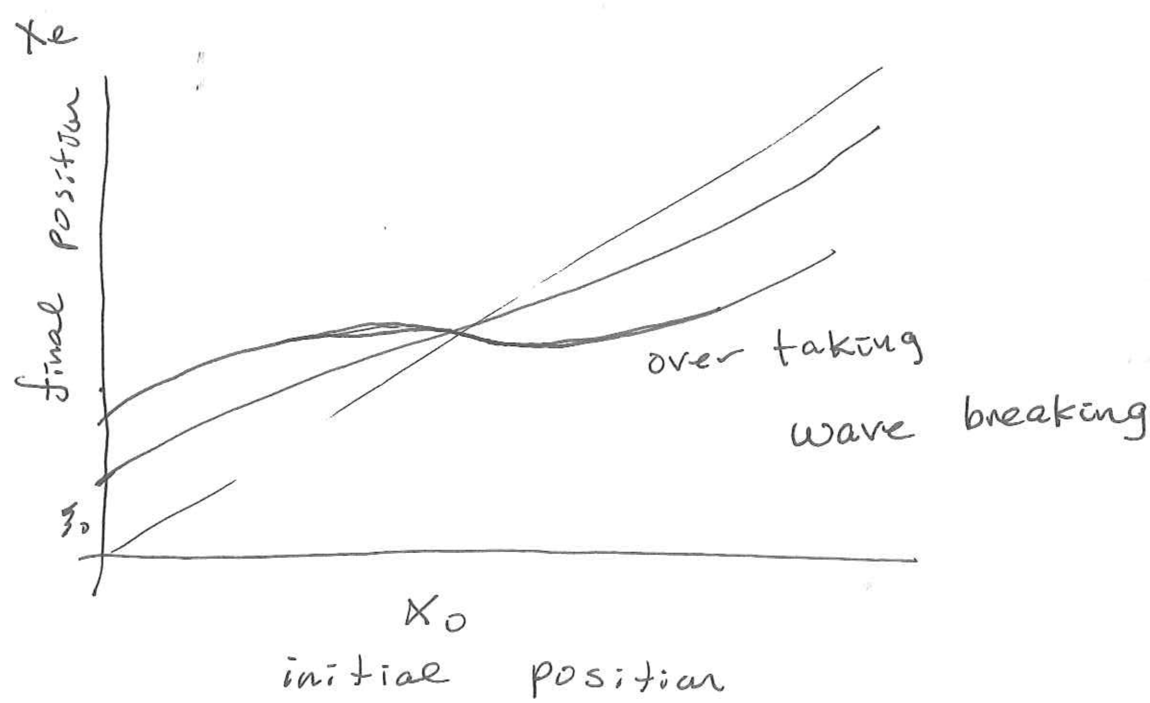
linear equation!

Solution corresponding to an initial displacement

$$\xi(x_0, 0) = \xi_0 \cos(k x_0)$$

$$\frac{\partial \xi}{\partial t}(x_0, t)\Big|_0 = 0$$

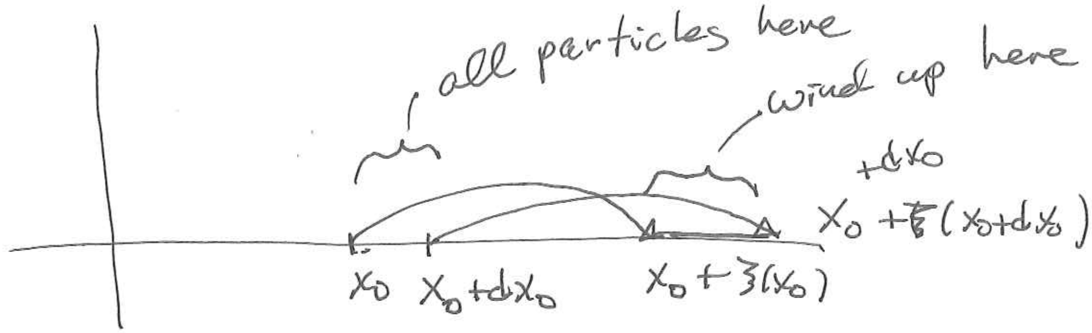
$$\xi(x_0, t) = \xi_0 \cos(k x - \omega_p t)$$



$$\frac{d x_e}{d x_0} < 0 \qquad \frac{d x_e}{d x_0} = 1 - k \xi_0 \sin(k x_0 - \omega_p t)$$

$$\boxed{\text{Requires } k\xi_0 > 1}$$

what is the density



$$\text{perturbed density} = \frac{n_{0e}\, dx_0}{x_0 + dx_0 + \xi(x_0 + dx_0) - (x_0 + \xi(x_0))}$$

$$= \frac{n_{0e}}{dx_e/dx_0}$$

$$n_n \quad \int dx_e\, n(x_e)\, e^{inkx_e}$$

density

$$\int \frac{dx_e}{dx_0} \cancel{d(\dots)}\, n(x_e)\, e^{ikx_e(k_0)}$$

$$n_0 \int dx_0\, e^{ink(x_0 + \xi_0 \cos(kx_0 - \omega_p t))}$$

$$= \frac{n_0}{k} e^{in\cancel{k}\omega_p t} \cancel{2\pi} \underbrace{\int d\theta\, e^{in(\theta + k\xi_0 \cos\theta)}}_{2\pi J_n(nk\xi_0)}$$

Resonant Three wave Interactions

Consider the interaction of three waves with frequencies and wave numbers

$$\omega_1, \omega_2, \omega_3$$

$$\underset{\sim}{k}_1, \underset{\sim}{k}_2, \underset{\sim}{k}_3$$

where: $\omega_1(\underset{\sim}{k}_1)$ $\omega_2(\underset{\sim}{k}_2)$ $\omega_3(\underset{\sim}{k}_3)$

are normal mode solutions of the type we have been studying EM waves or plasma waves.

"Resonant" means matching criteria for frequency and wave number are satisfied

$$\omega_1 = \omega_2 + \omega_3$$

$$\underset{\sim}{k}_1 = \underset{\sim}{k}_2 + \underset{\sim}{k}_3$$

In this way nonlinear terms have frequency and wave number satisfying the linear dispersion relation.

For example

$$\underset{\sim}{J} = q n \underset{\sim}{u}$$

perturbed density

$$n \cong n_0^{*} + n_1$$

$$\underset{\sim}{u} \cong \underset{\sim}{u}_1 = Re\Big\{ \hat{\underset{\sim}{u}}_{1,1}\, e^{i\underset{\sim}{k}_1\cdot\underset{\sim}{x} - \omega_1 t} + \hat{\underset{\sim}{u}}_{1,2}\, e^{i\underset{\sim}{k}_2\cdot\underset{\sim}{x} - \omega_2 t} + \hat{\underset{\sim}{u}}_{1,3}\, e^{i\underset{\sim}{k}_3\cdot\underset{\sim}{x} - \omega_3 t} \Big\}$$

$$\underset{\sim}{J}_2 = q n_1 \underset{\sim}{u}_1 = q\tfrac{1}{2}\Big(\sum_i \hat{n}_{1,i}\, e^{i\underset{\sim}{k}_i\cdot x - \omega_i t} + \hat{n}_{1,i}^{*}\, e^{-i\underset{\sim}{k}_i\cdot\underset{\sim}{x} + i\omega_i t}$$

$$\sum_j \hat{\underset{\sim}{u}}_{1,j}\, e^{i\underset{\sim}{k}_j\cdot\underset{\sim}{x} - \omega_j t} + \hat{u}_{1,j}^{*}\, e^{-i\underset{\sim}{k}_j\cdot x + \omega_j t}$$

$$\boxed{\begin{array}{ll} \underset{\sim}{k}_2 + \underset{\sim}{k}_3 = \underset{\sim}{k}_1 & \underset{\sim}{k}_1 - \underset{\sim}{k}_3 = \underset{\sim}{k}_2 \\ \underset{\sim}{k}_1 - \underset{\sim}{k}_2 = k_3 & \end{array}}$$

Resonantly excites normal modes

The three waves #1, #2, #3 will interact

The ineraction will conserve energy & momentum

$$\frac{d}{dt}\left(\sum_i W_i\right) = 0 \qquad \sum_i W_i = \text{const}$$

W_i = energy associated with wave i

* Suppose all waves are positive energy.

Then wave amplitudes have an upper bound

Suppose some waves are negative energy,
Then no upper bound

(Note if there exists a reference frame
in which all waves are positive energy
then upper bound)

$$\left.\begin{aligned} \omega_1 &= \omega_2 + \omega_3 \\ \underset{\sim}{k}_1 &= \underset{\sim}{k}_2 + \underset{\sim}{k}_3 \end{aligned}\right]$$

assumptions ω's >0

identifies ω_1 as highest freq.

$$\frac{d}{dt}\left(W_1 + W_2 + W_3\right) = 0$$

$$\omega_1 \frac{dN_1}{dt} + \omega_2 \frac{dN_2}{dt} + \omega_3 \frac{dN_3}{dt} = 0$$

$$\underset{\sim}{k}_1 \frac{dN_1}{dt} + \underset{\sim}{k}_2 \frac{dN_2}{dt} + k_3 \frac{dN_3}{dt} = 0$$

$$\omega_1 = \omega_2 + \omega_3 \qquad k_1 = \underset{\sim}{k}_2 + \underset{\sim}{k}_3$$

$$\omega_2 \left(\frac{d}{dt}(N_1 + N_2) \right) + \omega_3 \frac{d}{dt}(N_1 + N_3) = 0$$

$$\underset{\sim}{k}_2 \left(\frac{d}{dt}(N_1 + N_2) \right) + \underset{\sim}{k}_3 \frac{d}{dt}(N_1 + N_3) = 0$$

$$\frac{d}{dt}(N_1+N_2) = -\frac{\omega_3}{\omega_2}\frac{d}{dt}(N_1+N_3)$$

$$\cancel{\omega_3 \underset{\sim}{k}_2} \quad (\omega_2 \underset{\sim}{k}_3 - \omega_3 \underset{\sim}{k}_2)\frac{d}{dt}(N_1+N_3) = 0$$

either: $\omega_2 \underset{\sim}{k}_3 - \omega_3 \underset{\sim}{k}_2 = 0$ not likely actually 3 equations

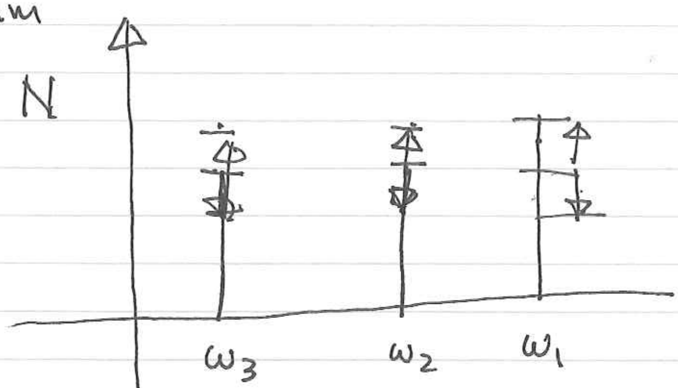
or $\frac{d}{dt}(N_1+N_3) = 0$

note $\frac{d}{dt}(N_2+N_3) \neq 0$

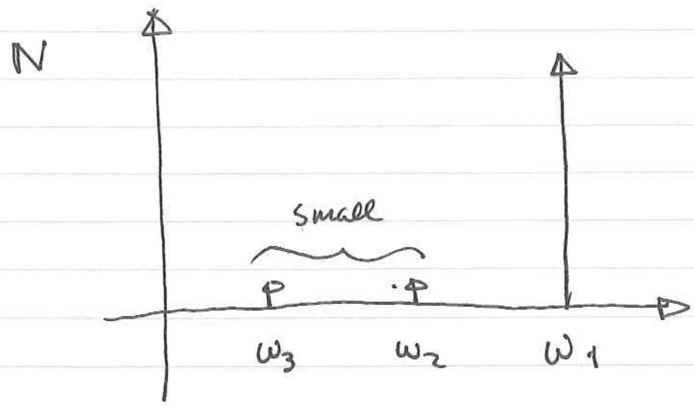
also $\frac{d}{dt}(N_1+N_2) = 0$

THUS if N_1 decreases N_2 & N_3 both ~~decrease~~ increase

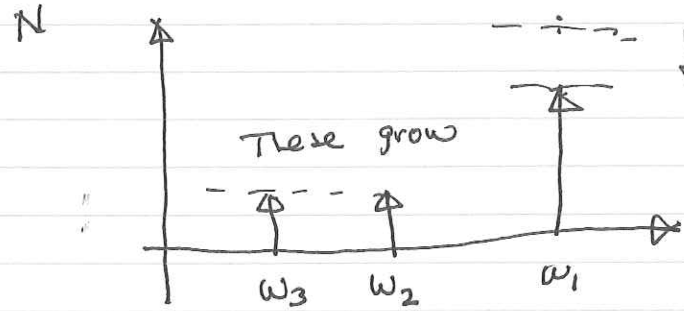
spectrum



Decay



initially mode #1 is big



this one decays

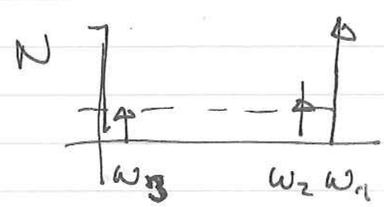
partitioning of energy is according to freq

~~ΔW~~

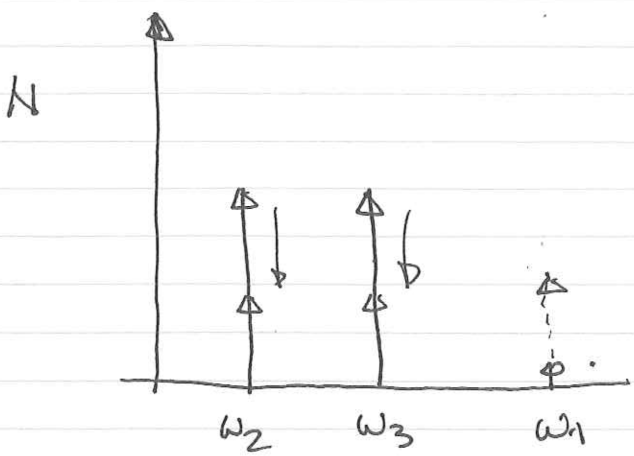
$$\Delta W_3 = \omega_3 \Delta N \uparrow$$

$$\Delta W_2 = \omega_2 \Delta N \uparrow$$

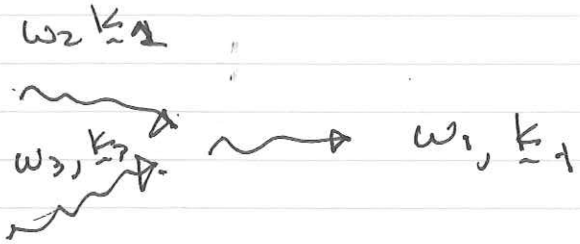
$$\Delta W_1 = \cancel{\Delta} - \omega_1 \Delta N$$



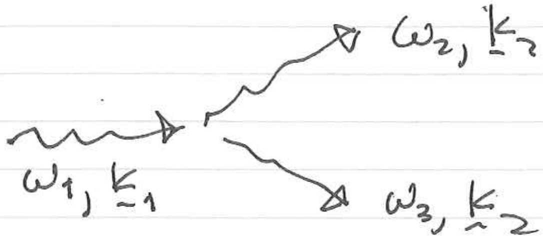
he gets all the energy



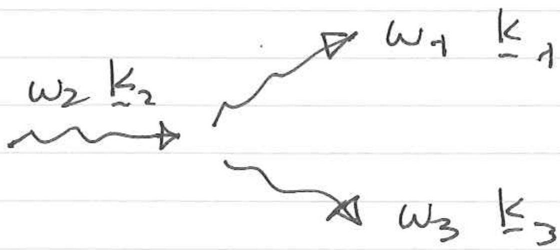
#2 and #3 can drop together and give their energy to #1.



allowed



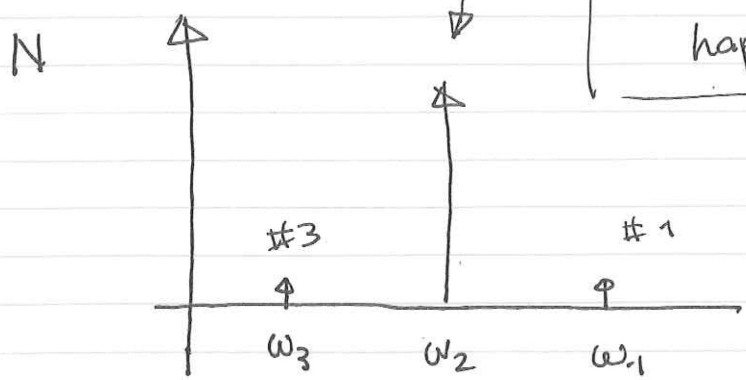
allowed



not allowed!!

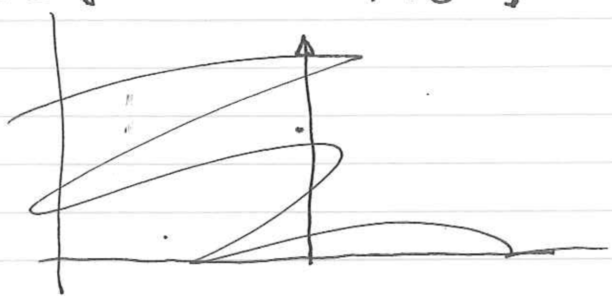
initially #2 is big

What can't happen



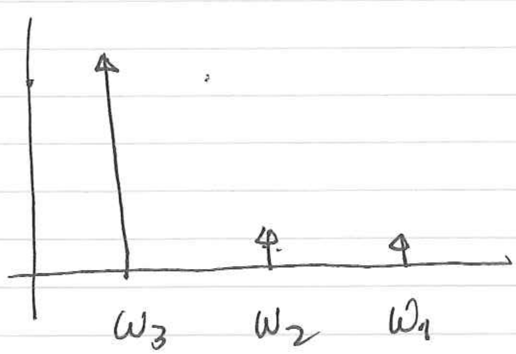
#1 & #3 are small

Can #1 get bigger by taking energy from #2? No!



For #2 to drop #3 mus drop by the same amount. But it can't because it is already small

Similarly



can't drop

* Generic Three wave mode coupling equations

$\hat{A}_{1,2,3}$ are complex amplitudes of modes 1, 2, 3

amplitudes defined such that

$$N_i = |\hat{A}_i|^2 \;\cancel{\$}\; > 0$$

---

mode coupling coefficient

$$i\frac{\partial A_1}{\partial t} = V_1 A_2 A_3 \qquad \omega_1 = \omega_2 + \omega_3$$

$$i\frac{\partial A_2}{\partial t} = V_2 A_1 A_3^* \qquad \omega_2 = \omega_1 - \omega_3$$

$$i\frac{\partial A_3}{\partial t} = V_3 A_1 A_2^* \qquad \omega_3 = \omega_1 - \omega_2$$

<u>Constraints</u> relate V_1, V_2, V_3 $\quad V_2 = V_3 = V_1^* = V$

$$i\frac{d}{dt}\left(\omega_1|A_1|^2 + \omega_2|A_2|^2 + \omega_3|A_3|^2\right) = 0$$

Action conservation $\quad \frac{d}{dt}(N_1 + N_2) = 0$

$$\frac{d}{dt}(N_1 + N_3) = 0$$

for mode i

basis vector give polarization

$$\hat{\underset{\sim}{E}} = A_i(t)\, \hat{\underset{\sim}{e}}_i \exp[i \underset{\sim}{k}_i \cdot \underset{-}{x} - i\omega_i t]$$

$A_i(t)$ varies in time due to nonlinear interaction

where

$$\underset{\approx}{G} \cdot \hat{\underset{\sim}{e}}_i = 0$$

determines ω_i for given $\underset{\sim}{k}_i$

as well as relative amplitudes of $\hat{\underset{\sim}{e}}_i$

How to pick normalization of $\underset{\sim}{e}_i$

one choice $|\hat{\underset{\sim}{e}}_i|^2 = 1$

another choice

freq

$$w_i = \frac{1}{16\pi} \hat{\underset{\sim}{e}}_i^* \cdot \frac{\partial}{\partial \omega} \omega \underset{\approx}{G} \cdot \hat{\underset{\sim}{e}}_i$$

THEN $W_i = \omega_i |A_i|^2 \qquad N_i = |A_i|^2$

$$i A_1^* \frac{\partial A_1}{\partial t} + i A_1 \frac{\partial A_1^*}{\partial t} = i \frac{d}{dt} N_1$$

$$= i V_1 A_1^* A_2 A_3$$

$$- i V_1^* A_1 A_2^* A_3^*$$

$$i A_2^* \frac{\partial A_2}{\partial t} + i A_2 \frac{\partial A_2^*}{\partial t} = i \frac{d N_2}{dt}$$

$$= i V_2 A_2^* A_1 A_3 - i V_2^* A_2 A_1^* A_3$$

<u>combine</u> $$i \frac{d}{dt}(N_1 + N_2) = 0$$

$$= i (V_1 - V_2^*) A_2 A_1^* A_3$$

$$- i (V_1^* - V_2) A_2^* A_1 A_3^*$$

must be true for any A_1, A_2, A_3

$\therefore \quad V_2 = V_1^*$ Likewise $V_3 = V_1^* =$

$$\boxed{\begin{aligned} i \frac{\partial A_1}{\partial t} &= V A_2 A_3 \qquad\qquad i \frac{\partial}{\partial t} A_3 = V A_1 A_2^* \\ i \frac{\partial A_2}{\partial t} &= V^* A_1 A_3^* \end{aligned}}$$

Can show that energy is conserved

VI3 $$\frac{d}{dt}\left(\omega_1|A_1|^2 + \omega_2|A_2|^2 + \omega_3|A_3|^2\right) = 0$$

Decay of MODE # 1

Let's assume initially that mode # 1 is big modes #2 & #3 are small

$$\underset{\text{pump}}{|A_1|} \gg \underset{\text{daughter waves}}{|A_2| \sim |A_3|}$$

Rate of change of A_1 will be small initially (product of two small numbers)

$$i\frac{\partial A_2}{\partial t} \simeq V_{2}^{*} \overset{\text{big}}{A_1} \overset{\text{small}}{A_3^*}$$

$$i\frac{\partial A_3}{\partial t} = V^* A_1 A_2^*$$

can treat A_1 as constant

$$i \frac{\partial^2 A_2}{\partial t^2} = V^* A_1 \frac{\partial A_3^*}{\partial t}$$

$$\frac{\partial A_3^*}{\partial t} = i V A_1^* A_2$$

$$\frac{\partial^2 A_2}{\partial t^2} = |V A_1|^2 A_2$$

$$A_2(t) = A_2(0)\, e^{\gamma_0 t} \qquad \gamma_0 = |V A_1|$$

rate of growth is proportional to $|V|^2$
coupling coefficient — depends on medium

and $|A_1|^2$ action in pump wave

This is interesting. Any wave will decay into other waves given enough time!

PROBABLY NOT TRUE

suppose mode #2 & #3 had linear damping rates ν_2 ν_3

γ_0 — linear growth rate

$$i\left(\frac{\partial}{\partial t}+\nu_2\right)A_2 = V^* A_1 A_3^*$$

$$\gamma_0^*$$

$$-i\left(\frac{\partial}{\partial t}+\nu_3\right)A_3^* = V A_1^* A_2$$

look for solution $\cancel{\frac{\partial A_2}{\partial t}}$ $A_2, A_3 \sim e^{\gamma t}$

$$i(\gamma+\nu_2)A_2 = \gamma_0 A_3^*$$

$$-i(\gamma+\nu_3)A_3^* = \gamma_0^* A_2$$

Product $(\gamma+\nu_3)(\gamma+\nu_2) = |\gamma_0|^2$

$$\gamma^2 + \gamma(\nu_2+\nu_3) + \nu_3\nu_2 - |\gamma_0|^2 = 0$$

$$\gamma = \frac{-(\nu_2+\nu_3) \pm \sqrt{(\nu_2+\nu_3)^2 - 4(\nu_2\nu_3 - |\gamma_0|^2)}}{2}$$

$$\gamma = \frac{-(\nu_2+\nu_3) \pm \sqrt{4|\gamma_0|^2 + (\nu_2-\nu_3)^2}}{2}$$

unstable if $\underline{|\gamma_0|^2 > \nu_2 \nu_3}$

* wave will decay if sufficient intense!
* if either wave is weakly damped (#2 or #3) unstable!