MAGNETIC HELICITY

$$K = \int_{\text{all space}} \underset{\sim}{A} \cdot \underset{\sim}{B} \, d^3x \qquad \& \qquad \underline{B} = \nabla \times \underline{A}$$

$$\underset{\sim}{A}' = \underset{\sim}{A} + \nabla \psi .$$

ψ must be single valued such that

Gauge dependent ?

$$\int_C d\underset{\sim}{l} \cdot \underset{\sim}{A}' = \int_C d\underset{\sim}{l} \cdot \underset{\sim}{A}$$

for any C

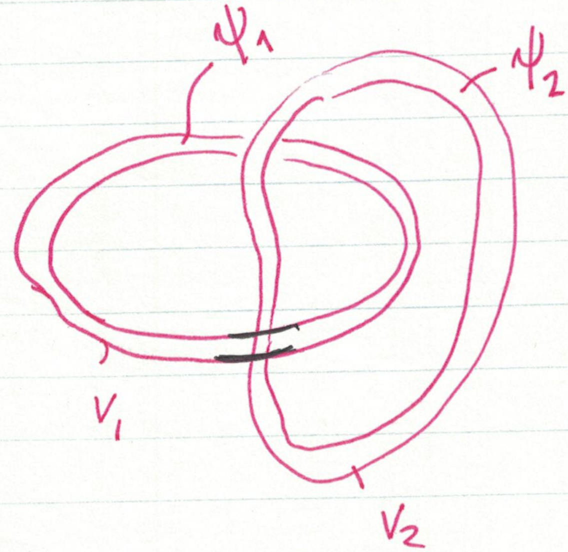
$$K' = \int \underset{\sim}{A}' \cdot \underset{\sim}{B} \, d^3x = K + \int \underset{\sim}{B} \cdot \nabla \psi \, d^3x = K + \int \psi \, \nabla \cdot \underset{\sim}{B} \, d^3x = K$$

gauge independent

Finn & Antonsen Comments on Plasma Physics 1985.

MAGNETIC HELICITY tells us the degree ~~of~~ to which magnetic fields are interlinked

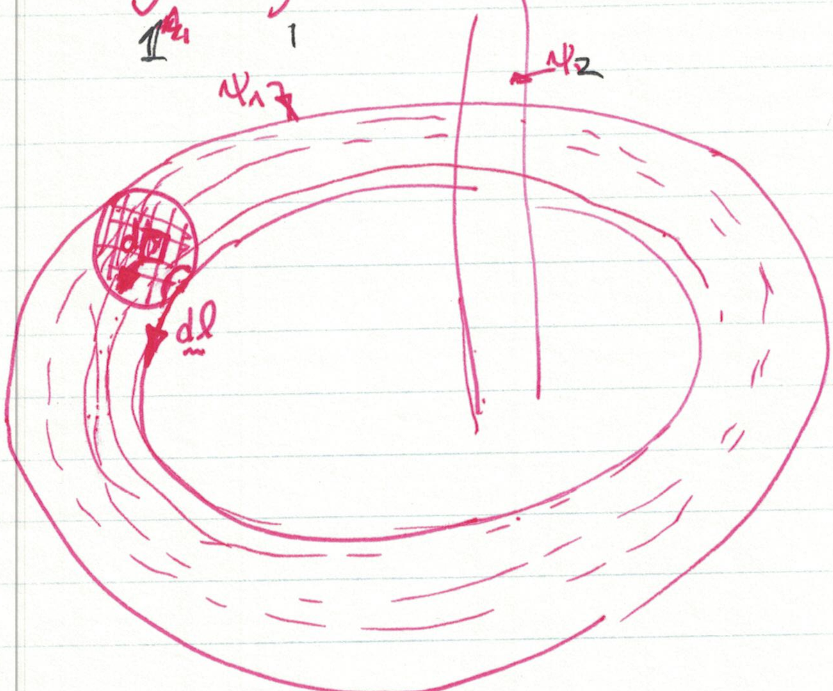
suppose two flux tubes are linked as shown



what is K ?

$$K = \int_{V_1} \underset{\sim}{A} \cdot \underset{\sim}{B}\, d^3x + \int_{V_2} \underset{\sim}{A} \cdot \underset{\sim}{B}\, d^3x$$

$$\int_{V_1} \underset{\sim}{A} \cdot \underset{\sim}{B}\, d^3x = \overbrace{\int \underset{\sim}{B} \cdot d\underset{\sim}{A}}^{\psi_1} \int_1 \underset{\sim}{A} \cdot d\underset{\sim}{\ell} = \psi_1 \psi_2$$

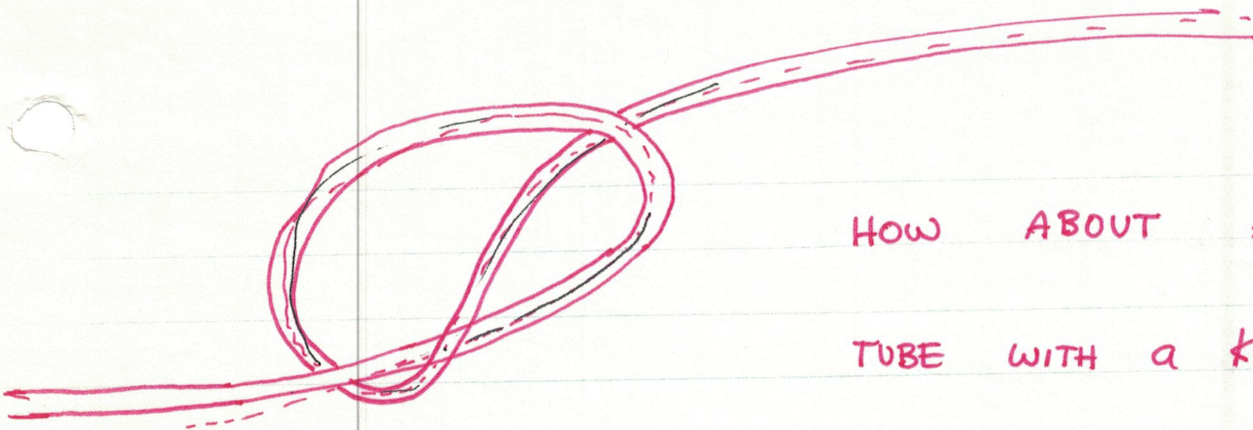


$$\int_{V_2} \underset{\sim}{A} \cdot \underset{\sim}{B}\, d^3x = \psi_1 \psi_2$$

$$K = 2\psi_1 \psi_2$$

IF THE FLUX tubes were not linked

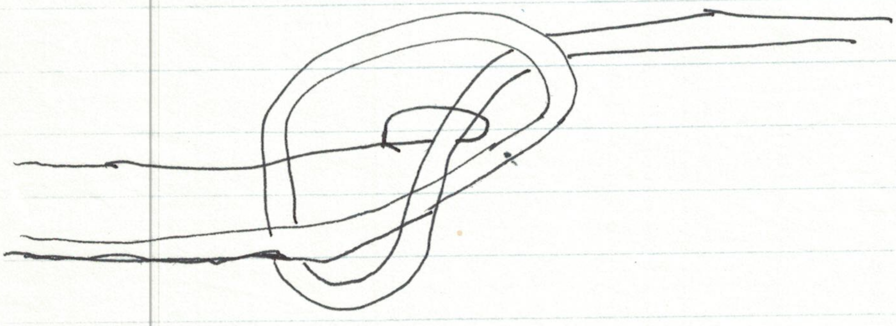
$$K = 0$$

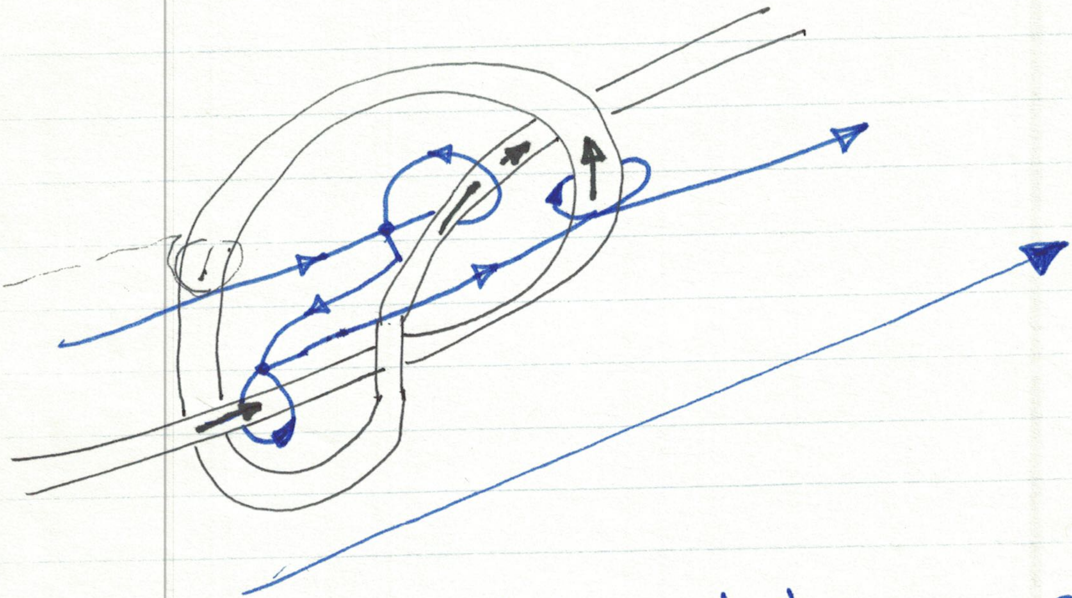


HOW ABOUT A FLUX TUBE WITH a knot?

$$K = \int B \cdot dA \int A \cdot dl$$

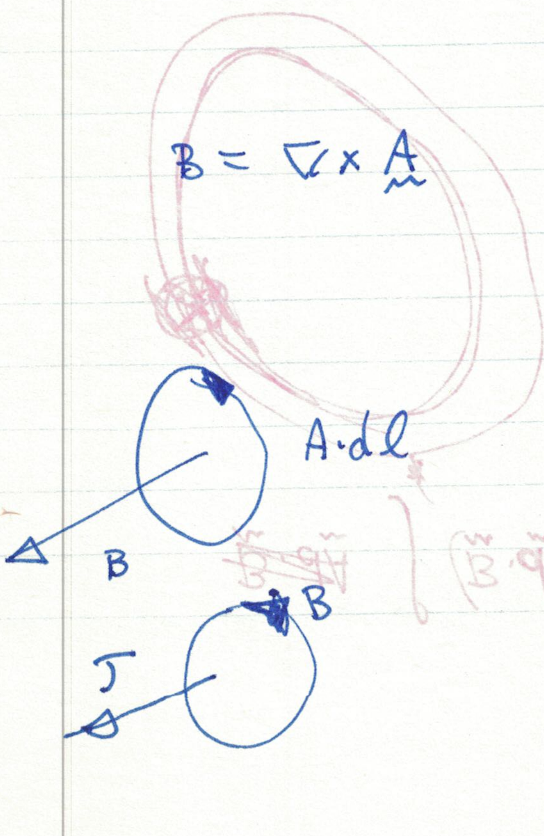
$$= \psi \int \underset{\sim}{A} \cdot d\underset{\sim}{l}$$

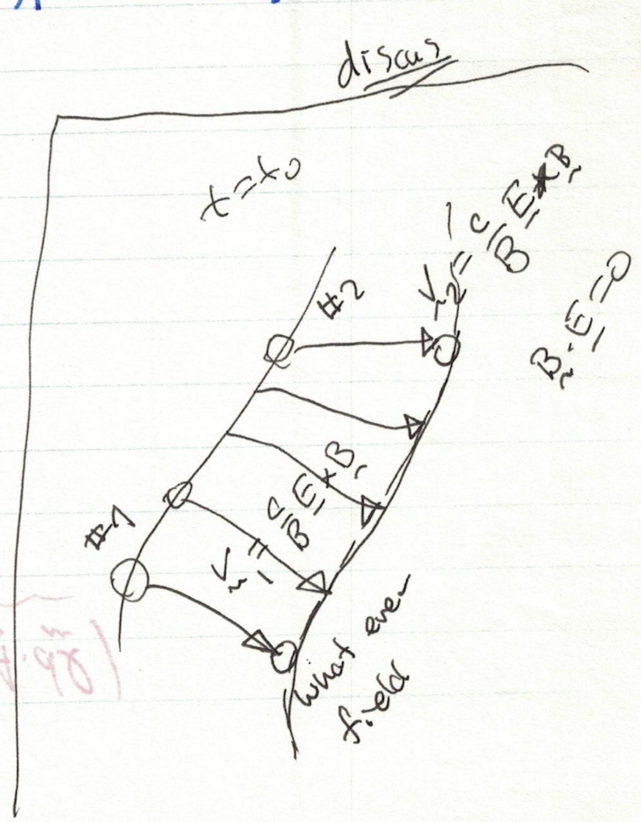


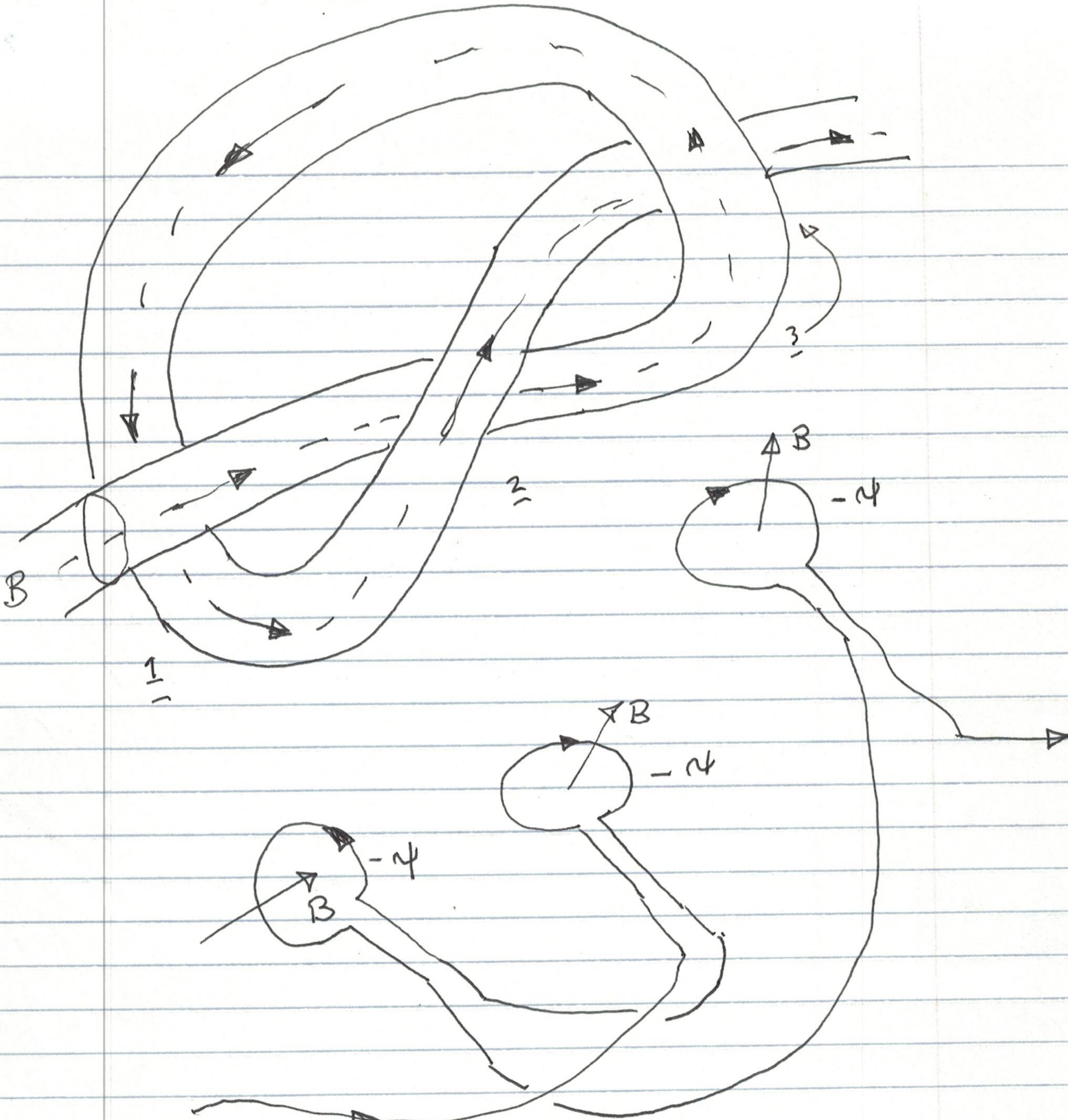


$$= \int d^3x\, \underset{\sim}{B} \cdot \underset{\sim}{A} = \int d\vec{a} \cdot \vec{B} \int d\vec{l} \cdot \vec{A} = -3\psi^2$$

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$$







$$K = -3\psi^2$$

$$K = \psi \int dl A$$

How Does Helicity Evolve?

$$\frac{dK}{dt} = \int_{F.T} d^3x \, \frac{\partial}{\partial t}(\mathbf{A}\cdot\mathbf{B}) + \int_S d\mathbf{a}\cdot\mathbf{v}(\mathbf{A}\cdot\mathbf{B})$$

$$= \int_{F.T} d^3x \left[\frac{\partial A}{\partial t}\cdot\mathbf{B} + \mathbf{A}\cdot\frac{\partial B}{\partial t} \right] + \sout{\nabla\cdot\mathbf{v}(\mathbf{A}\cdot\mathbf{B})} \int_S d\mathbf{a}\cdot\mathbf{v}(\mathbf{A}\cdot\mathbf{B}) = 0$$

$$-\frac{1}{c}\frac{\partial A}{\partial t} - \nabla\phi = E \qquad \frac{\partial B}{\partial t} = -c\nabla\times E$$

$$\mathbf{A}\cdot\nabla\times\mathbf{E} = \mathbf{E}\cdot\underbrace{\nabla\times\mathbf{A}}_{B} + \nabla\cdot(\mathbf{E}\times\mathbf{A})$$

$$\frac{dK}{dt} = -\int d^3x \left\{ c\mathbf{E}\cdot\mathbf{B} + c\nabla\phi\cdot\mathbf{B} + \mathbf{A}\cdot c\nabla\times\mathbf{E} \right\} + \int_S d\mathbf{a}\cdot\mathbf{v}(\mathbf{A}\cdot\mathbf{B})$$

$$\frac{dK}{dt} = -\int d^3x \left\{ \underbrace{2c\mathbf{E}\cdot\mathbf{B}}_{\to 0} + \underbrace{c\nabla\phi\cdot\mathbf{B}}_{\to 0} \right\} + \int_S d\mathbf{a}\cdot\left[\mathbf{v}(\mathbf{A}\cdot\mathbf{B}) - c\mathbf{E}\times\mathbf{A}\right]$$

$$\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c} = 0 \qquad \mathbf{E}\cdot\mathbf{B} = 0$$

BUT $c\mathbf{E} = -\mathbf{v}\times\mathbf{B}$ $\qquad c\mathbf{E}\times\mathbf{A} = -(\mathbf{v}\times\mathbf{B})\times\mathbf{A}$

$$= -\mathbf{B}\,\mathbf{v}\cdot\mathbf{A} + \mathbf{v}\,\mathbf{B}\cdot\mathbf{A}$$

$$\int_S d\mathbf{a}\cdot\left[\mathbf{v}(\mathbf{A}\cdot\mathbf{B}) + \underbrace{\mathbf{B}\,\mathbf{v}\cdot\mathbf{A}}_{B_n = 0} - \mathbf{v}(\mathbf{B}\cdot\mathbf{A})\right] = 0$$

THUS, the contribution to K from each flux tube is conserved during ideal MHD <u>motion</u>

$K = \int d^3x\, \underset{\sim}{A} \cdot \underset{\sim}{B}$ also a constant