MHD waves

Lets consider the linear waves that propagate in the infinite Homogeneous equilibrium

$$p = \text{constant} \quad , \quad \rho = \text{constant}$$

$$\underset{\sim}{B} = (0, 0, B_0) = \text{constant}$$

Linearize all equations

$$\underset{\sim}{B} = \underset{\sim}{B}_0 + \underset{\sim}{B}_1 \qquad \text{linear}$$

$$p = p_0 + p_1$$

$$\rho = \rho_0 + \rho_1$$

$$\underset{\sim}{v} = \underset{\sim}{v}_1$$

second order

$$(\cancel{\rho_1} + \rho_0)\left[\frac{\partial V_1}{\partial t} + \cancel{(V_1 \cdot \nabla V_1)}\right] = -\nabla p_1 + \underset{\sim}{J_1} \times \underset{\sim}{B_0}$$

$$J_1 = \frac{c}{4\pi} \nabla \times B_1$$

$$-\frac{1}{c}\frac{\partial B_1}{\partial t} = \nabla \times \underset{\sim}{E} = -\nabla \times \left(\frac{V_1 \times \underset{\sim}{B_0}}{c}\right) \qquad \cancel{V_1 \times B_1} \to 0$$

$$\frac{\partial p_1}{\partial t} + \cancel{(V_1 \cdot \nabla(p_0 + p_1))} + \gamma p_0 \nabla \cdot V_1 = 0$$

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot V_1 = 0$$

LINEARIZED EQUATIONS

Lets count time derivatives

(3)
+ (2) why
+ (1)
+ ~~(1)~~ → 0 ρ_1 does not couple to other equations — zero freq mode

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6 six modes

IN EACH EQUATION

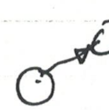
For each $\frac{\partial}{\partial t}$ there is a ∇

$$\frac{\partial}{\partial t} \sim \nabla \qquad \omega \sim k$$

For a given $\angle$ of propagation $\omega \propto |k|$

Let's introduce the concept of a fluid displacement $\underset{\sim}{\xi}$

such that $\underset{\sim}{V} = \frac{\partial \underset{\sim}{\xi}}{\partial t}$



motion of fluid elemen

(Nonlinearly $\boxed{\frac{d\underset{\sim}{\xi}}{dt} = \underset{\sim}{v}}$

<u>this has a magical result</u>

only momentum Balance involves $\frac{\partial}{\partial t}$

$$\rho_0 \frac{\partial^2}{\partial t^2} \underset{\sim}{\xi} = -\nabla p_1 + \frac{1}{4\pi} (\nabla \times \underset{\sim}{B}_1) \times \underset{\sim}{B}_0$$

$$\underset{\sim}{B}_1 = \nabla \times (\underset{\sim}{\xi} \times \underset{\sim}{B}_0)$$

$$p_1 + \gamma p_0 \nabla \cdot \underset{\sim}{\xi} = 0$$

$$\rho_1 + \rho_0 \nabla \cdot \underset{\sim}{\xi} = 0$$

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = \underset{\sim}{F}\{\xi\}$$

functional

Now lets assume all quantities vary as

$$\text{const} \exp(i \underset{\sim}{k} \cdot \underset{\sim}{x} - i\omega t)$$

$$\underset{\sim}{\xi} = \hat{\underset{\sim}{\xi}} \exp(i \underset{\sim}{k} \cdot \underset{\sim}{x} - i\omega t)$$

$$-\omega^2 \rho \hat{\underset{\sim}{\xi}} = -i\underset{\sim}{k}\hat{p}_1 + \frac{1}{4\pi}\left[\hat{\underset{\sim}{B}}_1\, i\underset{\sim}{k}\cdot\underset{\sim}{B}_0 - i\underset{\sim}{k}\,\hat{\underset{\sim}{B}}_1\cdot\underset{\sim}{B}_0\right]$$

$$\hat{\underset{\sim}{B}}_1 = i\left[\hat{\underset{\sim}{\xi}}\;\underset{\sim}{k}\cdot\underset{\sim}{B}_0 - \underset{\sim}{B}_0\;\underset{\sim}{k}\cdot\hat{\underset{\sim}{\xi}}\right]$$

$$p_1 + \gamma p_0\, i\underset{\sim}{k}\cdot\underset{\sim}{\xi} = 0 \qquad p_1 = -\gamma p_0\, i\underset{\sim}{k}\cdot\hat{\xi}$$

LETS TAKE THREE COMPONENTS OF

Momentum Balance

$\underset{\sim}{k}$ direction

$\underset{\sim}{B}_0$ direction

$\underset{\sim}{k}\times\underset{\sim}{B}_0$

$$-\omega^2\rho\,\underset{\sim}{k}\cdot\underset{\sim}{\xi} = -ik^2\hat{p}_1 + \frac{1}{4\pi}\left[\cancelto{0}{i\underset{\sim}{k}\cdot\underset{\sim}{B}_1}\, i\underset{\sim}{k}\cdot\underset{\sim}{B}_0 - ik^2\underset{\sim}{B}_1\cdot\underset{\sim}{B}_0\right]$$

$$\underset{\sim}{B}_1\cdot\underset{\sim}{B}_0 = i\left[\underset{\sim}{B}_0\cdot\hat{\underset{\sim}{\xi}}\;\underset{\sim}{k}\cdot\underset{\sim}{B}_0 - B_0^2\,\underset{\sim}{k}\cdot\hat{\underset{\sim}{\xi}}\right]$$

$$-\omega^2\rho\, \underset{\sim}{\xi}\cdot\underset{\sim}{B}_0 = -i\,\underset{\sim}{k}\cdot\underset{\sim}{B}_0\, \hat{P}_1$$

$$-\omega^2\rho\left(\underset{\sim}{k}\times\underset{\sim}{B}_0\cdot\underset{\sim}{\xi}\right) = \frac{1}{4\pi}\, i\,\underset{\sim}{k}\cdot\underset{\sim}{B}_0\left(\underset{\sim}{B}_1\cdot\underset{\sim}{k}\times\underset{\sim}{B}_0\right)$$

$$\underset{\sim}{B}_1\cdot\underset{\sim}{k}\times\underset{\sim}{B}_0 = i\,\underset{\sim}{k}\cdot\underset{\sim}{B}_0\left(\underset{\sim}{\xi}\cdot\underset{\sim}{k}\times\underset{\sim}{B}_0\right)$$

LET'S write A MATRIX

$$0 = \omega^2\rho\,\underset{\sim}{k}\cdot\underset{\sim}{\xi} - k^2\gamma p_0\,\underset{\sim}{k}\cdot\underset{\sim}{\xi} - \frac{k^2}{4\pi}\left(B_0^2\,\underset{\sim}{k}\cdot\underset{\sim}{\xi} - \underset{\sim}{B}_0\cdot\hat{\underset{\sim}{\xi}}\,\underset{\sim}{k}\cdot\underset{\sim}{B}_0\right)$$

$$0 = \omega^2\rho\,\underset{\sim}{\xi}\cdot\underset{\sim}{B}_0 - \underset{\sim}{k}\cdot\underset{\sim}{B}\,\gamma p_0\,\underset{\sim}{k}\cdot\underset{\sim}{\xi}$$

$$0 = \omega^2\rho\left(\underset{\sim}{k}\times\underset{\sim}{B}_0\cdot\underset{\sim}{\xi}\right) - \frac{1}{4\pi}\left(\underset{\sim}{k}\cdot\underset{\sim}{B}_0\right)^2\left(\underset{\sim}{k}\times\underset{\sim}{B}_0\cdot\underset{\sim}{\xi}\right)$$

NOTICE, #3 does not couple with

#1 and #2

one solution

$$\omega^2 \rho = \frac{1}{4\pi}(\underset{\sim}{k} \cdot \underset{\sim}{B}_0)^2 \qquad \text{and} \qquad \underset{\sim}{k} \cdot \underset{\sim}{\xi} = 0$$

$$\underset{\sim}{\xi} \cdot \underset{\sim}{B}_0 = 0$$

<u>shear</u> <u>alfven</u> <u>Wave</u> $\omega \propto |k|$

$\underset{\sim}{k} \cdot \underset{\sim}{\xi} = 0$ no compression

$\underset{\sim}{\xi} \cdot \underset{\sim}{B}_0 = 0$ no parallel velocity

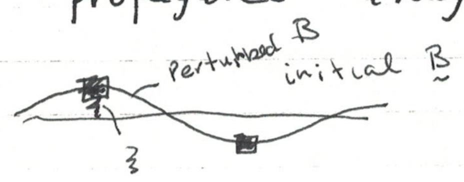
$\underset{\sim}{B}_1 = i \underset{\sim}{k} \cdot \underset{\sim}{B}_0 \, \hat{\xi}$ $\qquad \underset{\sim}{B}_1$ is $\perp$ TO $\underset{\sim}{B}_0$

$\tilde{\hat{p}}_1 = 0$

$$\boxed{\omega^2 = k_\parallel^2 V_a^2 \qquad V_a^2 = \frac{B_0^2}{4\pi\rho} = \frac{\text{Magnetic } \cancel{\text{pressure}} \text{ Tension}}{\text{Mass density}}}$$

group velocity

$$\frac{\partial \omega}{\partial \underset{\sim}{k}} = \pm V_a \underset{\sim}{b} \qquad \text{propagates along field}$$



CONSIDER THE OTHER TWO

$$\left[\omega^2\rho - k^2\left(\gamma p_0 + \frac{B_0^2}{4\pi}\right)\right]\underset{\sim}{k}\cdot\underset{\sim}{\xi} = -\frac{k^2}{4\pi}(\underset{\sim}{k}\cdot\underset{\sim}{B}_0)(\underset{\sim}{B}_0\cdot\underset{\sim}{\xi})$$

$$\omega^2\rho\,\underset{\sim}{\xi}\cdot\underset{\sim}{B}_0 = \underset{\sim}{k}\cdot\underset{\sim}{B}\,\gamma p_0\,\underset{\sim}{k}\cdot\underset{\sim}{\xi}$$

multiply together

$$\omega^2\rho\left[\omega^2\rho - k^2\left(\gamma p_0 + \frac{B_0^2}{4\pi}\right)\right] + \frac{k^2}{4\pi}\gamma p_0(\underset{\sim}{k}\cdot\underset{\sim}{B})^2 = 0$$

lets introduce $V_a^2 = \dfrac{B_0^2}{4\pi\rho}$ $C_s^2 = \dfrac{\gamma p_0}{\rho}$

$$\omega^2\left[\omega^2 - k^2(c_s^2 + V_a^2)\right] + k^2 k_\parallel^2 c_s^2 V_a^2 = 0$$

$$\omega^4 - \omega^2 k^2(c_s^2 + V_a^2) + k^2 k_\parallel^2 c_s^2 V_a^2 = 0$$

$$\omega^2 = \frac{k^2(c_s^2 + V_a^2) \pm \sqrt{\left[k^2(c_s^2 + V_a^2)\right]^2 - 4k^2 k_\parallel^2 c_s^2 V_a^2}}{2}$$

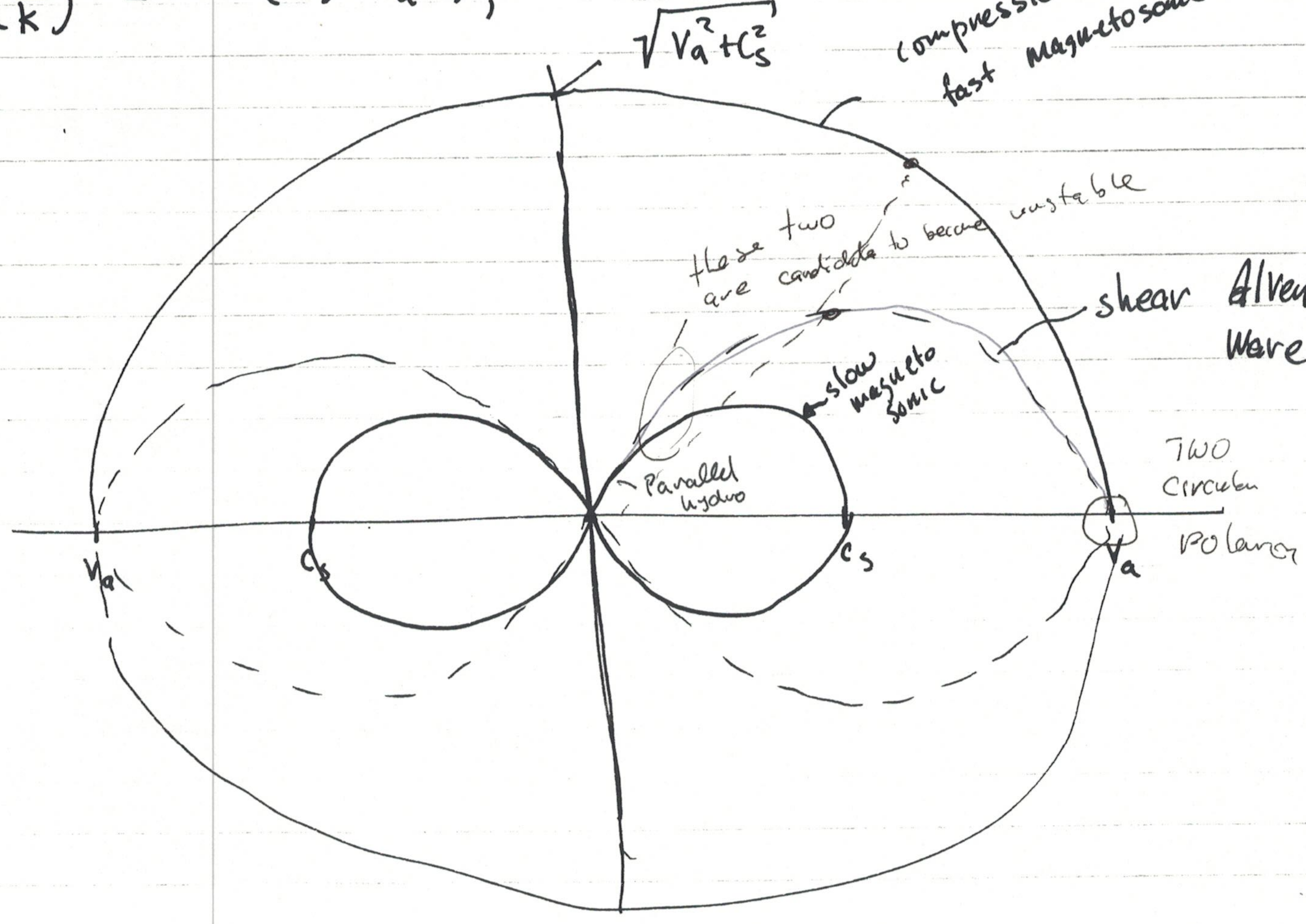
$$\left(\frac{\omega}{k}\right)^2 = \frac{C_s^2 + V_a^2 \pm \sqrt{(C_s^2+V_a^2)^2 - 4\cos^2\theta\, C_s^2 V_a^2}}{2}$$

$\cos\theta = 1$ Parallel propagation

$$\left(\frac{\omega}{k}\right)^2 = \frac{C_s^2 + V_a^2 \pm \sqrt{(V_a^2 - C_s^2)^2}}{2} = V_a^2,\ C_s^2$$

$\cos\theta = 0$

$$\left(\frac{\omega}{k}\right)^2 = (C_s^2 + V_a^2),\ 0$$



1 keep $k_{\parallel}$ finite but let $k_{\perp} \to \infty$

$\omega^2 \cong k_{\perp}^2 (c_s^2 + V_a^2)$ fast magneto sonic

$$\omega^2 \cong \frac{k_{\parallel}^2 C_s^2 V_a^2}{C_s^2 + V_a^2} \simeq \frac{k_{\parallel}^2 C_s^2}{1+\beta}$$ slow magneto sonic wave

ok!

$\omega^2 = k_{\parallel}^2 V_a^2$ shear Alfen

Two mode have finite ω

one mode has large ω

what is polarization

shear Alfven

for fast magneto sonic

$\underset{\sim}{k} \cdot \underset{\sim}{\xi} \neq 0$

for slow magneto sonic

$$\omega^2 \ll k^2 \left(\frac{\gamma p_0 + B_0^2}{4\pi}\right)$$

$$\underset{\sim}{k} \cdot \underset{\sim}{\xi} \sim \frac{\underset{\sim}{k} \cdot \underset{\sim}{B}_0 \; \underset{\sim}{B}_0 \cdot \underset{\sim}{\xi}}{(\gamma p_0 + B_0^2)}$$

as $k_\perp \to \infty$ $\quad \underset{\sim}{k} \cdot \underset{\sim}{\xi}$ is Finit

$\underset{\sim}{k} \cdot \underset{\sim}{\xi} \ll |\underset{\sim}{k}_\perp| \, |\xi_\perp|$

$$\boxed{\omega^2 \rho \, \underset{\sim}{\xi} \cdot \underset{\sim}{B}_0 = \frac{\underset{\sim}{k} \cdot \underset{\sim}{B}_0 \, \gamma p_0 \, \underset{\sim}{k} \cdot \underset{\sim}{B}_0}{(\gamma p_0 + B_0^2)} \underset{\sim}{\xi} \cdot \underset{\sim}{B}_0}$$

$\underset{\sim}{k} \cdot \underset{\sim}{\xi} \ll |\underset{\sim}{k}_\perp| \, |\underset{\sim}{\xi}_\perp|$ quasi incompressible

fast $\quad k$