MHD Equilibria

MHD Equations

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\underset{\sim}{V} = 0$$

$$\frac{\partial P}{\partial t} + \underset{\sim}{V}\cdot\nabla P + \Gamma P\,\nabla\cdot\underset{\sim}{V} = 0$$

$$\rho\left(\frac{\partial \underset{\sim}{V}}{\partial t} + \underset{\sim}{V}\cdot\nabla\underset{\sim}{V}\right) = -\nabla P + \frac{\underset{\sim}{J}\times\underset{\sim}{B}}{c}$$

$$\nabla\times\underset{\sim}{B} = \frac{4\pi}{c}\underset{\sim}{J}$$

$$\nabla\cdot\underset{\sim}{B} = 0$$

$$-\frac{1}{c}\frac{\partial B}{\partial t} = \nabla\times\underset{\sim}{E}$$

$$\underset{\sim}{E} + \frac{\underset{\sim}{V}\times\underset{\sim}{B}}{c} = 0$$

Lets look for solutions with

a) $\partial/\partial t = 0$ This means all quantities are steady in time

b) $\underset{\sim}{V} = 0$ no flow

Equilibria

$\cancel{\underset{\sim}{v} = 0 \qquad \frac{\partial}{\partial t} = 0}$

~~First two equations tell us nothing~~

$\underset{\sim}{v} = 0$

momentum balance

$$\begin{cases} 0 = -\nabla p + \dfrac{\underset{\sim}{J} \times \underset{\sim}{B}}{c} \\ \nabla \times \underset{\sim}{B} = \dfrac{4\pi}{c} \underset{\sim}{J} \\ \nabla \cdot \underset{\sim}{B} = 0 \end{cases}$$

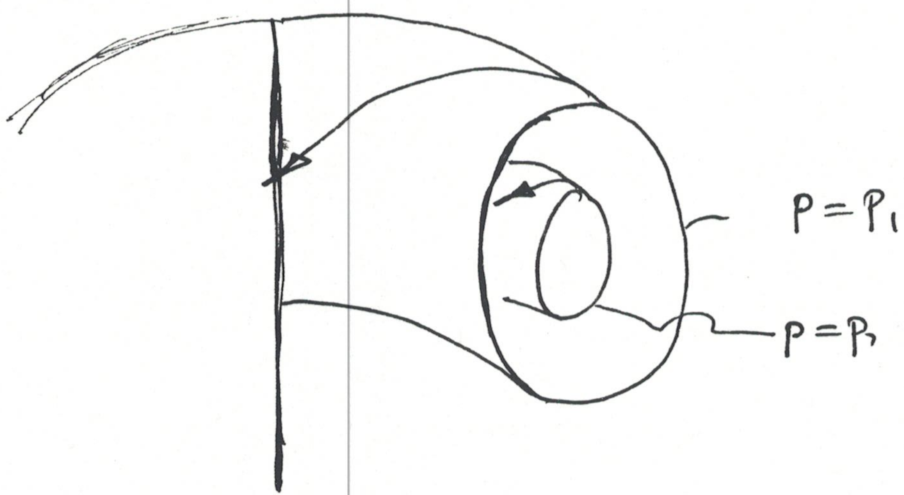
consider first equation

$$\underset{\sim}{B} \cdot \left(-\nabla P + \frac{\underset{\sim}{J} \times \underset{\sim}{B}}{c} \right) = -\underset{\sim}{B} \cdot \nabla p = 0$$

P is constant along a field line

'so $\underset{\sim}{J} \cdot \nabla p = 0$ no current flows in direction of pressure gradient

1) magnetic field lines lie in surfaces of constant p, called flux surface



2) no current flows <u>across</u> flux surfaces

$$\underset{\sim}{J} = \frac{c}{4\pi} \nabla \times \underset{\sim}{B}$$

$$0 = -\nabla p + \frac{1}{4\pi} (\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B}$$

$$(\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B} = -\nabla \frac{B^2}{2} + \underset{\sim}{B} \cdot \nabla \underset{\sim}{B}$$

$$0 = -\nabla \left(P + \frac{B^2}{8\pi} \right) + \frac{1}{4\pi} \underset{\sim}{B} \cdot \nabla \underset{\sim}{B}$$

This can be ~~or~~ written in a number of ways

lets look at the component perpendicular $\underset{\sim}{B}$

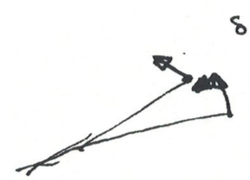
$$\underset{\sim}{B}\cdot\nabla\underset{\sim}{B} = B^2\, \underset{\sim}{b}\cdot\nabla\underset{\sim}{b} + \underset{\sim}{b}\, \underset{\sim}{b}\cdot\nabla \frac{B^2}{2}$$

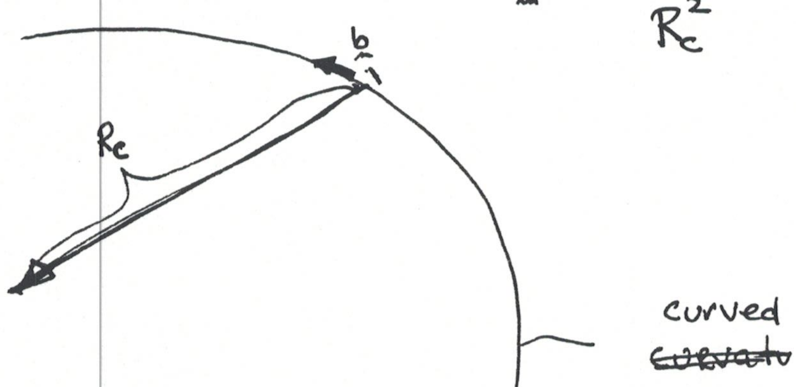
where $\underset{\sim}{B} = B\,\underset{\sim}{b}$

$$(\underset{\sim}{B}\cdot\nabla\underset{\sim}{B})_\perp = B^2(\underset{\sim}{b}\cdot\nabla\underset{\sim}{b})_\perp = B^2\,\underset{\sim}{\kappa}$$

$\underset{\sim}{b}\cdot\nabla\underset{\sim}{b}$ is the curvature of the magnetic field

$$\underset{\sim}{\kappa} = \frac{\vec{R}_c}{R_c^2}$$

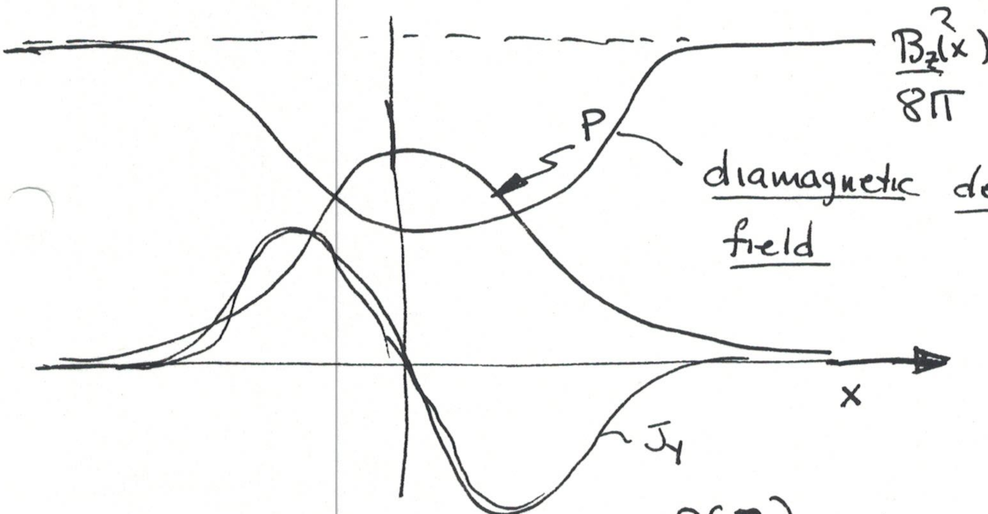




$$\nabla_\perp \left(p + \frac{B^2}{8\pi} \right) = \frac{B^2}{4\pi} \underset{\sim}{\kappa}$$

for straight fields $\underset{\sim}{\kappa} = 0$

$$p + \frac{B^2}{8\pi} = \text{constant},$$ plasma pressure + magnetic pressure = constant



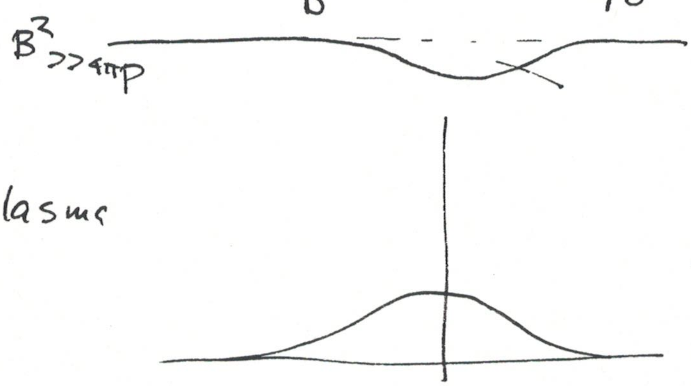
J_y

$p(r)$

$B = (B_z(r), B_\theta = 0, B_r =$

Example

$$\beta = \frac{8\pi P}{B^2} = \frac{\text{ratio of plasma pressure}}{\text{to magnetic field pres}}$$



low β plasma

$\underset{\sim}{J} = \frac{c}{4\pi} \nabla \times \underset{\sim}{B}$

$J_\theta = -\frac{c}{4\pi} \frac{\partial}{\partial x} B_z$

$\frac{(J_\theta B_z)}{c} = F_r$

for curved field lines

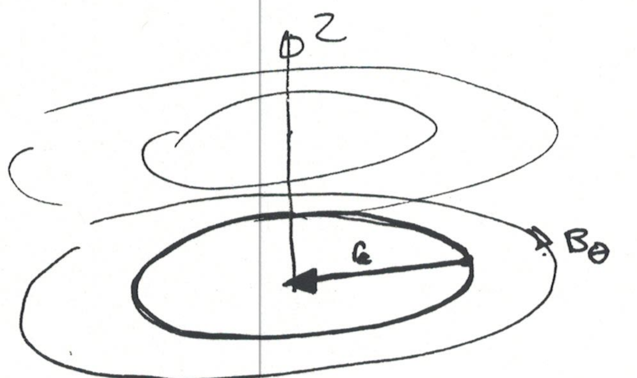
(r, θ, z)

$p = p(r)$

$\underset{\sim}{B} = (0, B_\theta(r), 0)$

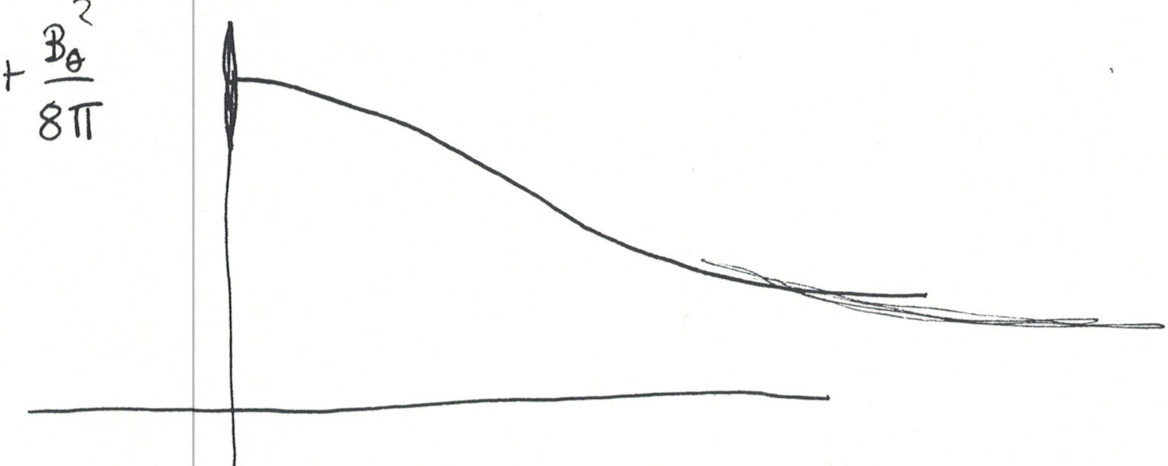
$$\frac{\partial}{\partial r}\left(p + \frac{B_\theta^2}{8\pi}\right) = -\frac{B_\theta^2}{4\pi r}$$

Inward force



J

$p + \frac{B_\theta^2}{8\pi}$



$\nabla\left(p + \frac{B_\theta^2}{8\pi}\right) < r$ $\quad -\frac{B_\theta^2}{4\pi r}$

If you think of the field lines ~~a~~ as having tension

pressure (magnetic + plasma) is balanced by tension in fields

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Equilibrium Currents

$$0 = -\nabla p + \frac{\underset{\sim}{J} \times \underset{\sim}{B}}{c}$$

first suppose $p \cong 0$ low β plasma

$$0 = \frac{\underset{\sim}{J} \times \underset{\sim}{B}}{c}$$

$\underset{\sim}{J}$ is parallel to $\underset{\sim}{B}$ $\quad \underset{\sim}{J} = \frac{c}{4\pi} k(x) \underset{\sim}{B}$

force free equilibrium

k is constant on field lines

$$\nabla \cdot \underset{\sim}{J} \Rightarrow \frac{c}{4\pi} \nabla \cdot k \underset{\sim}{B} = \frac{c}{4\pi} \underset{\sim}{B} \cdot \nabla k = 0$$

~~$\underset{\sim}{J} = \nabla \times$~~

Now, allow for p $\qquad \underset{\sim}{J}_\perp = - \dfrac{c \nabla p \times \underset{\sim}{B}}{B^2}$

$$\underset{\sim}{J} = J_\parallel \underset{\sim}{b} - \frac{c(\nabla p \times \underset{\sim}{B})}{B^2}$$

BUT $\nabla \cdot \underset{\sim}{J}$ must vanish $\qquad$ no Build up of charge

this leads to the following differential equation for $J_\parallel$

$$\nabla \cdot J_\parallel \underset{\sim}{b} = \nabla \cdot \left(\frac{c \nabla p \times \underset{\sim}{B}}{B^2} \right)$$

$$= -c \nabla p \cdot \nabla \times \left(\frac{\underset{\sim}{B}}{B^2} \right)$$

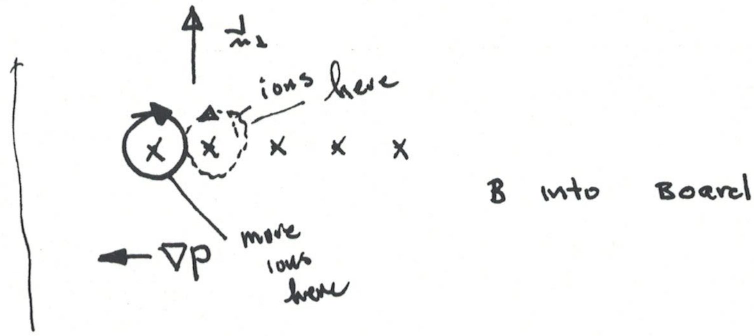
$$\cancel{= -c \nabla p \cdot \left[\nabla \left(\frac{1}{B} \right) \times \underset{\sim}{b} + \frac{1}{B^2} \nabla \times \underset{\sim}{b} \right]}$$

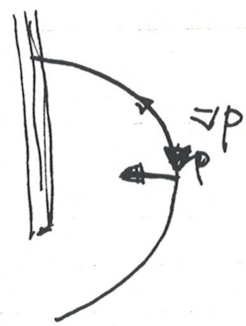
$$= - \frac{2c \nabla p \cdot \underset{\sim}{b} \times \underset{\sim}{K}}{B}$$

curvature

8'

Diamagnetic current





$$p = p(z)$$

$$J_r = \frac{c}{B_\theta(r)} \frac{\partial p}{\partial z}$$

$$\frac{1}{r}\frac{\partial}{\partial r} r J_r \neq 0$$

evaluate

$$-c\,\nabla p \cdot \nabla \times \left(\frac{\underset{\sim}{B}}{B^2}\right) = -c\,\nabla p \cdot \frac{\nabla \times \underset{\sim}{B}}{B^2} + c\,\nabla p \cdot \underset{\sim}{B} \times \nabla B^{-2}$$

$$= -\cancel{\frac{4\pi \nabla p \cdot \overset{0}{\underset{\sim}{J}}}{B^2}} + \cancel{\ \ }\ -c\,\frac{\nabla p}{B^4} \cdot \underset{\sim}{B} \times \nabla B^2 \qquad (B^2)^{-1}$$

BUT $\nabla B^2 = -8\pi \nabla p + 2B^2 \underset{\sim}{k}$

$$\nabla p \cdot B \times \nabla B^2 = 2B^2\,\nabla p \cdot \underset{\sim}{B} \times \underset{\sim}{k}$$

$$-c\,\nabla p \cdot \nabla \times \frac{\underset{\sim}{B}}{B^2} = -c\,\nabla p \cdot \frac{\nabla \times \underset{\sim}{B}}{B^2} + c\,\nabla p \cdot \underset{\sim}{B} \times \nabla \frac{1}{B^2}$$

$$c\,\nabla p \cdot \underset{\sim}{B} \times \nabla\left(\frac{1}{B^2}\right) = -c\,\frac{\nabla p \cdot B \times \nabla B^2}{B^4}$$

$\left(\nabla B^2 = 8\pi \nabla p + 2B^2 \underset{\sim}{k} \right)$ *equilibrium force balance*

$$\nabla \cdot J_{\parallel} \underset{\sim}{b} = \nabla \cdot \left(\frac{J_{\parallel}}{B}\right) \underset{\sim}{B} = \underset{\sim}{B} \cdot \nabla \left(J_{\parallel}/B\right)$$

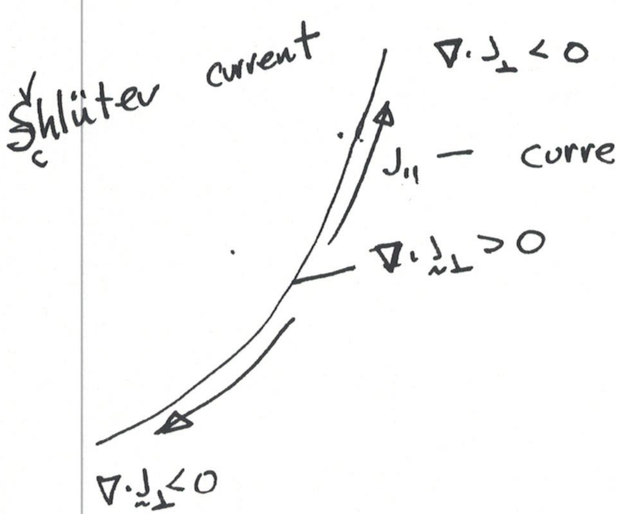
$$\left(\frac{J_{\parallel}}{B}\right) = \text{constant} - \int^{\ell} \frac{d\ell'}{B} \frac{2c}{B} \nabla p \cdot \underset{\sim}{b} \times \underset{\sim}{\kappa}$$

"force free current"

Shlüter current – $\frac{\partial p}{\partial \psi}$

thus, the parallel current will have two contributions

one of these, which is proportional to the pressure gradient will be present whenever $\underset{\sim}{\kappa}$ and ∇p are not in the same direction. (e.g. a tokamak, RFP, spheromak etc.)



$\underset{\sim}{J}_{\perp}$ is determined by force Balance

however $\nabla \cdot \underset{\sim}{J}_{\perp}$ may not vanish so a $J_{\parallel}$ is required to maintain charge neutrality the ~~ext~~ integration constant represents a force free ~~fa~~ current which has the form

could be different
~~d~~ but ~~d~~ on each field line

$$\underset{\sim}{J} = J_{\parallel} \underset{\sim}{b} = \text{constant } \underset{\sim}{B}$$

$J_{\parallel} \propto B$ constant

~~THE MAGNETIC~~