Particle Orbits

Because collisions are so infrequent the long time motion of individual particles is important.

Example : Landau-damping comes from resonant dedominator

$$\omega_d = \omega - kV = 0$$ Doppler shifted freq.

$$f_1 = \frac{q/m \; i k \varphi \cdot \partial f_0/\partial \underset{\sim}{v}}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v} + i\nu}$$

$\llcorner$ collision freq

Smallest denom can get is $\nu = \frac{1}{\tau}$
τ = time between collisions

Denominator comes from integrated effect of perturbing force along particle orbit

VE $$\frac{\partial f}{\partial t} + \underset{\sim}{v}\cdot\nabla f - \frac{q}{m}\nabla\phi\cdot\frac{\partial f}{\partial \underset{\sim}{v}} = 0$$

Linearized VE $\quad f = f_0 + f_1 \quad \phi = \phi_1$

$$\underbrace{\frac{\partial f_1}{\partial t} + \underset{\sim}{v} \cdot \nabla f_1} = \frac{q}{m} \nabla \phi_1 \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}} .$$

convective derivative along unperturbed ($\phi = 0$) orbit

$$f_1 = \int_0^t dt' \, \frac{q}{m} \nabla \phi_1 (\underset{\sim}{x}', t') \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}} + f_1(0, \underset{\sim}{v}, \cancel{x})$$

$\underset{\sim}{x}', t'$ are evaluated on unperturbed orbit

$\underset{\sim}{x}' = \underset{\sim}{x} + \underset{\sim}{v}(t' - t) \leftarrow$ straight lines

$\underset{\sim}{x}$ is position at time t

assume $\phi_1 = Re\{ \hat{\phi}_1 e^{i \underset{\sim}{k} \cdot \underset{\sim}{x}' - \omega t'} \}$

$$f_1 = Re\left\{ f_1(\underset{\sim}{v}, \underset{\sim}{x}, 0) + \int_0^t dt' \, \frac{q}{m} i \underset{\sim}{k} \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}} \hat{\phi}_1 \exp(i \underset{\sim}{k} \cdot \underset{\sim}{x} - i\omega t) \exp(-i(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})(t' - t)) \right\}$$

$$f_1 = Re\{ f_1(\underline{v}, \underline{x}, 0) - \exp(i\underline{k}\cdot\underline{x} - i\omega t)$$

$$\frac{q}{m} \frac{i\underline{k}\cdot\frac{\partial f_0}{\partial \underline{v}} \; e^{-i(\omega - \underline{k}\cdot\underline{v})(t'-t)}}{i(\omega - \underline{k}\cdot\underline{v})}$$

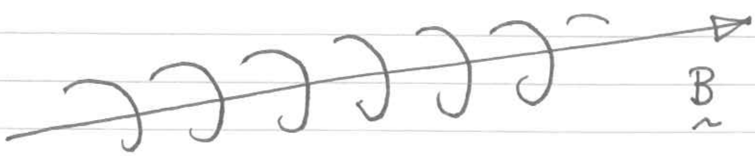
denominator

Perturbed f_1 is integral of time history along unperturbed orbit.

For unmagnetized plasma unperturbed orbits are straight lines particles maintain constant $\underset{\sim}{v}$.

Trivial

What are the orbits in a magnetized plasma?



particles spiral around B

at rate $\Omega = \dfrac{qB}{mc}$ = gyro freq.

orbit can be considered straight on time scales $T << \Omega^{-1}$

Examples $\Omega_e = 1.76 \times 10^7 B(\text{Gauss}) \frac{\text{rad}}{\text{sec}}$

$\Omega_i = \dfrac{q_i}{m_i} \dfrac{m_e}{q_e} \Omega_e << \Omega_e$

* Ionosphere at equator $B \sim 0.35 \times 10^{-4} T$

$\sim .35$ Gauss

$\Omega_e = 6.\cancel{\ } \times 10^6 \Rightarrow$ 1 MHZ (AM Band)

* Sunspot $B \sim 3 \times 10^3$ G

$\Omega_e = 5.3 \times 10^{10} \sim 10$ GHz

* Fusion Exp $B \sim 5 \times 10^4$ G

$\Omega_e \sim 8.8 \times 10^{11} \sim 100$ GHz

How big is the radius of spiral

ρ = gyroradius

$$V_t \approx \sqrt{\frac{2T}{m}}$$

$$\rho = \frac{V_t}{\Omega} \qquad \frac{V_{te}}{\Omega_e}, \quad \frac{V_{ti}}{\Omega_i}$$

$$\rho_i \sim \sqrt{\frac{T_i}{T_e}\frac{m_i}{m_e}}\,\rho_e >> \rho_e$$

Ionosphere $T_e \sim 10^{-2}$ eV $\qquad V_{te} \sim C\sqrt{\frac{T_e}{511 \times 10^3}}$

$V_{te} \sim 4 \times 10^6$ cm/sec $\qquad \sim 4 \times 10^7\, T^{1/2}(eV)$

$$\rho_e \sim \frac{4 \times 10^6 \text{ cm/sec}}{6 \times 10^6 - 1/\text{sec}} \sim 1 \text{ cm}$$

very small
on scale of ionosph

Sun $\quad T_e \sim 10^4$ eV $\quad V_{the} \sim 4\times10^9$ cm/sec

$$\rho_e \sim \frac{4\times10^9}{5.3\times10^{10}} \simeq .1 \text{ cm}$$

very small

Fusion Machine $\quad T \sim 10^4$ eV

$$\rho_e \sim \frac{4\times10^9}{8.8\times10^{11}} = \underline{\underline{5\times10^{-3}\text{ cm}}}$$

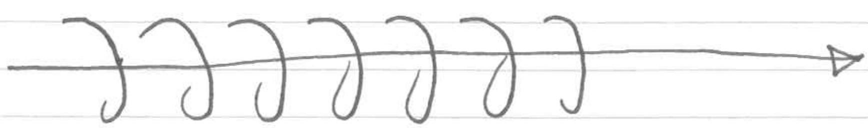
small

Conclusion $\quad$ magnetic field strongly modifies orbits

* ORBITS IN Electric and Magnetic fields

Simplest case: straight constant $\underset{\sim}{B}$

$$\frac{d\underset{\sim}{V}}{dt} = \frac{q}{m}\underset{\sim}{V} \times \underset{\sim}{B}$$



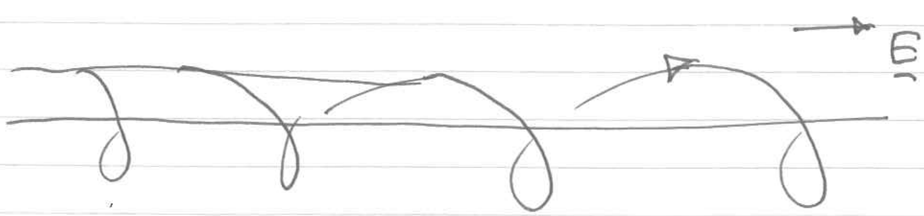
$V_{\parallel}$ = component of $\underset{\sim}{V} \parallel \underset{\sim}{B}$ = const

$\underset{\sim}{V}_{\perp}$ = components $\perp$ B oscillate (gyrate)

<u>ORBit is a spiral</u>

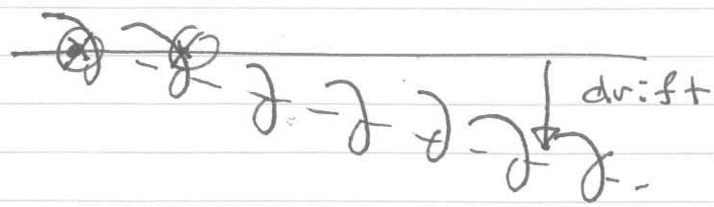
* what happens if we add an $\underset{\sim}{E}$?

if $\underset{\sim}{E}$ is $\parallel$ to $\underset{\sim}{B}$ particle accelterates



if $\underset{\sim}{E}$ is $\perp$ to $\underset{\sim}{B}$

$\underset{\sim}{E}$ into board



center of spiral "drifts" in direction $\underline{E} \times \underline{B}$

$$\underset{\sim}{V}_D = \frac{c}{B^2} \underset{\sim}{E} \times \hat{B}$$

* if $\underset{\sim}{E}$ is ~~timeske~~ time dependent (slow)

drift in direction of $d\underset{\sim}{E}/dt$

– Polarization drift

* what if $\underset{\sim}{B}$ depends on space

converging field lines

B-high

B-Low

$V_\perp$ increases

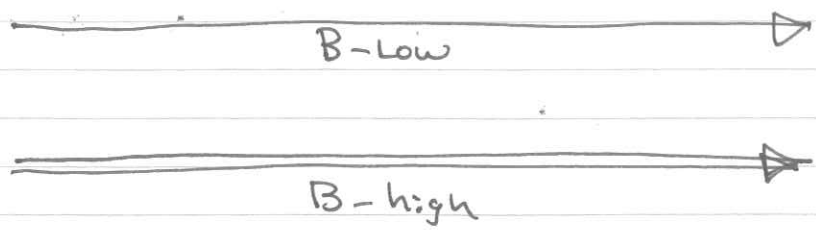
$V_\parallel$ decreases

$$\mu = \frac{\frac{1}{2}mV_\perp^2}{B} \underline{\simeq \text{const}}$$

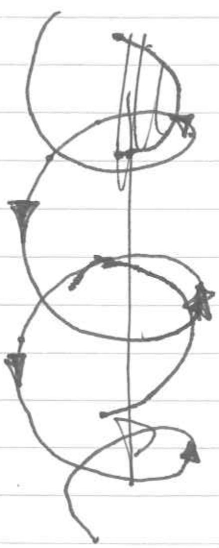
particle reflects

magnetic-mirror

* what if $\underset{\sim}{B}$ is straight but depends on space



B_{low}

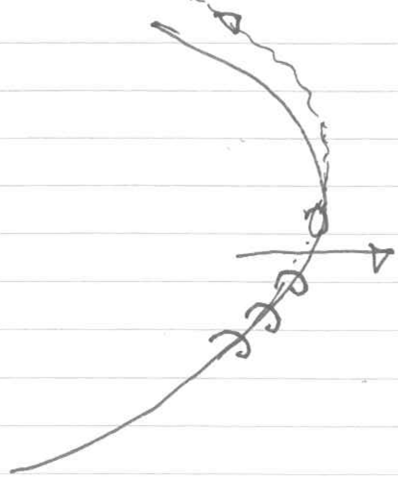


B high

$\nabla B \rightarrow$ <u>gradient B drift</u>

drift of center

* what happens if $\underset{\sim}{B}$ is curve



drift $\perp$ to radius

centrifugal force

$$\frac{\underset{\sim}{F} \times B}{B^2}$$

<u>curvature drift</u>

Let's begin with the simplest case

$$\underset{\sim}{B} = B\,\underline{a}_z \qquad B = \text{const}$$

$$\underset{\sim}{E} = \underset{\sim}{E}_\perp = E_x\,\underline{a}_x + E_y\,\underline{a}_y \qquad E_x, E_y \sim \text{const}$$

$$m\frac{d\underset{\sim}{V}_\perp}{dt} = q\left[\underset{\sim}{E}_\perp + \frac{\underset{\sim}{V}_\perp \times \underline{a}_z B}{c}\right]$$

$$\text{Let} \quad \underset{\sim}{V}_\perp = \overbrace{\frac{c}{B}\underset{\sim}{E}_\perp \times \underline{a}_z}^{\text{const}} + \tilde{\underset{\sim}{V}}_\perp$$

$$\underset{\sim}{V}_\perp \times \underline{a}_z = \tilde{\underset{\sim}{V}}_\perp \times \underline{a}_z - \frac{c}{B}\underset{\sim}{E}_\perp$$

$$m\frac{d\underset{\sim}{V}_\perp}{dt} = m\frac{d\tilde{\underset{\sim}{V}}_\perp}{dt} = -\frac{q}{c}\tilde{\underset{\sim}{V}}_\perp \times \underline{a}_z B$$

$E_\perp$ has dissapeared

$\tilde{\underset{\sim}{V}}_\perp$ corresponds to gyromotion

IN Reference frame moving with velocity

$$\underset{\sim}{V}_d = \frac{c}{B} \underset{\sim}{E}_\perp \times \underset{\sim}{a}_z$$

Electric field is zero!

Generalization to arbitrary direction of $\underset{\sim}{B}$

$$\underset{\sim}{V}_d = \frac{c}{B^2} \underset{\sim}{E} \times \underset{\sim}{B} \quad \text{or} \quad \frac{c}{|B|} \underset{\sim}{E} \times \underset{\sim}{b} \qquad \underset{\sim}{b} = \frac{\underset{\sim}{B}}{|B|}$$

Insert

Polarization drift

$$\underset{\sim}{E}_\perp = \text{function of time (slow)}$$

$$\frac{d\underset{\sim}{V}_\perp}{dt} = \frac{q}{m}\left[\underset{\sim}{E}_\perp^{(t)} + \frac{\underset{\sim}{V}_\perp \times \underset{\sim}{B}}{c}\right]$$

let $$V_\perp = \frac{c}{B^2} \underset{\sim}{E} \times \underset{\sim}{B} + \tilde{\underset{\sim}{V}}_\perp$$

Next

$$\frac{d\tilde{V}_\perp}{dt} + \frac{c}{B^2}\left(\frac{dE}{dt} \times \underset{\sim}{B}\right) = \frac{q}{m}\tilde{\underset{\sim}{V}}_\perp \times \underset{\sim}{B}$$

OR

$$\frac{d\tilde{V}_\perp}{dt} = \boxed{-\frac{c}{B^2}\left(\frac{dE}{dt} \times \underset{\sim}{B}\right)} + \frac{q}{m}\tilde{\underset{\sim}{V}}_\perp \times \underset{\sim}{B}$$

treat like a new force $\underset{\sim}{g}$

$$\underset{\sim}{V}_D = \frac{1}{\Omega}\underset{\sim}{g} \times \underset{\sim}{b} = \frac{1}{(q|B|/mc)}\left(-\frac{c}{B^2}\frac{dE}{dt} \times \underset{\sim}{B}\right) \times b$$

$$= \frac{mc}{q|B|^2}\, c\, \frac{d\underset{\sim}{E}_\perp}{dt}$$

$$\boxed{= \frac{1}{\Omega}\frac{c}{|B|}\frac{dE_\perp}{dt}}$$

Polarization drift

Extra force (like gravity)

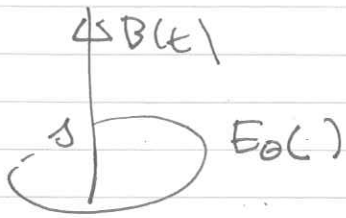
$$\frac{F}{m}$$

$$\frac{d\underset{\sim}{V}_\perp}{dt} = \underset{\sim}{g} + \frac{q}{m}\frac{\underset{\sim}{V}_\perp \times \underset{\sim}{B}}{c}$$

$$\underset{\sim}{V}_D = \frac{\underset{\sim}{g} \times \hat{b}}{\Omega} \qquad \Omega = \frac{q|B|}{mc}$$

Constancy of $\mu = \frac{\frac{1}{2} m v_\perp^2}{B}$

Two scenario's ① $\underset{\sim}{B}(t) \Rightarrow \underset{\sim}{E}$ via Faraday's Law increases $V_\perp$



② $\underset{\sim}{B}(\underset{\sim}{x})$ also can change $V_\perp$

both lead to $\mu = \frac{1}{2} m v_\perp^2 / B \simeq$ const

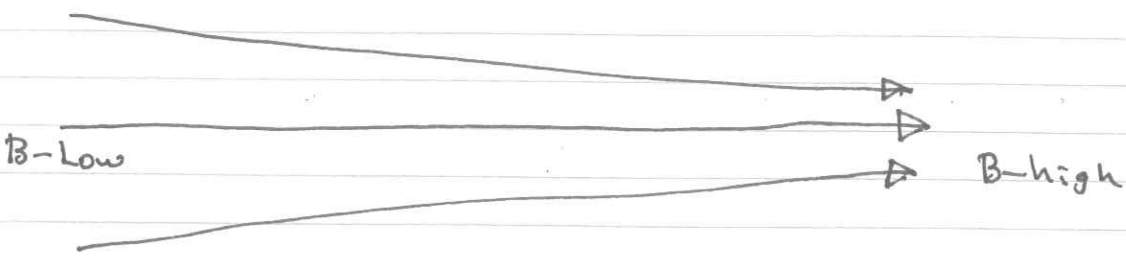
Let's suppose $\underset{\sim}{B} = B(z)\,\underset{\sim}{a}_z$ $\qquad \underset{\sim}{E} = 0$

$\frac{dV_z}{dt} = 0$ $\qquad V_z = \text{const} \Rightarrow V_\perp^2 = \text{const}$

for motion in magnetic field $\qquad E = \frac{1}{2} m v^2 = \text{const}$

What is wrong with this picture?

$\nabla \cdot \underset{\sim}{B} \neq 0$



must also be B_r

$$\frac{1}{r}\frac{\partial}{\partial r} r B_r + \frac{\partial B_z}{\partial z} = 0$$

so, if $B_z = B(z)$

then $B_r = -\frac{r}{2}\frac{\partial B}{\partial z}$

$$\underset{\sim}{B} = \underset{\sim}{a}_z B_z + \underset{\sim}{B}_\perp$$

$$\underset{\sim}{B}_\perp = -\frac{r}{2}\frac{\partial B}{\partial z}\underset{\sim}{a}_r$$

Solve

$$\frac{d\underset{\sim}{V}}{dt} = \frac{q}{m}\left(\underset{\sim}{V} \times \underset{\sim}{B}\right)$$

in cylindrical coordinates

$$\cancel{\frac{d\underset{\sim}{V}}{dt} = q\left(\frac{\partial \underset{\sim}{V}}{\partial t} + \underset{\sim}{V}\cdot\nabla \underset{\sim}{V}\right)}$$

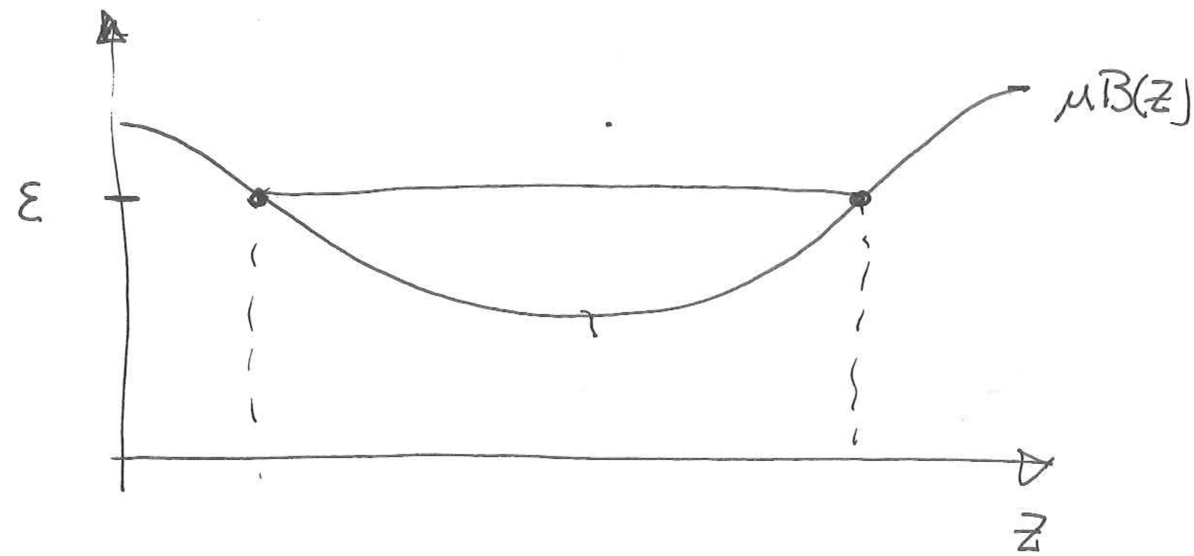
TOTAL ENERGY

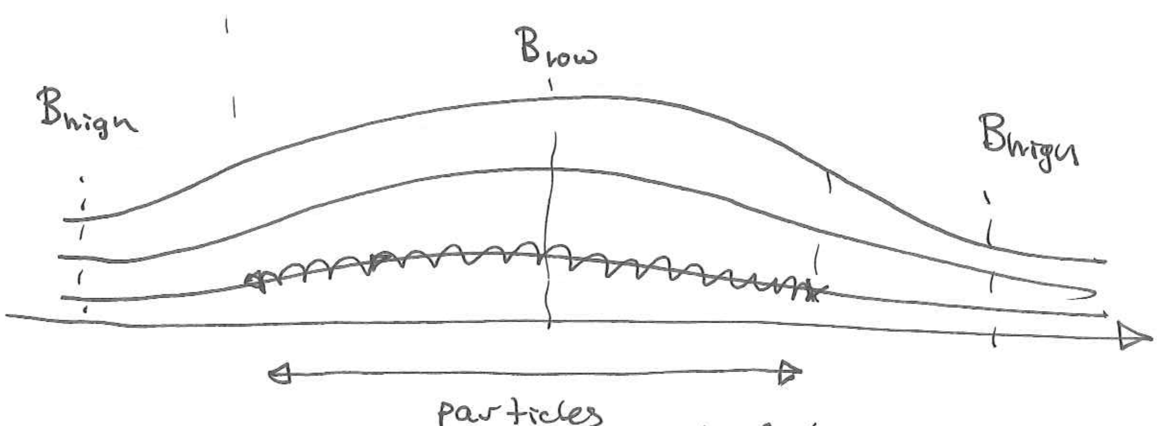
$$E = \frac{1}{2}m(V_\perp^2 + V_z^2) \quad \text{is conserved}$$

for motion in $\underset{\sim}{B}$

$$\frac{1}{2}mV_z^2 = \mathcal{E} - \frac{1}{2}mV_\perp^2 = \mathcal{E} - \mu B$$

$$V_z = \pm\sqrt{\frac{2}{m}(\mathcal{E} - \mu B)}$$

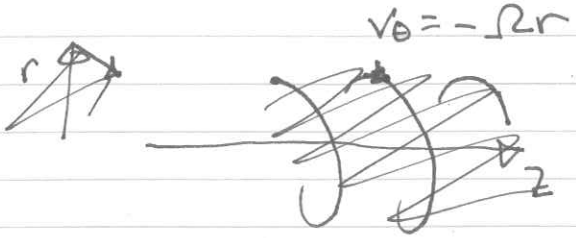




To lowest order B is straight

$$-\frac{V_\theta^2}{r} = \frac{q}{mc} V_\theta B \qquad V_\theta = -\frac{q}{mc} r B$$

$$\boxed{V_\theta = -r\Omega}$$



$\underset{\sim}{V}_\perp \sim V_\theta \underset{\sim}{a}_\theta$

$$\underset{\sim}{V} = \underset{\sim}{V}_\perp + \underset{\sim}{a}_z V_z$$

$$\underset{\sim}{V}_\perp \cdot \frac{d}{dt} \underset{\sim}{V}_\perp = \frac{d}{dt} V_\perp^2/2 = \frac{q}{mc} \underset{\sim}{V}_\perp \cdot (\underset{\sim}{V} \times \underset{\sim}{B})$$

$$= \frac{q}{m} \underset{\sim}{V}_\perp \cdot (V_z \underset{\sim}{a}_z \times \underset{\sim}{B})$$

θ z r

$$= \frac{q}{mc} V_\theta V_z B_r$$

$$B_r = -\frac{r}{2} \frac{\partial B_z}{\partial z}$$

$$r = -\frac{V_\theta}{\Omega}$$

$$\frac{d}{dt}\frac{V_\theta^2}{2} = \frac{q}{mc}V_z \frac{V_\theta^2}{2\Omega}\frac{\partial}{\partial z}B = \cancel{V_z}\frac{V_\theta^2}{2}V_z\frac{1}{\Omega}\frac{\partial\Omega}{\partial z}$$

$$= \frac{V_\theta^2}{2}\frac{1}{\Omega}\frac{d\Omega}{dt}$$

thus $\frac{d}{dt}\left(\frac{V_\theta^2}{2\Omega}\right) = 0$

$$\boxed{\mu = \frac{mV_\perp^2}{2B} = \text{const}}$$

what about V_z

$$\frac{dv_z}{dt} = \frac{q}{mc}\,\underline{a}_z\cdot\left(\underset{\sim}{V}_\perp\times\underset{\sim}{B}\right)$$

$$-V_\theta B_r$$

$$= \frac{q}{mc}\,V_\theta + \frac{r}{2}\frac{\partial B}{\partial z} = -\frac{V_\theta^2}{2}\frac{1}{\Omega}\frac{\partial\Omega}{\partial z}$$

$\frac{V_\theta^2}{\Omega} = \text{const}$

effective potential

$$\boxed{F_z = -m\mu\frac{\partial B}{\partial z}}$$

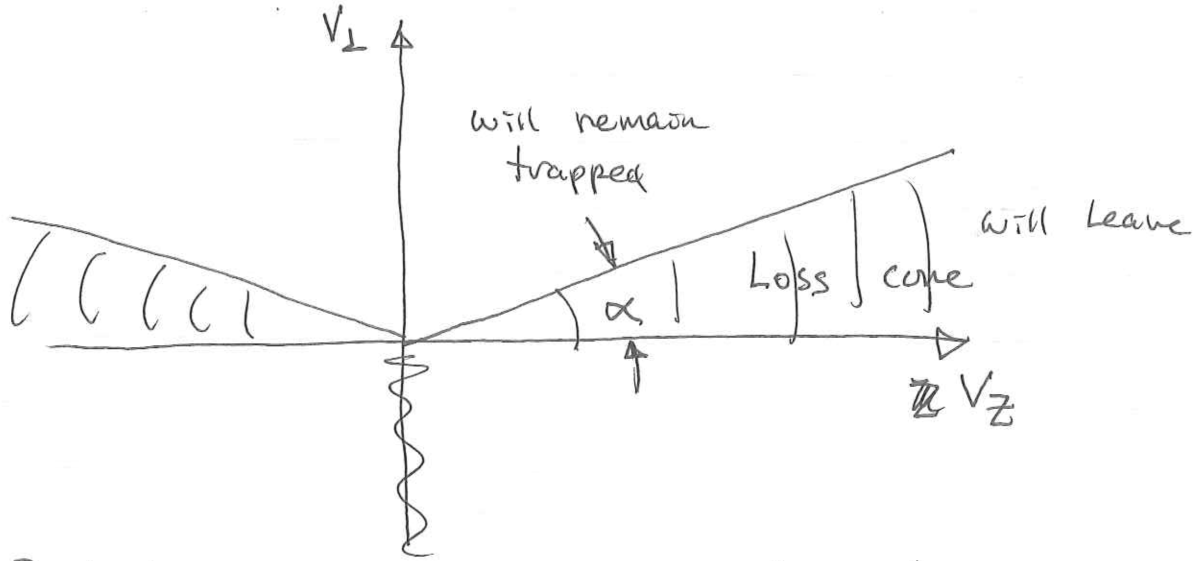
Some particles make it over the top of the hill.

if $\mathcal{E} > \mu B_{max}$ particles escape

Let's draw velocity space for particles as they pass point where $B = B_{min}$

at B_{min} $\qquad \frac{1}{2} m v_\perp^2 = \mu B_{min}$

$$\frac{1}{2} m v_z^2 = \mathcal{E} - \mu B_{min}$$



Particles will escape if

$\mathcal{E} > \mu B_{max}$ $\qquad \dfrac{\mathcal{E}}{\mu B_{min}} > \dfrac{B_{max}}{B_{min}}$

$$\left.\frac{V_\perp^2 + V_z^2}{V_\perp}\right|_{B_{min}} > \frac{B_{max}}{B_{min}}$$

OR

$$\frac{V_z^2}{V_\perp^2} > \left(\frac{B_{max}}{B_{min}} - 1\right)$$

$$\tan \alpha = \sqrt{\frac{B_{max}}{B_{min}} - 1}$$

$\frac{B_{max}}{B_{min}}$ = mirror ratio

What happens

(1) prompt loss of particles in ~~cone~~ loss cone

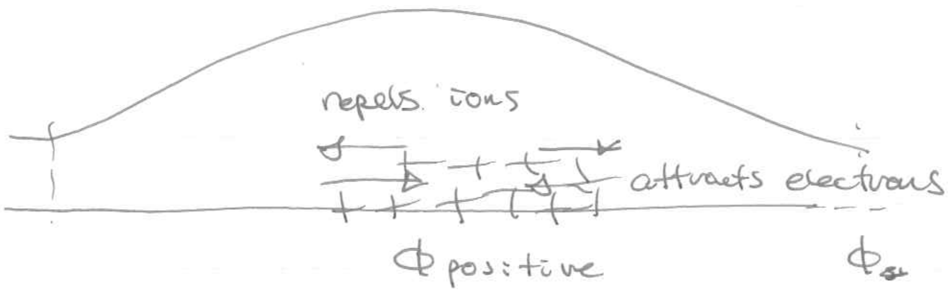
(2) slow loss of particles due to scattering in pitch angle (classical)

(3) Electrons scatter faster than ions, preferentially lost

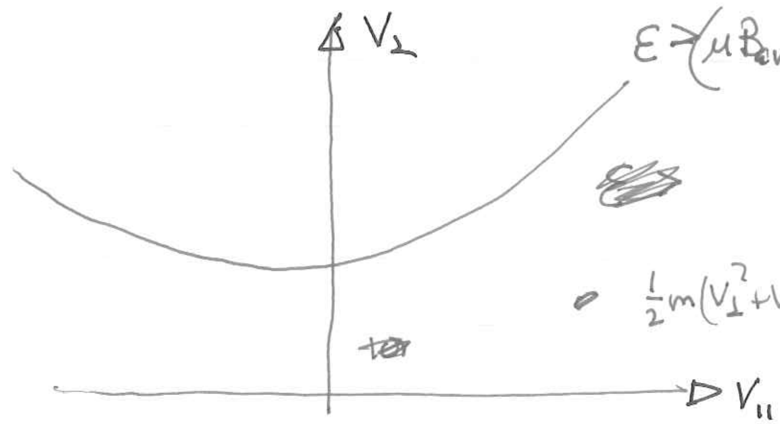
$n_i > n_e$

electro-static potential is built up

$$\frac{1}{2}mV_\parallel^2 + \frac{1}{2}mV_\perp^2 + q\phi = \mathcal{E} = \text{const}$$



$$V_\parallel = \pm\sqrt{\frac{2}{m}\left(\mathcal{E} - \mu B - q\phi\right)}$$



Particles lost if

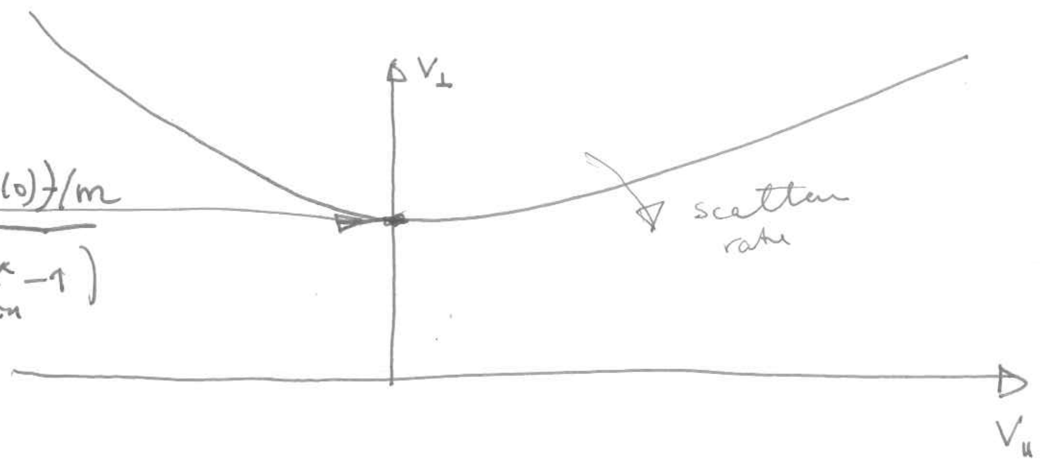
$$\mathcal{E} > \left(\mu B_{max} + q\phi_{end}\right)$$

$$\frac{1}{2}m\left(V_\perp^2 + V_\parallel^2\right) > \frac{1}{2}mV_\perp^2\frac{B_{max}}{B_{min}} + q\left(\phi_{end} - \underset{cent}{\phi(0)}\right)$$

$$V_\parallel^2 > V_\perp^2\left(\frac{B_{max}}{B_{min}} - 1\right) + \underbrace{\frac{2}{m}q\left(\underset{0}{\phi_{end}} - \underset{P}{\phi(0)}\right)}_{\substack{\text{Positive for electr}\\ \text{negative for ions}}}$$

Ions

$$V_{\perp c}^2 = \frac{2q_i(\phi(0))/m}{\left(\frac{B_{max}}{B_{min}} - 1\right)}$$



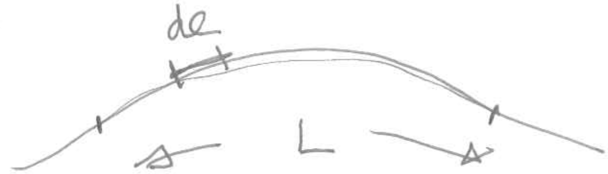
electrons

$$V_{\parallel c}^2 = -\frac{2q_e}{m}\phi(0)$$

Typically $e\phi \sim T_e$ to suppress electron loss rate

Ions lost in a collision time

Bounce Time



$$\tau_b = \oint \frac{dl}{v_\parallel} \sim \frac{L}{v_{th}}$$

Integral taken between turning points

$$\tau(\varepsilon, \mu)$$

Introduce $\quad J(\varepsilon, \mu) = \oint dl\, m v_\parallel$

$$= \oint dl\, m \sqrt{(\varepsilon - \mu B - q\Phi)\frac{2}{m}}$$

$B(l) \quad \phi(l)$

$$\boxed{\frac{\partial J}{\partial \varepsilon} = \tau(\varepsilon, \mu)}$$

~~Supp~~ $J(\varepsilon, \mu)$ is an adiabatic invariant if B, ϕ depend ~~on~~ slowly enough on time

μ is still constant

ε changes according to $J(\varepsilon, \mu) = \text{const}$

TRANSVERSE DRIFTS

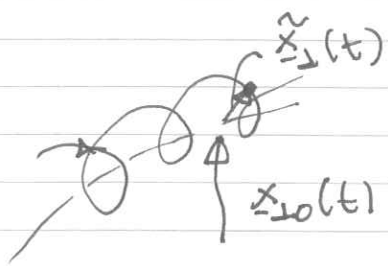
Gradient - B drift

$$\underset{\sim}{B} = B(\underset{\sim}{x}_\perp)\, \underset{\sim}{a}_z$$

$$\frac{d\underset{\sim}{V}_\perp}{dt} = \frac{q}{mc}\, \underset{\sim}{V}_\perp \times B(\underset{\sim}{x}_\perp)\, \underset{\sim}{a}_z$$

$$\underset{\sim}{x}_\perp = \underset{\sim}{X}_{\perp 0} + \tilde{\underset{\sim}{X}}_\perp$$ – sinusoidal oscillat (fast)

center of spiral (slow)



$$\underset{\sim}{V}_\perp = \frac{d}{dt}\underset{\sim}{x}_{\perp 0} + \frac{d}{dt}\tilde{\underset{\sim}{x}}_\perp$$

$$= \underset{\sim}{V}_d + \tilde{\underset{\sim}{V}}_\perp$$

$$\frac{d}{dt}\underset{\sim}{V}_\perp = \frac{q}{mc}\left[(\underset{\sim}{V}_d + \tilde{\underset{\sim}{V}}_\perp) \times \left(B(\underset{\sim}{x}_{\perp 0}) + \tilde{\underset{\sim}{x}}_\perp \cdot \nabla B \right) \underset{\sim}{a}_z \right]$$

Average over rapid oscillations.

$$\frac{q}{mc}\Big[V_d \times B(x_{\perp 0})\, a_z + \cancelto{0}{\langle \tilde{V}_\perp \times B(x_{\perp 0})\, a_z \rangle}$$

$$+ \cancelto{0}{\langle V_d \times (\tilde{x}_\perp \cdot \nabla B)\, a_z \rangle} + \langle (\tilde{V}_\perp \times a_z)\, \tilde{x}_\perp \rangle \cdot \nabla B$$

fast oscillation

$$\frac{d\tilde{V}_\perp}{dt} = \frac{qB}{mc}(\tilde{V}_\perp \times a_z)$$

$$\langle (\tilde{V}_\perp \times a_z)\, x_\perp \rangle = \left\langle \frac{d\tilde{V}_\perp}{dt}\, \tilde{X}_\perp \right\rangle \frac{mc}{qB}$$

$$= \left\langle \frac{d}{dt}(\tilde{V}_\perp \tilde{X}_\perp) - \tilde{V}_\perp \frac{d}{dt}\tilde{X}_\perp \right\rangle \frac{mc}{qB}$$

$$= - \langle \tilde{V}_\perp \tilde{V}_\perp \rangle \frac{mc}{qB}$$

$$= - \underline{\underline{I}}\, \frac{1}{2} v_\perp^2 \frac{mc}{qB}$$

$$\frac{q}{mc}\left[V_d \times B\, a_z - \underline{\underline{I}} \cdot \nabla B \left(\frac{1}{2} m v_\perp^2 \frac{c}{qB} \right) \right] \qquad \frac{\mu c}{q}$$

$$V_d = \frac{c}{q} \frac{a_z \times \mu \nabla B}{B} \quad \cancel{\frac{1}{2} m v_\perp^2}$$

For static ~~F~~ Fields or slowly varying

$$\boxed{\underset{\sim}{V}_d = \frac{c}{q|B|}\,\underset{\sim}{b} \times \Big[\overbrace{q \nabla \phi}^{E \times B} + \overbrace{m v_\parallel^2 \underset{\sim}{\kappa}}^{\text{curvature}} + \overbrace{\mu \nabla |B|}^{\text{grad } B} \Big]}$$

$$\frac{d\underset{\sim}{x}}{dt} = v_\parallel \underset{\sim}{b} + \underset{\sim}{V}_d + \cancel{V_p}$$

$$\frac{d\mathcal{E}_k}{dt} = q\left(v_\parallel \underset{\sim}{b} + \underset{\sim}{V}_d\right) \cdot \underset{\sim}{E}$$

$\mu = \text{const}$ $\qquad$ $v_\parallel = \pm \frac{2}{m}\sqrt{\mathcal{E}_k - \mu B}$

$$\cancel{\frac{d\mathcal{E}_k}{dt} = \frac{d}{dt}\left(\frac{1}{2} m v_\parallel^2 + \mu B\right)}$$

$$\cancel{= q\, v_\parallel \left(-\underset{\sim}{b} \cdot \nabla \phi\right) + \frac{c}{|B|}\, \underset{\sim}{b} \times \left(m v_\parallel^2 \underset{\sim}{\kappa} + \mu \nabla B\right) \cdot (-\nabla \phi)}$$

$$\cancel{\mu\, \underset{\sim}{V}_d \cdot \nabla B}$$

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unit vector

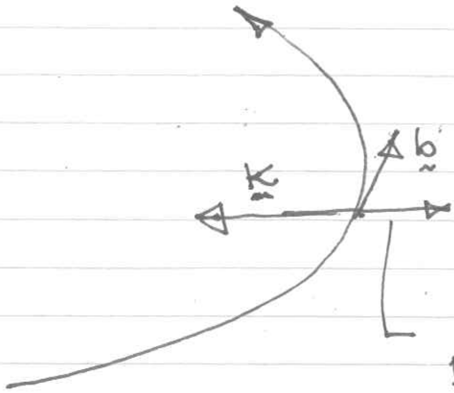
$$\underset{\sim}{b} = \frac{\underset{\sim}{B}}{B}$$

unit vector

Curvature drift

$$\underset{\sim}{\kappa} = \underset{\sim}{b} \cdot \nabla \hat{b}$$

$$|\underset{\sim}{\kappa}| = \frac{1}{R_c}$$



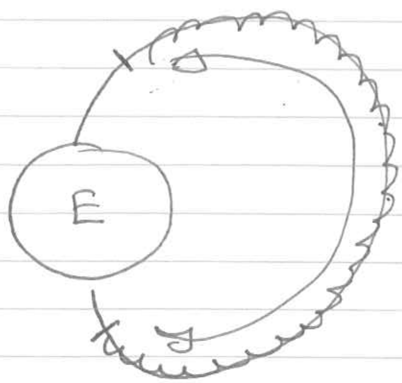
$$\frac{m V_\parallel^2}{R_c}$$ ~ radius of curvature

$$\underset{\sim}{F}_{ct.} = - m V_\parallel^2 \underset{\sim}{\kappa}$$

$$\underset{\sim}{V}_d = \frac{c}{q|B|} \underset{\sim}{b} \times \underset{\sim}{\kappa} \, m V_\parallel^2$$

$$\underset{\sim}{V}_d = \frac{c}{q|B|} \underset{\sim}{F} \times \underset{\sim}{b} =$$

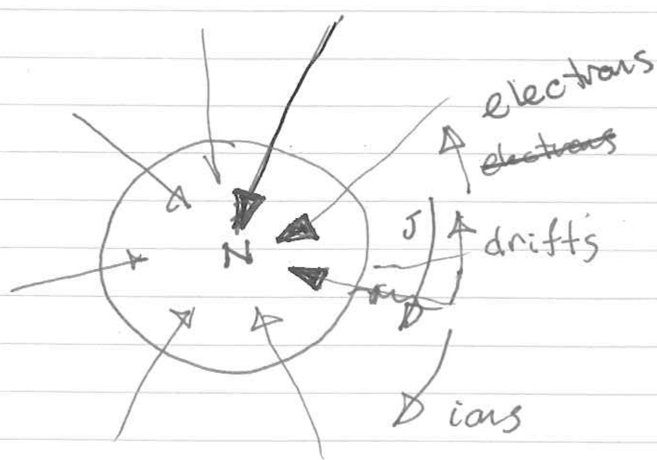
Motion



Lowest order
bounce back and
forth between
poles

$$\frac{c}{q|B|} b \times \mu \nabla B$$

Top View



$\underset{\sim}{k} = \underset{\sim}{b} \cdot \nabla \underset{\sim}{b}$ inward

towards
earth

∇B inward

$$\underset{\sim}{V_d} = \frac{c}{q|B|} \underset{\sim}{b} \times (\mu \nabla B + m v_{\parallel}^2 \underset{\sim}{k})$$

makes a current

Time scales / Length scales

$$\Omega_e = 1.76 \times 10^7 \; B$$

at equator

earth radius

$$B = 0.35 \; \text{Gauss} \left(\frac{R_e}{R}\right)^3$$

$$\Omega_e = 6 \times 10^6 \left(\frac{R_e}{R}\right)^3$$

for $\mathcal{E} \sim 1$ mev

$$\rho_e = \frac{V_{the}}{\Omega_e} =$$

$$= \frac{3 \times 10^{10} \text{ cm/sec}}{6 \times 10^6 \text{ sec}^{-1}} = .5 \times 10^4 \text{ cm}$$

$$\rho_e = 5 \times 10^3 \text{cm} \left(\frac{R}{R_e}\right)^3 \qquad R_e = 6.4 \times 10^8 \text{ cm}$$

$$\frac{\rho_e}{R} = \left(\frac{5 \times 10^3}{R_e}\right) \left(\frac{R}{R_e}\right)^2$$

$$\boxed{\frac{\rho_e}{R} = 7.81 \times 10^{-6} \left(\frac{R}{R_e}\right)^2 \ll 1}$$

Bounce time

assume $V_{te} \sim c$ ok for 1 MeV

$$T_e \sim \cancel{3\times10^{10}\ cm/sec} \left(\frac{R}{R_e}\right)\frac{R_e}{c} = \frac{6.4\times10^{8}\ cm}{3\times10^{10}\ cm/sec}$$

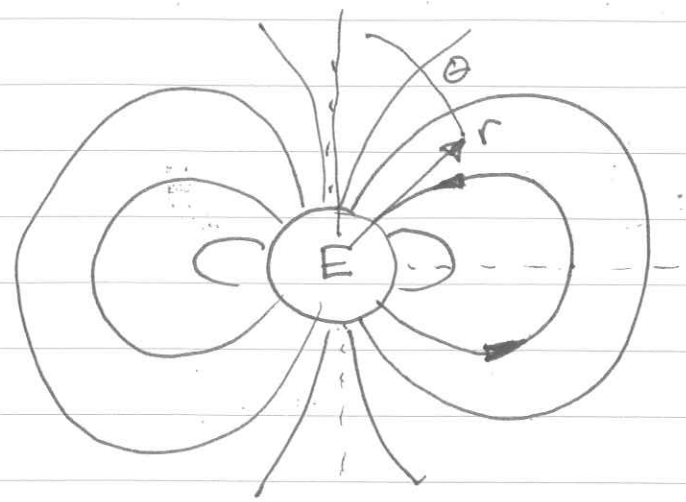
$$\sim \left(\frac{R}{R_e}\right) 2\times10^{-2}\ \underline{sec}$$

electrons // $\sim$ 1-10 sec ions $\sim$ 1 meV

DRIFT Time $\sim$ hours

$V_{dm} \sim \frac{\rho}{R} V_{th}$ $\sim$ hours (1 meV) particles

~~Notes~~ Example: Bounce motion and drifts in magneto-sphere



Dipole Field

~~$\underset{\sim}{B} = \frac{M_E}{r^3} ($~~

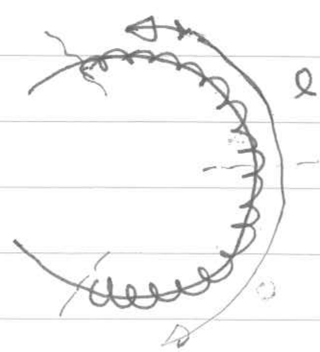
$$\underset{\sim}{B} = B_r \underset{\sim}{a}_r + B_\theta \underset{\sim}{a}_\theta$$

$$B_r = -\frac{M}{r^3} 2\cos\theta$$

$$B_\theta = -\frac{M}{r^3} \sin\theta$$

Particles bounce back and forth on a field line.

M = dipole moment



we need to find a coordinate system where field lines are labeled.

Magnetic ~~Clebsch~~ coordinates

$$\nabla \cdot \underset{\sim}{B} = 0$$

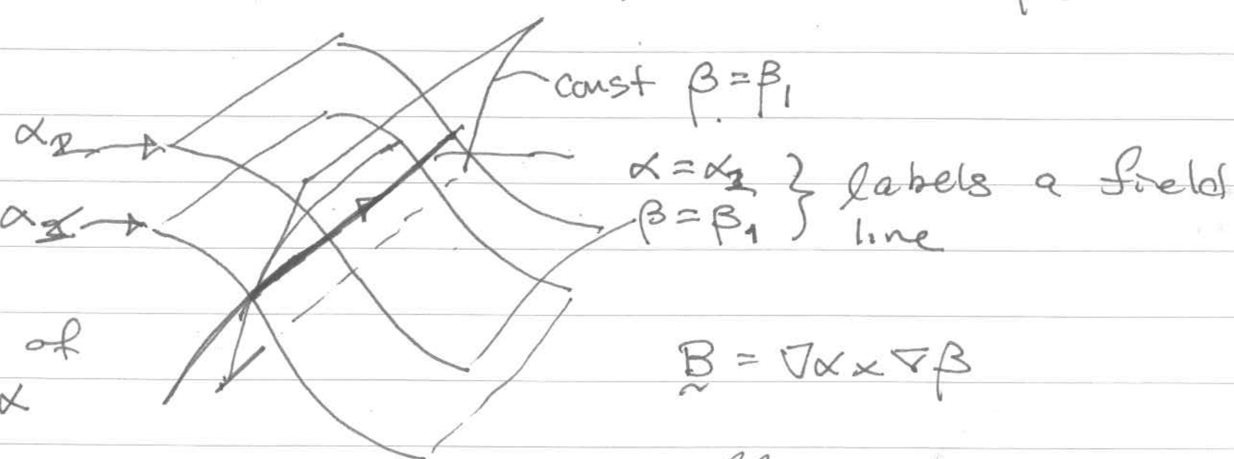
$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A} \qquad \text{or} \qquad \underset{\sim}{B} = \nabla\alpha \times \nabla\beta$$

α & β are two scalar functions

Note: $\underset{\sim}{B} \cdot \nabla\alpha = 0 \qquad \underset{\sim}{B} \cdot \nabla\beta = 0$

$\alpha = \beta = \text{const}$ following a field line

Also by design $\nabla \cdot \underset{\sim}{B} = \nabla \cdot (\nabla\alpha \times \nabla\beta) = 0$



$$\underset{\sim}{B} = \nabla\alpha \times \nabla\beta$$

~~α & β = const~~

Find α & β for dipole field

$$\underset{\sim}{B} = \nabla\alpha \times \nabla\beta \qquad \nabla\beta = \nabla\varphi$$

$\angle$ longitudinal angle

Precessional Drift

Let's assume $\nabla \times \underset{\sim}{B} = 0$

THEN $\nabla \times (B\underset{\sim}{b}) = \nabla B \times \underset{\sim}{b} + B\, \nabla \times \underset{\sim}{b} = 0$

IT FOLLOWS $\underset{\sim}{b} \cdot \nabla \times \underset{\sim}{b} = 0$ for vacuum field

Vector identity (for any $\underset{\sim}{b}$)

$$\underset{\sim}{b} \times \nabla \times \underset{\sim}{b} + \underset{\sim}{b} \cdot \nabla \underset{\sim}{b} = \frac{1}{2} \nabla (\underset{\sim}{b} \cdot \underset{\sim}{b}) = 0 \quad \text{if } |\underset{\sim}{b}| = 1$$

THUS ~~$\underset{\sim}{b} \times \nabla$~~ $\boxed{\underset{\sim}{b} \times \underset{\sim}{\kappa} = + \nabla \times \underset{\sim}{b} = \cancel{\quad}}$

$$= \nabla \times \frac{\underset{\sim}{B}}{B} = \frac{1}{B} \underbrace{\nabla \times \underset{\sim}{B}}_{\frac{4\pi}{c} \underset{\sim}{J}} + \underbrace{\nabla \frac{1}{B} \times \underset{\sim}{B}}_{\frac{\underset{\sim}{b}}{B} \times \nabla B}$$

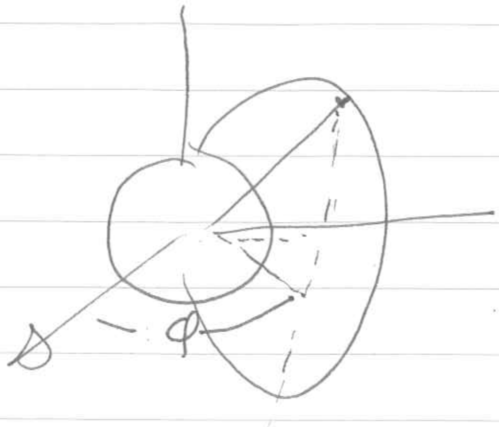
0 for vacuum field

$$\underset{\sim}{V}_{dm} = \frac{c}{qB} \underset{\sim}{b} \times \left(\mu \nabla B + \frac{m V_{\parallel}^2}{B} \nabla B \right)$$

$$\underset{\sim}{V}_{dm} = \frac{c}{qB} b \times \left(\mu B + m V_{\parallel}^2 \right) \frac{\nabla B}{B}$$

curvature drift and grad B drift are parallel to each other.

<u>at equator</u> $\nabla B / B = -\frac{3}{r} \hat{r} \quad \underset{\sim}{b} \parallel \hat{z}$



$$\alpha = \alpha(r, \theta)$$

$$\nabla\alpha = \mathbf{a}_r \frac{\partial \alpha}{\partial r} + \mathbf{a}_\theta \frac{1}{r}\frac{\partial \alpha}{\partial \theta}$$

$$\nabla\beta = \nabla\varphi = \frac{\mathbf{a}_\varphi}{r\sin\theta}$$

$$\nabla\alpha \times \nabla\beta = \underbrace{\mathbf{a}_r \times \mathbf{a}_\varphi}_{-\mathbf{a}_\theta} \frac{1}{r\sin\theta}\frac{\partial \alpha}{\partial r} + \underbrace{\mathbf{a}_\theta \times \mathbf{a}_\varphi}_{\mathbf{a}_r} \frac{1}{r^2\sin\theta}\frac{\partial \alpha}{\partial \theta}$$

$$\frac{1}{r^2\sin\theta}\frac{\partial \alpha}{\partial \theta} = B_r = -\frac{m}{r^3} 2\cos\theta$$

$$-\frac{1}{r\sin\theta}\frac{\partial \alpha}{\partial r} = -\frac{m}{r^3}\sin\theta$$

$$\frac{\partial \alpha}{\partial r} = \frac{m}{r^2}\sin^2\theta$$

$$\alpha = -\frac{m}{r}\sin^2\theta + C$$

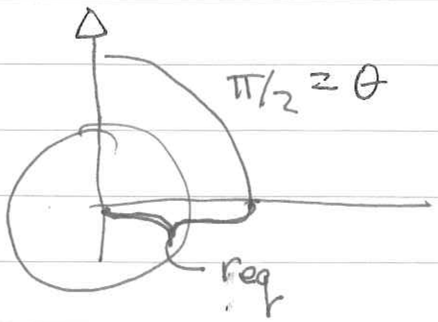
irrelevant

$$\boxed{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(-\frac{m}{r}\sin^2\theta\right) = -\frac{2m\cos\theta}{r^3} \quad \underline{\underline{\text{OK}}}}$$

Equations for field lines

$$\phi = \text{cons}$$

$$\alpha = -\frac{m}{r}\sin^2\theta = \text{const}$$

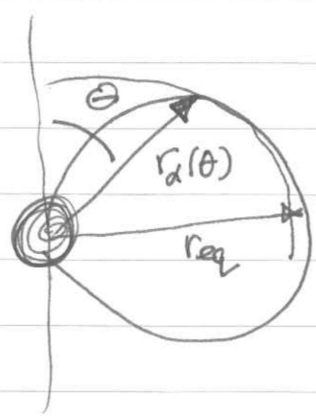


r_{eq} = radius of field line at equator

$\theta = \pi/2 \quad \sin\theta = 1$

$$\text{const} = -\frac{m}{r_{eq}}$$

$$r_\alpha = r_{eq}\sin^2\theta$$



on a field line

$$|B| = \sqrt{B_r^2 + B_\theta^2}$$

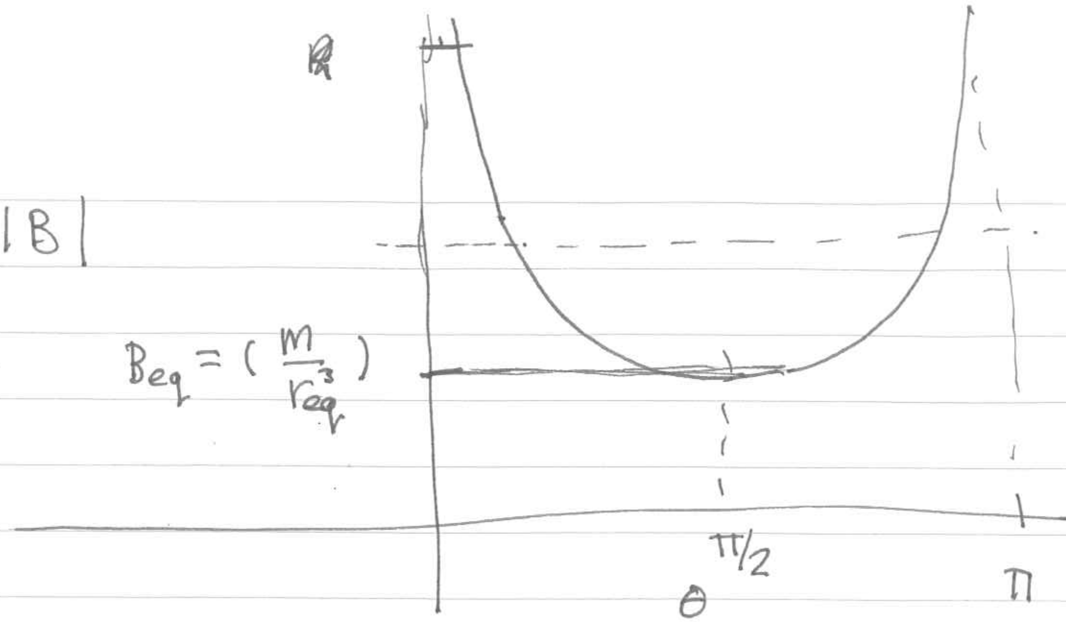
$$= \sqrt{\left(\frac{m}{r^3}\sin\theta\right)^2 + \left(\frac{m}{r^3}2\cos\theta\right)^2}$$

$$= \frac{m}{r^3}\left(\sin^2\theta + 4\cos^2\theta\right)^{1/2}$$

$$r = r_{eq}\sin^2\theta$$

$$|B| = \frac{m}{r_{eq}^3}\,\frac{\left(\sin^2\theta + 4\cos^2\theta\right)^{1/2}}{\sin^6\theta}$$

Plot $|B|$



$$V_{\parallel} = \pm\sqrt{\frac{2}{m}(\varepsilon-\mu B)} = \pm\sqrt{\frac{2\varepsilon}{m}}\left(1-\frac{\mu B}{\varepsilon}\right)^{1/2}$$

~~B_{min} =~~ $$B = B_{eq} \frac{(\sin^2\theta + 4\cos^2\theta)^{1/2}}{\sin^6\theta}$$

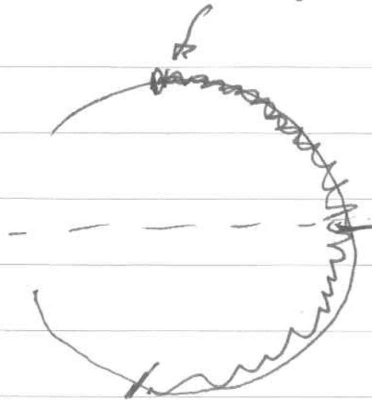
~~consider a particle with perpen~~

define the pith angle δ

$$\boxed{\tan\delta = \frac{v_\perp}{v_\parallel}}$$

δ will vary as a particle moves along a field line

let δ_{eq} = value of pitch angle at the equator



$\alpha = \alpha_{eq}$

$$\tan^2\alpha_{eq} = \left.\frac{V_\perp^2}{V_\parallel^2}\right|_{eq}$$

$$= \frac{V_\perp^2}{\sqrt{V^2 - V_\perp^2}}$$

$$\mu = \frac{\frac{1}{2}mV_\perp^2}{B}$$

at equator $\quad \frac{\mu}{\mathcal{E}} = \frac{V_\perp^2}{B_{eq}\,V^2}$

$$\mathcal{E} = \frac{1}{2}mV^2$$

$$= \frac{\sin^2\alpha_{eq}}{B_{eq}}$$

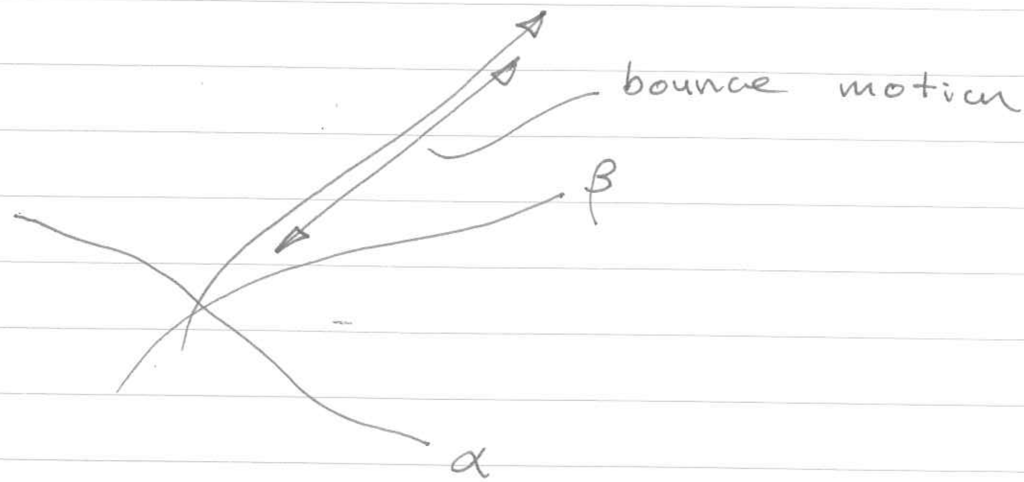
$$\frac{V_\parallel}{V} = \cos\alpha = \left(1 - \sin^2\alpha_{eq}\frac{B}{B_{eq}}\right)^{1/2}$$

turning point $\quad \frac{B}{B_{eq}} = \frac{1}{\sin^2\alpha_{eq}} \geq 1$

$$\boxed{\sin^2\alpha = \sin^2\alpha_{eq}\frac{B}{B_{eq}}}$$

<u>skip</u>

DRIFT in $\alpha-\beta$ coordinates



In absence of drift particle remains at constant α & β.

How do α & β change as a result of drift?

$$\frac{d\alpha}{dt} = \underset{\sim}{V}_d \cdot \nabla\alpha$$

CAN WRITE

$$\underset{\sim}{V}_d = \frac{c}{qB}\left(V_\parallel \nabla\times\left(\underset{\sim}{b}\, m V_\parallel\right)\right)_\perp$$

$$\nabla\times\left(\underset{\sim}{b}\, m V_\parallel\right)_\perp = m V_\parallel \left(\nabla\times\underset{\sim}{b}\right)_\perp - \underset{\sim}{b}\times m\nabla V_\parallel \qquad V_\parallel = \sqrt{\frac{2}{m}\left(\mathcal{E} - \mu B - q\phi\right)}$$

$$\nabla V_\parallel = \frac{1}{m V_\parallel}\left(-\mu\nabla B - q\nabla\phi\right)$$

$\underline{\text{so}}$

$$\boxed{\underset{\sim}{V}_d = \frac{c}{qB}\,\underset{\sim}{b} \times \left(\mu \nabla B + q \nabla \phi + m V_{\parallel}^2 \underset{\sim}{\kappa}\right)}$$

$$\underset{\sim}{V}_d \cdot \nabla \alpha = \frac{c}{qB} V_{\parallel} \nabla \alpha \cdot \nabla \times \left(m V_{\parallel} \underset{\sim}{b}\right)$$

$$= \frac{c}{qB} V_{\parallel} \nabla \cdot \left(m V_{\parallel} b \times \nabla \alpha\right)$$

in $\alpha - \beta$ coordinates

$\cancel{b \times \nabla \alpha} =$

$$\nabla \cdot \underset{\sim}{F} = \frac{1}{J}\left[\frac{\partial}{\partial \alpha}\left(\overset{\text{Jacobian}}{J} \nabla \alpha \cdot F\right) + \frac{\partial}{\partial \beta}\left(J \nabla \beta \cdot F\right) + \frac{\partial}{\partial l} J \underset{\sim}{b} \cdot F\right]$$

$\nabla \alpha \cdot \underset{\sim}{F} = 0 \qquad \underset{\sim}{b} \cdot \underset{\sim}{F} = 0 \qquad \nabla \beta \cdot m V_{\parallel} \underset{\sim}{b} \times \nabla \alpha$

$$= m V_{\parallel} \underset{\sim}{b} \cdot \left(\nabla \alpha \times \nabla \beta\right)$$

$$= m B V_{\parallel}$$

$$J^{-1} = \left(\nabla \alpha \times \nabla \beta\right) \cdot \underset{\sim}{b} = B$$

$$V_d \cdot \nabla \alpha = \frac{c}{q} V_{\parallel} \frac{\partial}{\partial \beta} m V_{\parallel}$$

average drift over one bounce

$$\overline{\frac{d\alpha}{dt}} = \frac{1}{\tau}\oint \frac{dl}{|V_\parallel|}\frac{c}{q} V_\parallel \frac{\partial}{\partial\beta} mV_\parallel = \frac{1}{\tau}\frac{c}{q}\oint dl\, m|V_\parallel|$$

$$= \frac{1}{\tau}\frac{c}{q}\frac{\partial J}{\partial\beta}$$

$$\overline{\frac{d\beta}{dt}} = -\frac{1}{\tau}\frac{c}{q}\frac{\partial J}{\partial\alpha} \qquad \tau = \frac{\partial J}{\partial H}$$

$J(\alpha, \beta, \varepsilon, \mu)$ is like HAMILTONIAN!

$\cancel{\frac{dJ}{dt} = \frac{\partial J}{\partial \varepsilon}}$

$$J(\alpha, \beta, H, \mu) = J_0$$

$\cancel{\tau = \frac{\partial J}{\partial \varepsilon}}$ defines $H(\alpha, \beta, \mu J_0)$

$$J(\alpha, \beta, H(\alpha,\beta,\mu), \mu) = J_0 \text{ constant}$$

$$\boxed{\frac{dJ}{dt} = \overset{0}{\frac{d\varepsilon}{dt}}\frac{\partial J}{\partial \varepsilon} + \frac{d\alpha}{dt}\frac{\partial J}{\partial \alpha} + \frac{d\beta}{dt}\frac{\partial J}{\partial \beta} = 0}$$