Stability Theorems

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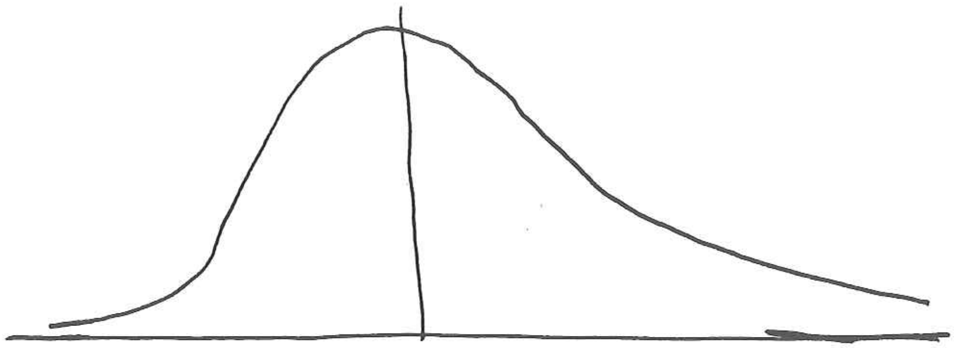


we saw for $kv_e \ll \omega \simeq \omega_p$ this distribution is stable. Can we show this more generally?

is this distribution unstable?

take $\underset{\sim}{k} = k\hat{z}$

$$\epsilon(k,\omega) = 1 + \frac{4\pi q^2}{m k^2} \int \frac{dv_z \; k \frac{\partial \cancel{f_0} \bar{f}(v_z)}{\partial v_z}}{\omega - k v_z}$$

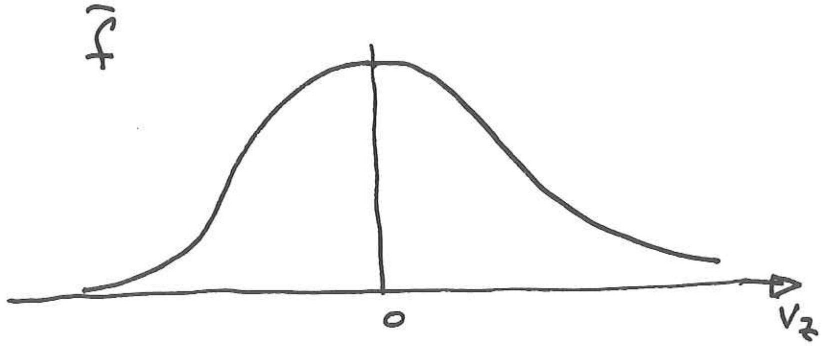
$$\bar{f}(v_z) = \int dv_x dv_y \, f(v_x, v_y, v_z)$$

This is for one specie

recall we assumed ions were infinitely massive and did not respond.

~~suppose~~
Theorem:

IF $\bar{f}(v_z)$ is monotonically decreasing function of $|v_z|$



Then plasma is stable.

Prove by contradiction

* assume that plasma is unstable ~~when~~ and $\bar{f}(v_z)$ is monotonically decreasing function of $|v_z|$.

* obtain a contradiction

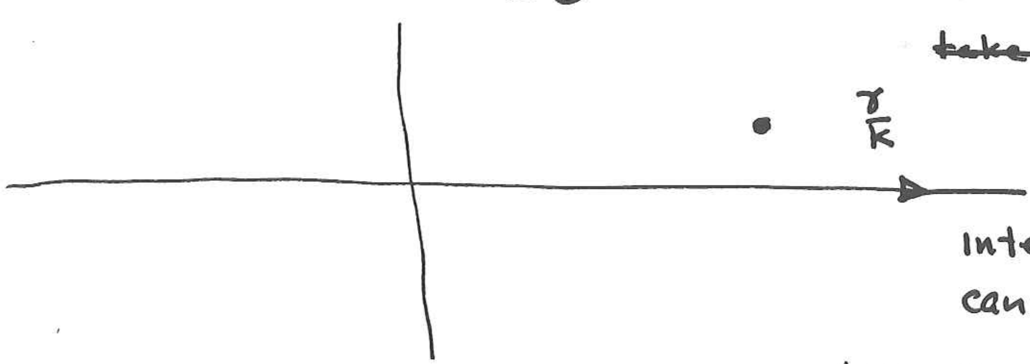
$\cancel{\epsilon(k,\omega) =}$

$$\epsilon(k, \omega = \omega_r + i\gamma) = 1 + \frac{4\pi q^2}{mk^2} \int \frac{dv_z \, k}{(\omega_r + i\gamma - kv_z)} \frac{\partial \bar{f}}{\partial v_z}$$

$\gamma > 0$

$$= 0$$

~~take $k \to 0$~~



Integration carried out along v_z axis

$$\frac{1}{\omega_r + i\gamma - kV_z} = \frac{(\omega_r - kV_z) - i\gamma}{(\omega_r - kV_z)^2 + \gamma^2}$$

$$\mathrm{Im}(\epsilon) = -i\gamma \frac{4\pi q^2}{mk^2} \int \frac{dV_z \quad k}{(\omega_r - kV_z)^2 + \gamma^2} \frac{\partial \bar{f}}{\partial V_z} = 0$$

$$\gamma > 0 \quad \therefore \quad \int \frac{dV_z}{(\omega_r - kV_z)^2 + \gamma^2} \frac{\partial \bar{f}}{\partial V_z} = 0$$

$$\mathrm{Re}(\epsilon) = 1 + \frac{4\pi q^2}{mk^2} \int \frac{dV_z \, k(\omega_r - kV_z)}{(\omega_r - kV_z)^2 + \gamma^2} \frac{\partial \bar{f}}{\partial V_z}$$

$$= 1 - \frac{4\pi q^2}{m} \int \frac{dV_z \quad V_z}{(\omega_r - kV_z)^2 + \gamma^2} \frac{\partial \bar{f}}{\partial V_z}$$

$$V_z \frac{\partial \bar{f}}{\partial V_z} \leq 0 \qquad \frac{1}{(\omega_r - kV_z)^2 + \gamma^2} > 0$$

$1 +$ positive $\# = 0$ contradiction!

$$-\nabla^2\phi = 4\pi q \tilde{n}$$

$$k^2\phi = 4\pi q \tilde{n}$$

$$k^2\left[\hat{\phi} - \frac{4\pi q}{k^2}\hat{n}\right] = 0$$

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2}$$

fluid theory

$$\frac{\partial n_1}{\partial t} + \nabla\cdot n_0 u_1 = 0$$

$$m n_0 \frac{\partial u_1}{\partial t} = - n_0 q \nabla\phi_1 - \nabla P_1$$

First order correction

"adiabatic" compression

$$\frac{P_1}{P_0} = \gamma_s \frac{n_1}{n_0}$$

cold Fluid (neglect pressure)

$$\hat{n} = \frac{k^2 q n_0}{\omega^2 m}\hat{\phi}$$

$$-i\omega\hat{u} = -i\underset{\sim}{k}\frac{q}{m}\hat{\phi}$$

$$-i\omega\hat{n} + i\underset{\sim}{k}\cdot n_0\hat{\underset{\sim}{u}} = 0$$

valid if $\frac{\omega}{kV_t} \gg 1$

Large arg of Z

~~Valid if $\frac{\omega}{kV_t}$~~

Opposite limit $\frac{kV_t}{\omega} \ll 1$

* ~~Energety~~ Energy transported at thermal velocity

* infinite thermal conductivity

perturbed pressure

$$P_1 = (n_1 T_0 + n_0 T_1)$$

infinite thermal conductivity $T_1 = 0$

$$P_1 = n_1 T_0$$

momentum balance

$$\underset{\substack{\uparrow \\ \text{small}}}{\frac{\omega}{k}} \tilde{u} n_0 = -ik(q n_0 \varphi_1 + n_1 T_0)$$

~~$n_1 = -q\phi_1$~~ $\boxed{n_1 = -\frac{q\phi_1}{T_0} n_0}$

also called "adiabatic"

$$n_{\text{TOTAL}} = n_0 \exp\left(-\frac{q\phi_1}{T_0}\right)$$

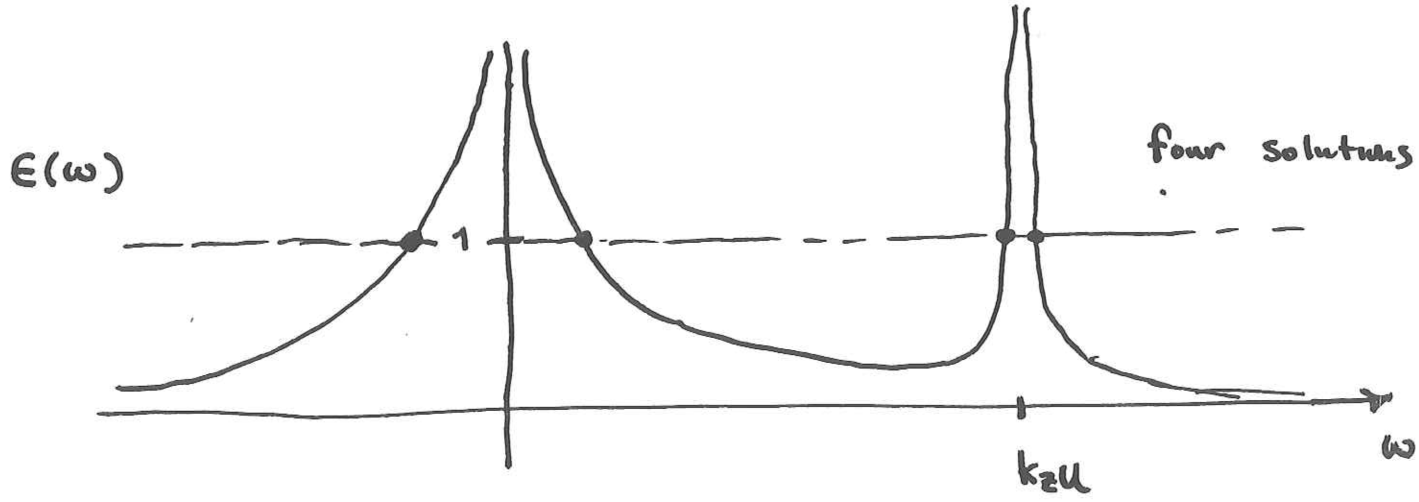
$$\epsilon = 1 + \frac{4\pi q^2 n_0}{k^2 T} = 1 + \frac{1}{k^2 \lambda_D^2}$$

For our problem

$$\epsilon(\omega, \underset{\sim}{k}) = 1 - \overbrace{\frac{\omega_{pe}^2}{\omega^2}}^{\sim \text{electrons}} - \overbrace{\frac{\omega_{pi}^2}{(\underbrace{\omega - k_z u}_{\text{doppler shifted frequency}})^2}}^{\sim \text{ions}}$$

$\epsilon(\omega, k_z) = 0$ how many solutions ?

4th order polynomial with real coeficients



all ω's real no instability

Instabilities caused by non monotonic distribution functions

$$\epsilon(k,\omega) = 1 + \sum_q \frac{4\pi q^2}{mk^2} \int d^3v \frac{\underset{\sim}{k} \cdot \frac{\partial f_q}{\partial \underset{\sim}{v}}}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})}$$

dielectric constant

Lets assume we have two species (electrons and ions) and they are both cold, but the ions are streaming with a velocity $\underset{\sim}{u} = u\hat{z}$ with respect to the electrons

$$f_{eo} = n_e\, \delta(\underset{\sim}{v})$$

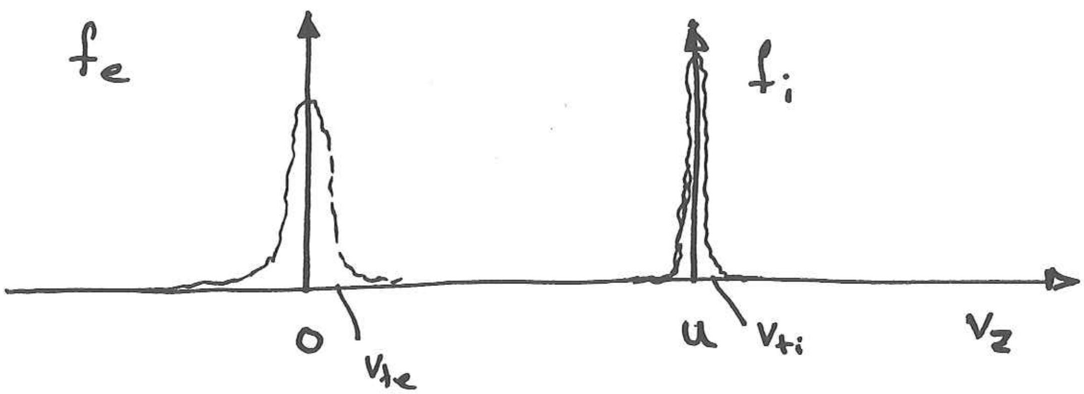
assume $|Z|$ charge $= 1$ ~~atomic~~

$$f_{io} = n_e\, \delta(\underset{\sim}{v} - \underset{\sim}{u})$$

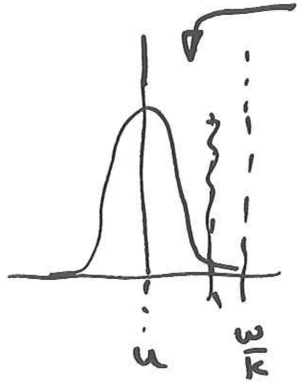
$q_i n_i + q_e n_e = 0$

$n_i = n_e$

distribution functions



actually V_{te} and V_{ti} will be finite. what is the condition for validity of the cold approximation?



difference between particle velocity and phase velocity $\gg V_t$

$$\left| \frac{\omega}{k} - u \right| \gg V_{ti}$$

$$\left| \frac{\omega}{k} \right| \gg V_{te}$$

$$\frac{4\pi q^2}{m_q k^2} \int d^3v \; \frac{\underset{\sim}{k} \cdot}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})} \frac{\partial}{\partial \underset{\sim}{v}} f_{oq}$$

do by parts

$$- \frac{4\pi q^2}{m_q k^2} \int d^3v \; f_{oq} \quad \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{v}} \frac{1}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})}$$

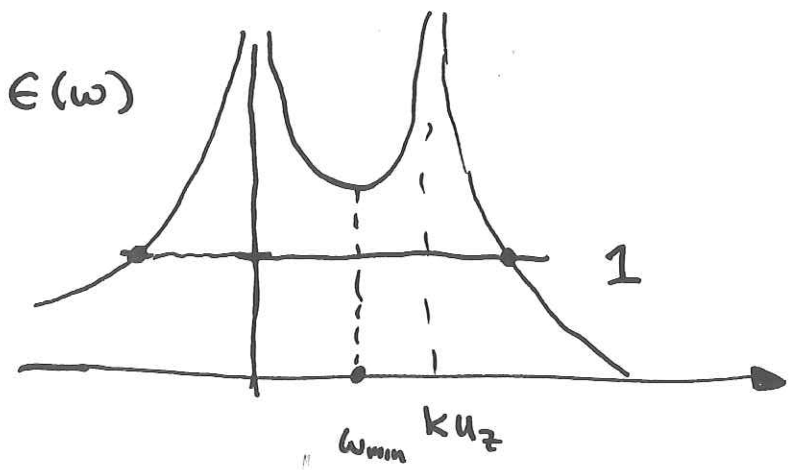
$$= - \frac{4\pi q^2}{m_q k^2} \int d^3v \; f_{oq} \quad \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{v}} \underset{\sim}{k} \cdot \underset{\sim}{v} \quad \frac{1}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})^2}$$

$$= - \frac{4\pi q^2}{m_q} \int d^3v \; \frac{f_{oq}}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})^2}$$

$$f_{oq} = n_{oq} \; \delta(\underset{\sim}{v} - \underset{\sim}{u}_q)$$

$$= - \frac{4\pi q^2 n_{oq}}{m_q (\omega - \underset{\sim}{k} \cdot \underset{\sim}{u}_q)^2} = \frac{- \omega_{pq}^2}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{u}_q)^2}$$

what happens when $k_z u$ gets smaller



only two real solutions.

other two solutions are complex (complex conjugates)

$\therefore$ unstable.

Buneman Instability

$\boxed{\text{mean flow of particles} \quad \text{Source of free energy}}$

unstable if $\epsilon_{min} > 1$

$$\frac{d\epsilon(\omega)}{d\omega} = 0 \quad \text{determines } \omega_{min}$$

$$\epsilon(\omega_{min}) > 1 \quad \text{gives critical } ku_z$$

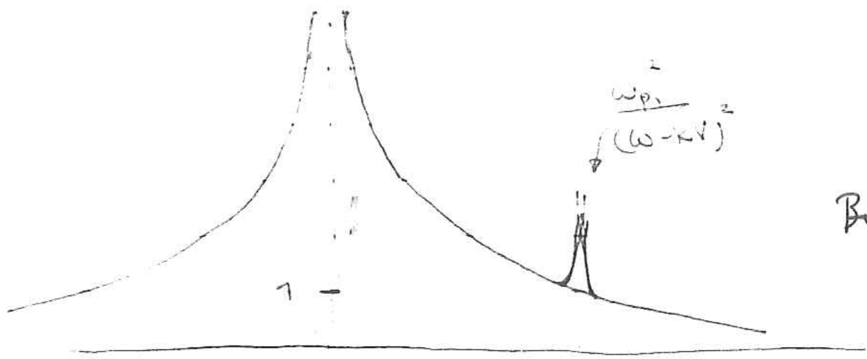
$$|ku_z|_{crit} = \omega_{pe}\left(1 + \left(\frac{m_e}{m_i}\right)^{1/3}\right)^{3/2} = \left(\omega_{pe}^{\frac{2}{3}} + \omega_{pi}^{\frac{2}{3}}\right)^{3/2}$$

IF $|ku_z| < \omega_{pe}\left(1 + \left(\frac{m_e}{m_i}\right)^{1/3}\right)^{3/2}$ then unstable

Thus if $kv \sim \omega_{pe}$ [illegible] unstable

How to find growth rate

actually $\omega_{pe}^2 \gg \omega_{pi}^2$



$\omega_m \simeq k_z v \sim \omega_{pe}$

~~B~~

$\omega \simeq \omega_{pe}$

$\simeq \omega_m$

unstable

Look for modes with $\omega \simeq \omega_{pe}$

$k_z u \sim \omega_{pe}$ **$k_z u \sim \omega_{pe}$**

$\omega = \omega_r$

$$1 = \frac{\omega_{pe}^2}{\omega^2} + \frac{\omega_{pi}^2}{(\omega - k_z u)^2}$$

$$(\omega - k_z u)^2 (\omega^2 - \omega_{pe}^2) = \omega^2$$

$\omega = \omega_{pe} + \delta\omega$

$$1 = 1 - \frac{2\delta\omega}{\omega_{pe}} + \frac{\omega_{pi}^2}{(\delta\omega + \underbrace{\omega_{pe} - k_z v}_{\text{small}})^2}$$

Lets check on valitity of cold model

$$\omega \simeq \qquad k \simeq \frac{\omega_{pe}}{u}$$

~~for electrons~~

$$\frac{\omega}{k} \simeq \frac{\omega_{pe} + \delta\omega}{(\omega_{pe}/u)} \simeq u\left(1 + \frac{\delta\omega}{\omega_{pe}}\right)$$

electrons wave has he

$$\left|\frac{\omega}{k}\right| > V_{te} \qquad \text{thus} \qquad u > V_{te}$$

BIG velocity

required for cold model

$$\frac{\delta\omega}{\omega_{pe}} \sim \left(\frac{m_e}{m_i}\right)^{1/3}$$

Ions

$$\left|\frac{\omega}{k} - u\right| = u\,\underbrace{\left|\frac{\delta\omega}{\omega_{pe}}\right|}_{\text{small}} \gg V_{ti}$$

$$V_{ti} < \left(\frac{m_e}{m_i}\right)^{1/3} u$$

for ions to be considered cold.

$$\delta\omega(\delta\omega + \omega_{pe} - k_z u)^2 = \frac{1}{2}\omega_{pi}^2\omega_{pe}$$

cubic maximum growth will occur

if $\omega_{pe} - k_z u = 0$

$$\delta\omega^3 = \frac{1}{2}\omega_{pi}^2\omega_{pe}$$

$$\delta\omega = e^{i\frac{2\pi}{3}}\left(\frac{\omega_{pi}^2\omega_{pe}}{2}\right)^{1/3}$$

$$\delta\omega_i = \frac{\sqrt{3}}{2^{3/2}}\,\omega_{pi}^{2/3}\omega_{pe}^{1/3}$$

$$\omega_{pi} \ll \delta\omega_i \ll \omega_{pe}$$

$$\delta\omega_r = -\frac{1}{2^{3/2}}\,\omega_{pi}^{2/3}\omega_{pe}^{1/3}$$

$$\frac{\delta\omega}{\omega_{pe}} \sim \frac{\omega_{pi}^{2/3}}{\omega_{pe}^{2/3}} \sim \left(\frac{m_e}{m_i}\right)^{1/3}$$

$$\omega = \omega_{pe} - \frac{1}{2^{3/2}}\,\omega_{pi}^{2/3}\omega_{pe}^{1/3} + i\frac{\sqrt{3}}{2^{3/2}}\,\omega_{pi}^{2/3}\omega_{pe}^{1/3}$$

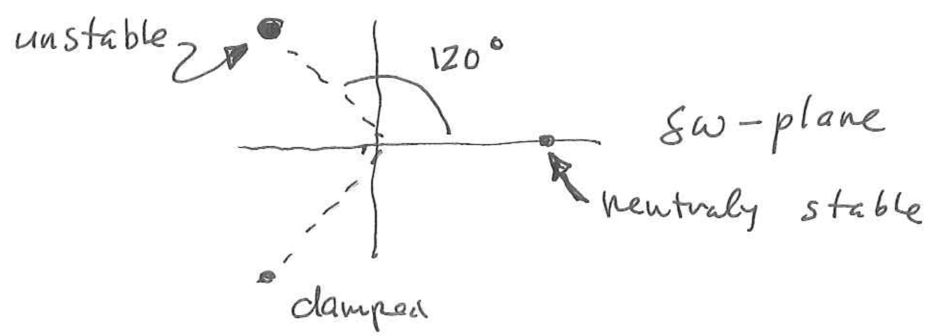
$k = \dfrac{\omega_{pe}}{V}$ $\quad V =$ speed of ion beam

$$\delta\omega\left(\delta\omega + \omega_{pe} - ku\right)^2 = \frac{1}{2}\omega_{pi}^2\omega_{pe} \equiv \gamma_0^3$$

suppose $ku - \omega_{pe} >$ $\qquad \gamma_0 = \left(\frac{1}{2}\omega_{pi}^2\omega_{pe}\right)^{1/3}$

if $\omega_{pe} - ku = 0$ $\qquad \delta\omega^3 = \gamma_0^3$

$$\delta\omega = \gamma_0\left[1, e^{i2\pi/3}, e^{-i2\pi/3}\right]$$



what happens if $|ku - \omega_{pe}| >> \gamma_0$

$$\delta\omega\left(\delta\omega - (ku - \omega_{pe})\right)^2 = \gamma_0^3$$

one possibility $\delta\omega$ is small

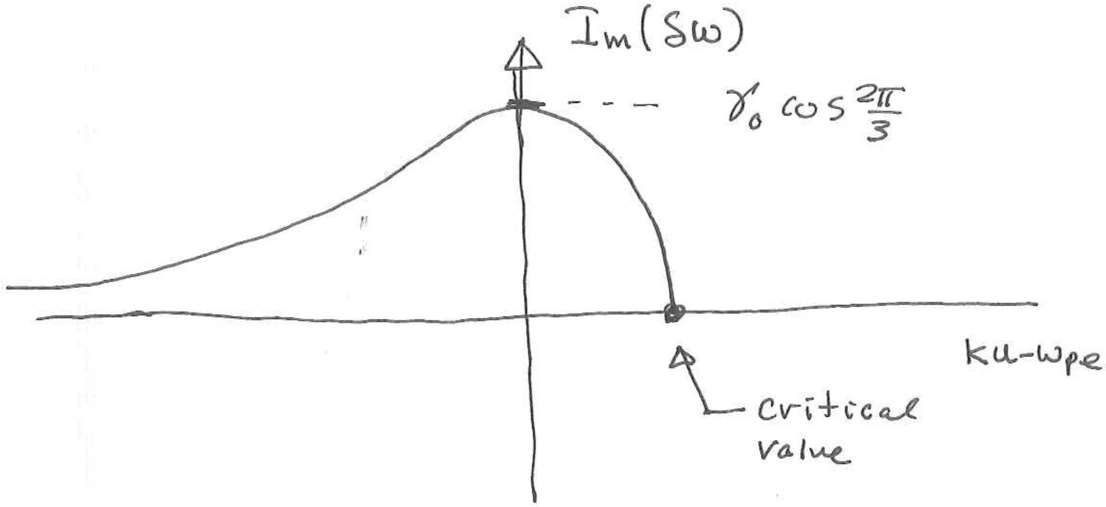
$$\delta\omega = \frac{\gamma_0^3}{(ku - \omega_{pe})^2}$$ (neutraly stable)

other possibility $\delta\omega - (ku - \omega_{pe})$ is small

$$(\delta\omega - (ku - \omega_{pe}))^2 = \frac{\gamma_0^3}{\delta\omega} \simeq \frac{\gamma_0^3}{(ku - \omega_{pe})}$$

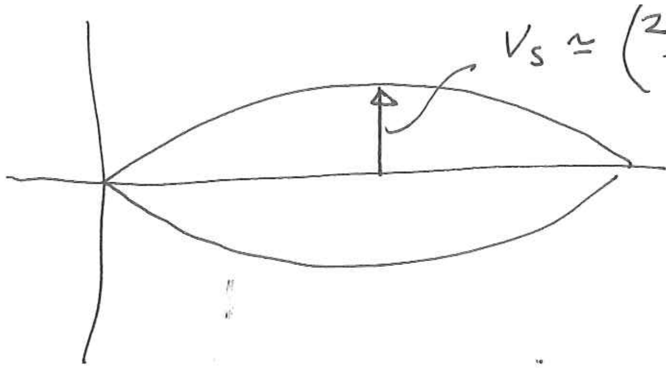
$$\delta\omega = ku - \omega_{pe} \pm \sqrt{\frac{\gamma_0^3}{(ku-\omega_{pe})}}$$

unstable if $ku < \omega_{pe}$



Saturation by trapping

$$\frac{1}{2} m V^2 + q\phi_0 \cos(kz) = H$$



Ions saturate if $\left(\frac{2q\phi_0}{m_i}\right)^{1/2} > \left(\frac{m_e}{m_i}\right)^{1/3} u$

electrons saturate if $\left(\frac{2q\phi_0}{m_e}\right)^{1/2} > u$

Ions $\quad e q\phi_0 > u^2 m_i \left(\frac{m_e}{m_i}\right)^{2/3}$

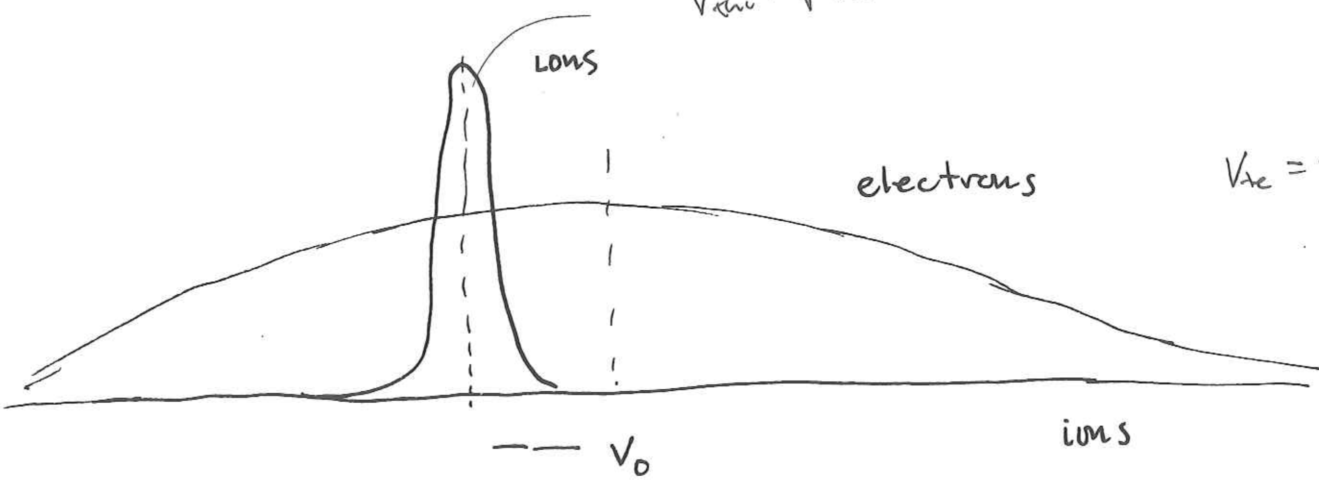
$m_i^{1/3} m_e^{2/3}$

Electrons

$$e\phi_0 > u^2 m_e$$

Ion Acoustic Waves

$V_{thi} = \sqrt{\frac{T_i}{m_i}}$



$V_{te} = \sqrt{\frac{T_e}{m_e}}$

consider a plasma that is carrying a current.

~~Ions~~ &

Ions $\quad \bar{f}_{io} = \frac{n_0}{\pi^{1/2} V_{thi}} \exp\left(-\frac{v_z^2}{V_{thi}^2}\right) \qquad V_{thi} = \sqrt{\frac{2T_i}{m_i}}$

Electrons $\quad \bar{f}_{eo} = \frac{n_0}{\pi^{1/2} V_{the}} \exp\left(-\frac{(v_z - V_0)^2}{V_{the}^2}\right) \qquad V_{the} = \sqrt{\frac{2T_e}{m_e}}$

u = relative speed

$V_{the} \gg V_{thi}$

What is the Dielectric Constant?

$$\epsilon = 1 + \sum_q \frac{4\pi q^2}{m_q k^2} \int \frac{d^3v \; k_z \frac{\partial}{\partial v_z} \bar{f}_q(v_z)}{\omega - k_z v_z}$$

$$= 1 + \overbrace{\frac{4\pi q_i^2 n_i}{T_i k^2}\left[1 + \frac{\omega}{k_z v_{ti}} Z\left(\frac{\omega}{k_z v_{ti}}\right)\right]}^{\text{ions}}$$

$$+ \overbrace{\frac{4\pi e^2 n_e}{T_e k^2}\left[1 + \frac{\omega - k_z u}{k_z v_{te}} Z\left(\frac{\omega - k_z u}{k_z v_{te}}\right)\right]}^{\text{electrons}}$$

doppler shifted frequency

Lets assume

$$V_{thi} \ll \frac{\omega}{k_{z}}, u \ll V_{the}$$

FOR ions

$$\frac{4\pi q_i^2 n_i}{T_i k^2}\left[1 + \frac{\omega}{k_z V_{thi}} Z\left(\frac{\omega}{k_z V_{thi}}\right)\right] \cong -\frac{\omega_{pi}^2}{\omega^2}\left[\right.$$

FOR electrons $Z \simeq i\pi^{1/2}$

$$\frac{4\pi e^2 n_e}{T_e k_z^2}\left[1 + i\pi^{1/2}\frac{\omega - k_z u}{k_z V_{he}}\right]$$

4.

$$= \frac{1}{k_z^2 \lambda_d^2}$$

"adiabatic response" $\tilde{n}_e = n_0\left(-\frac{q_e\tilde{\phi}}{T_e}\right)$

dielectric constant

$$1 - \frac{\omega_{pi}^2}{\omega^2} + \frac{1}{k_z^2\lambda_{de}^2}\left[1 + \underbrace{i\pi^{1/2}\frac{\omega - k_z u}{k_z V_{the}}}_{\text{small}}\right] = \epsilon = 0$$

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$$\epsilon = \epsilon_R(\omega) + i\,\epsilon_I(\omega)$$

$\cancel{\omega^2 =}$ $\epsilon_I \ll \epsilon_R$

$$\omega = \omega_r + i\,\delta\omega$$

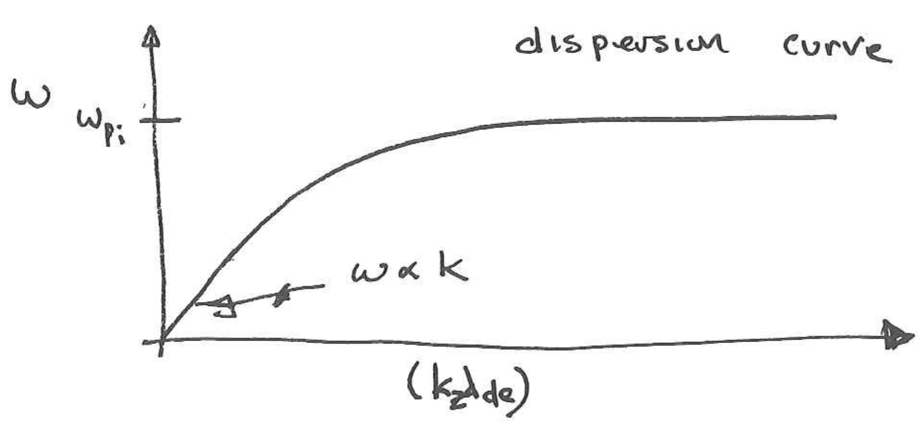
$$1 - \frac{\omega_{pi}^2}{\omega_r^2} + \frac{1}{k_z^2\lambda_{de}^2} = 0 \qquad \epsilon(\omega_r)$$

$$\frac{\partial \epsilon_R}{\partial \omega_r}\delta\omega + i\,\epsilon_I'' = 0$$

$$\delta\omega = -\frac{i\pi^{1/2}}{k_z^2\lambda_{de}^2}\,\frac{\omega_r - k_z u}{k_z V_{the}} \Big/ \left(\frac{2\omega_{pi}^2}{\omega_r^3}\right)$$

unstable if $(\omega_r - k_z u) < 0$

$$\omega_r^2 = \frac{k_z^2\lambda_{de}^2}{1 + k_z^2\lambda_{de}^2}\,\omega_{pi}^2$$



$\frac{\omega}{k} > V_{ti}$

requires $T_e \gg T_i$

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proton plasm

$q_i = e \quad n_{0i} = n_e$

$$\frac{\omega^2}{k_z^2} = \lambda_{de}^2 \omega_{pi}^2 = \frac{T_e}{4\pi e^2 n_{oe}} \frac{4\pi q_i^2 n_{oi}}{m_i} = \frac{T_e}{m_i}$$

$$\frac{\omega}{k_z} = \sqrt{\frac{T_e}{m_i}}$$ ion acoustic speed

$$\frac{\omega}{k_z} = \sqrt{\frac{\gamma T}{m}}$$ for a sound wave

temp

mass

ratio of specific heats

we have

$$\gamma = \frac{n+2}{n} = 1 \qquad \left(\frac{P}{P_0}\right) = \left(\frac{n}{n_0}\right)^{\gamma} = \frac{n}{n_0}$$

$P = nT$

$T =$ const isothermal !

electron temperature

ion mass

$k^2\lambda_d^2 \ll 1$ means quasineutral $n_e = n_i = n$

$$m_i n_0 \frac{\partial u_i}{\partial t} = + e n_0 E$$

$$\cancel{m_e n_0 \frac{\partial u_e}{\partial t}}^{\,0} = \underbrace{- e n E - \frac{\partial}{\partial z} P_e}_{\text{force balance}} \qquad P_e = \overset{\text{const}}{T_e}\, n$$

neglect

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z} n u = 0$$

add together momentum equ

Linearize

$$n = n_0 + n_1$$

$$m_i n_0 \frac{\partial u_i}{\partial t} = - \cancel{n} T_e \frac{\partial}{\partial z} n_1$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z} n u \simeq \frac{\partial n_1}{\partial t} + n_0 \frac{\partial u_i}{\partial z}$$

~~$m_i n_0 \partial^2$~~ combine

$$\frac{\partial^2 n_1}{\partial t^2} = \frac{T_e}{m_i} \frac{\partial^2}{\partial z^2} n_1 \qquad \frac{\omega^2}{k_z^2} = \frac{T_e}{m_i}$$

Why should electrons be isothermal

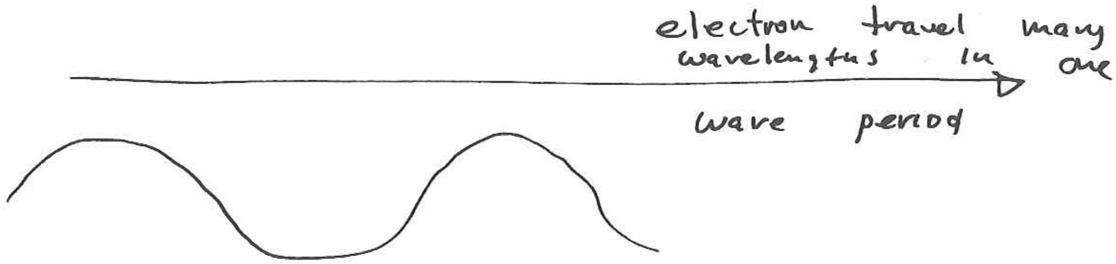
$$\frac{\omega}{k_z} \ll V_{the}$$

How far does an electron go in one wave period L_e

$$L_e = 2\pi V_{the} / \omega$$

$$\frac{L_e}{\lambda} = \frac{L_e k_z}{2\pi} = \frac{k_z V_{the}}{\omega} \gg 1$$

compare with a wavelength



equivalent thermal conductivity $= \infty$

isothermal.

* Ion acoustic Waves driven unstable be electron current

$$f_i = \frac{n_{i0}}{\left(\frac{2\pi T_i}{m_i}\right)^{1/2}} \exp\left(-\tfrac{1}{2} m_i v^2 / T_i\right)$$

$$f_e = \frac{n_{e0}}{\left(\frac{2\pi T_e}{m_e}\right)^{1/2}} \exp\left(-\tfrac{1}{2} m (v-u)^2 / T_e\right)$$

$$\epsilon(\omega, k) = 1 + \sum_{e,i} \frac{4\pi n_{e,i} q_{e,i}^2}{T_{e,i} k^2} \left[1 + \zeta_{e,i} Z(\zeta_{e,i}) \right]$$

$$\zeta_e = \frac{\omega - ku}{k v_{te}} \qquad v_{te} = \sqrt{2T_e/m_e}$$

$$\zeta_i = \frac{\omega}{k v_{ti}} \qquad v_{ti} = \sqrt{2T_i/m_i}$$

$$\mathrm{Im}(Z) = \pi^{1/2} e^{-\xi^2}$$

$\mathrm{Im}(\omega, k)$

$$\pi^{1/2}\left[\frac{4\pi n_i q_i^2}{T_i k^2}\,\frac{\omega}{kV_{ti}}\,e^{-\frac{\omega^2}{k^2V_{ti}^2}} + \frac{4\pi n_e q_e^2}{T_e k^2}\,\frac{(\omega - ku)}{kV_{te}}\,e^{-\frac{(\omega-ku)^2}{k^2V_{te}^2}}\right.$$

~~dam~~ Landau damping on ions

Landau growth from electrons

put $\quad \dfrac{\omega}{k} = c_s = \sqrt{\dfrac{T_e}{m_i}} \quad$ (assumes $T_e \gg T_i$)

unstable if $\quad \mathrm{Im}(\epsilon(\omega, k)) < 0 \quad$ negative dissipation

Requires (assume $q_e^2 = q_i^2$ $n_e = n_i$ hydrogen)

small

$$u > C_s + \frac{V_{the}}{V_{thi}} \frac{T_e}{T_i} \exp\left(- \frac{C_s^2}{V_{ti}^2}\right) \exp\left(+ \frac{(C_s - u)^2}{V_{te}^2}\right)$$

Big #

assume $C_s, u << V_{the}$ 1

must be small

$C_s >> V_{ti}$

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* waves grow

momentum exchange electrons → ions

* Anomalous Resistivity

There is collisional momentum exchange between ions & electrons