CLASS NOTES: LECTURE I

REVIEW OF LANDAU DAMPING (INITIAL VALUE SOLUTION OF THE VLASOV EQUATION)

$$\boxed{\text{ASSUME } \underset{\sim}{B} = 0}$$

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \nabla f + \frac{q}{m} \underset{\sim}{E} \cdot \frac{\partial}{\partial \underset{\sim}{v}} f = 0 \qquad \text{VLASOV EQ.}$$

$$\nabla \cdot \underset{\sim}{E} = 4\pi q \int d^3v\, f - 4\pi q n_0 \qquad \text{POISSON EQ.}$$

LINEARIZE ABOVE

$$\underset{\sim}{E} = -\nabla \phi_1 \qquad \phi_1 = \phi_1(t) \exp(i \underset{\sim}{k} \cdot \underset{\sim}{x})$$

$$f = f_0 + f_1 \qquad f_0 = f_0(\underset{\sim}{v})$$

$$f_1 = f_1(t) \exp(i \underset{\sim}{k} \cdot \underset{\sim}{x})$$

$$\frac{\partial f_1(t)}{\partial t} + i\,\underset{\sim}{k}\cdot\underset{\sim}{v}\,f_1 = \frac{q}{m}\,\phi_1(t)\; i\,\underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_0$$

$$k^2\phi_1(t) = 4\pi q \int d^3v\, f_1$$

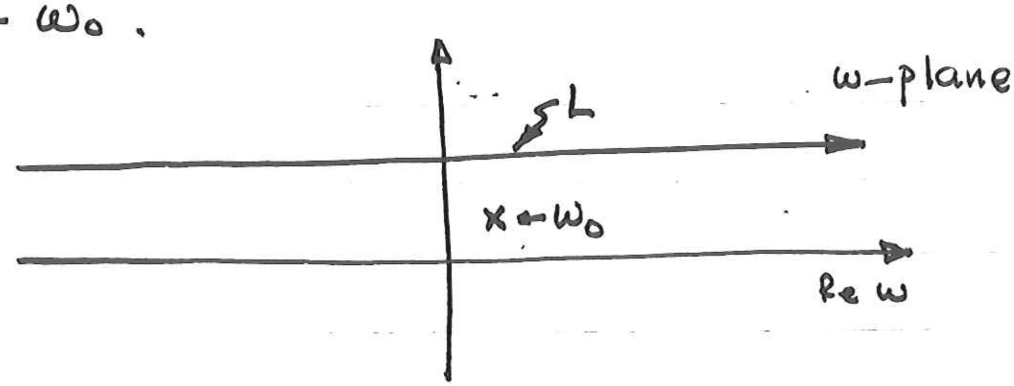
Introduce Laplace Transforms

$$\bar{g}(\omega) = \int_0^\infty dt\; g(t)\exp(i\omega t) \qquad (a)$$

$$g(t) = \int_L \frac{d\omega}{2\pi}\,\bar{g}(\omega)\exp(-i\omega t) \qquad (b)$$

INTEGRAL (a) IS DEFINED ONLY FOR PARTICULAR VALUES OF ω SUCH THAT AS $t \to \infty$ $g(t)\exp(i\omega t) \to 0$ SUFFICIENTLY FAST. FOR

EXAMPLE, IF $g(t) = g_0 \exp(i\omega_0 t)$, WE MUST HAVE $Im(\omega) > Im(\omega_0)$ INORDER THAT THE INTEGRAL BE WELL DEFINED. IN GENERAL THIS WILL BE THE CASE IF $Im(\omega)$ IS SUFFICIENTLY LARGE. INTEGRAL (b) IS CARRIED OUT ALONG THE CONTOUR L WHICH PASSES FROM $-\infty$ TO $+\infty$ THROUGH REGIONS OF the ω-plane WHERE $\bar{g}(\omega)$ IS DEFINED. FOR THE CASE OF $g(t) = g_0 \exp(i\omega_0 t)$ THE CONTOUR MUST PASS ABOVE THE POINT $\omega = -\omega_0$.



LAPLACE TRANSFORM EQUATIONS

$$\int_0^\infty dt \exp(i\omega t) \frac{\partial f_1}{\partial t} = - f_1(t=0) - i\omega \int_0^\infty dt \exp(i\omega t) f_1(t)$$

$$-i(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}) \bar{f}_1(\omega) = \frac{q}{m} \bar{\phi}_1(\omega)\, i\, \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{v}} f_0 + f_1(t=0)$$

$$k^2 \bar{\phi}_1(\omega) = 4\pi q \int d^3v\, \bar{f}_1(\omega)$$

Tom → up to here (Ed.)

SOLVING FOR $\bar{f}_1(\omega)$ AND SUBSTITUTING IN

POISSONS EQ. GIVES

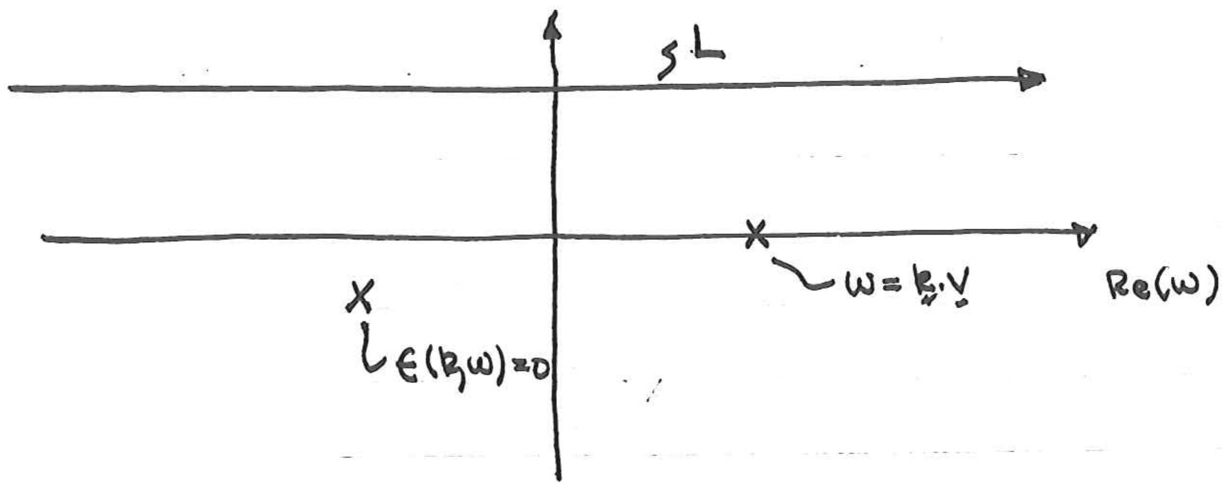
$$k^2 \epsilon(k,\omega) \bar{\phi}_1(\omega) = 4\pi q\, i \int d^3v \frac{f_1(t=0)}{\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}}$$

WHERE:

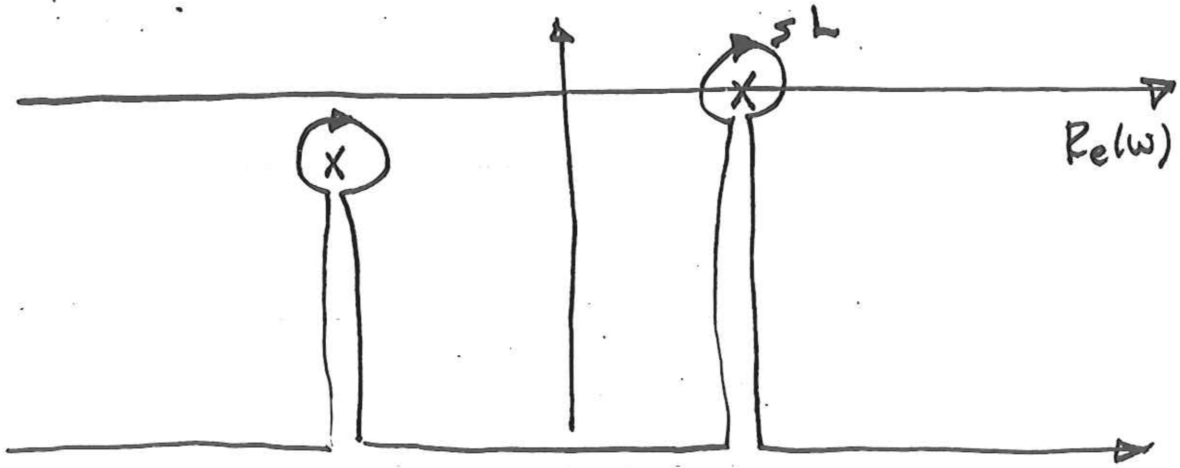
$$\epsilon(k,\omega) = 1 + \frac{4\pi q^2}{m k^2} \int d^3v \frac{\underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{v}} f_0}{\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}}$$

INVERTING $\bar{\phi}(\omega)$

$$\phi_1(t) = \int_L \frac{d\omega}{2\pi}\, 4\pi q i \int \frac{d^3v\ f(0)\ \exp(-i\omega t)}{(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v})\ k^2 \epsilon(\underset{\sim}{k},\omega)}$$



TO EVALUATE INTEGRAL DEFORM CONTOUR AND ANALYTICALLY CONTINUE INTEGRAND INTO LOWER HALF ω-plane.



ON LOWER PORTIONS OF CONTOUR $\exp(-i\omega t) \to 0$ $(t>0)$. THEREFORE $\phi_1(t) = -2\pi i$ TIMES RESIDUES AT SINGULARITIES.

$$\epsilon(\underset{\sim}{k}, \omega_0) = 0$$

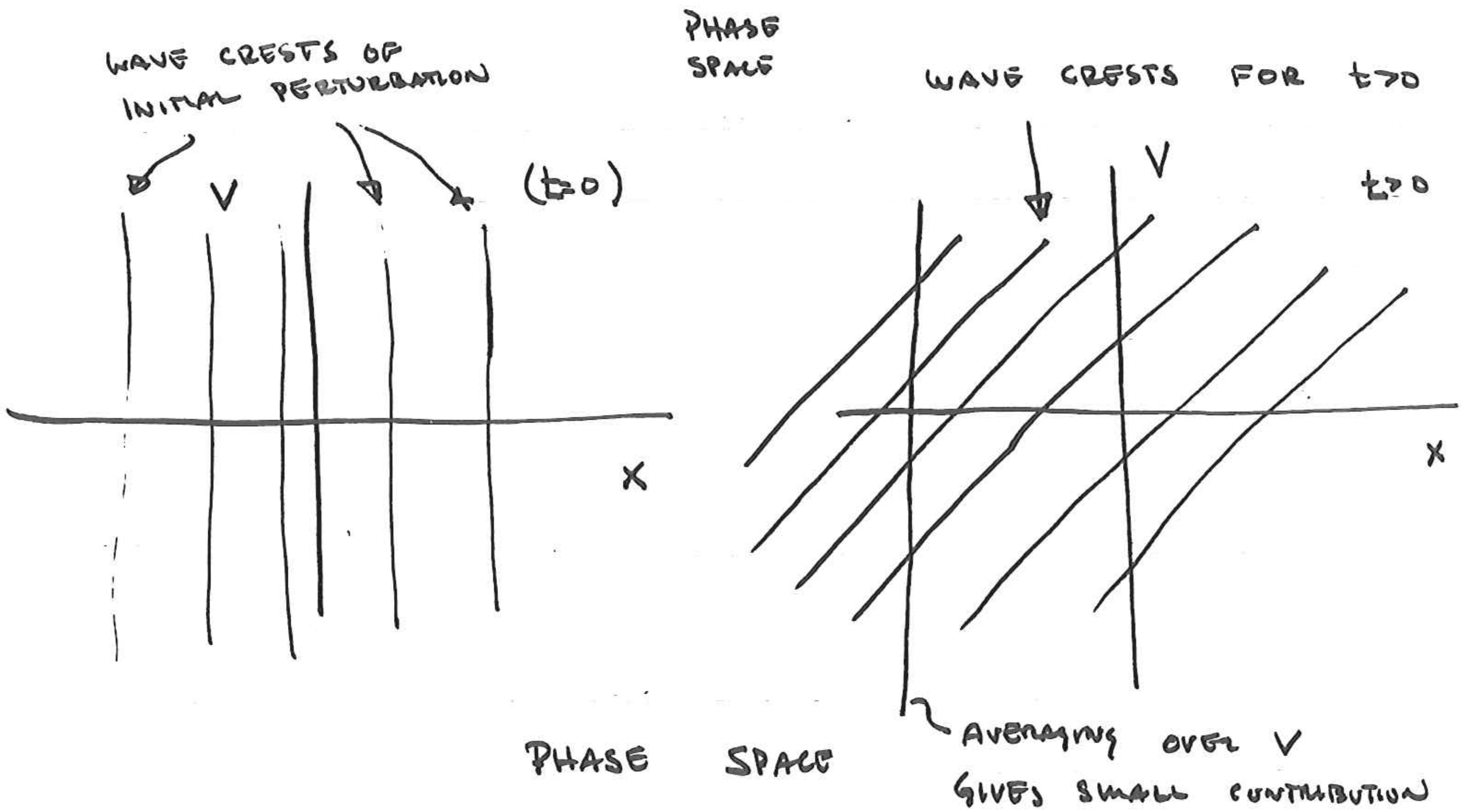
$$\phi_1(t) = -i \Big[\exp(-i\omega_0 t)\, 4\pi q_i \int d^3v \frac{f(0)}{(\omega_0 - \underset{\sim}{k}\cdot\underset{\sim}{V})\, \frac{\partial\epsilon}{\partial\omega_0}}$$

$$+ 4\pi q_i \int d^3v \frac{f(0)\exp(-i\underset{\sim}{k}\cdot\underset{\sim}{V}t)}{\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{V})} \Big]$$

THE FIRST TERM OSCILLATES AT FREQUENCY ω_0 AND REPRESENTS THE EXCITATION OF A NATURAL MODE OF THE SYSTEM BY THE INITIAL CONDITIONS. THE SECOND TERM HAS A COMPLICATED TIME DEPENDENCE

AND REPRESENTS THE WAY IN WHICH THE INITIAL DISBURANCE EVOLVES DUE TO THE FREE STREAMING OF THE PARTICLES.

As t GETS LARGE, THE RAPID OSCILLATIONS IN V OF THE FACTOR $\exp(-i \underset{\sim}{k} \cdot \underset{\sim}{v} t)$ CAUSE THIS CONTRIBUTION TO BECOME SMALL.



PHASE SPACE

NATURAL MODE OF THE SYSTEM OCCURS FOR $\epsilon(\underset{\sim}{k}, \omega_0) = 0$

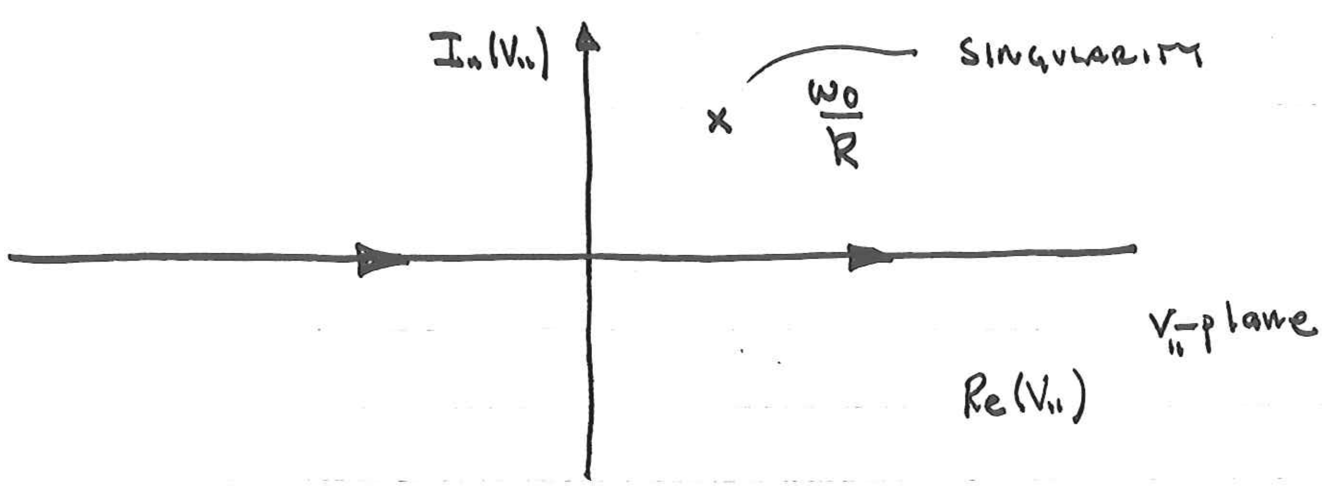
$$\epsilon(k, \omega_0) = 1 + \frac{4\pi q^2}{m k^2} \int d^3v \, \frac{\underset{\sim}{k} \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}}}{\omega_0 - \underset{\sim}{k} \cdot \underset{\sim}{v}}$$

do $v_\perp$ integral

NOTE THAT IN THE INTEGRAL OVER VELOCITY THERE IS A SINGULARITY AT $\omega_0 = \underset{\sim}{k} \cdot \underset{\sim}{v} = k v_\parallel$

($v_\parallel$ IS THE COMPONENT OF $\underset{\sim}{v}$ PARALLEL TO $\underset{\sim}{k}$)

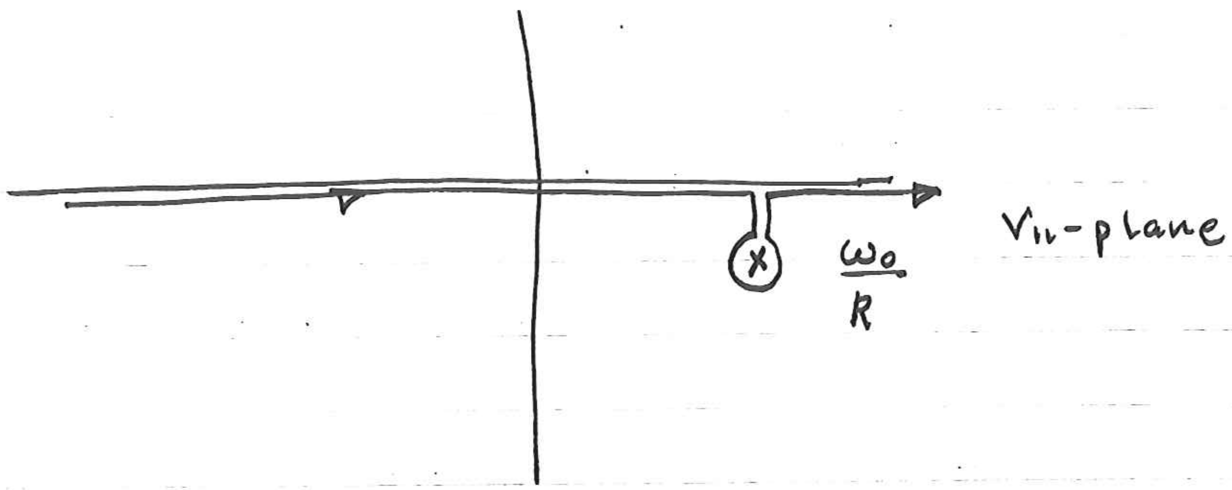
ASSUME $k > 0$



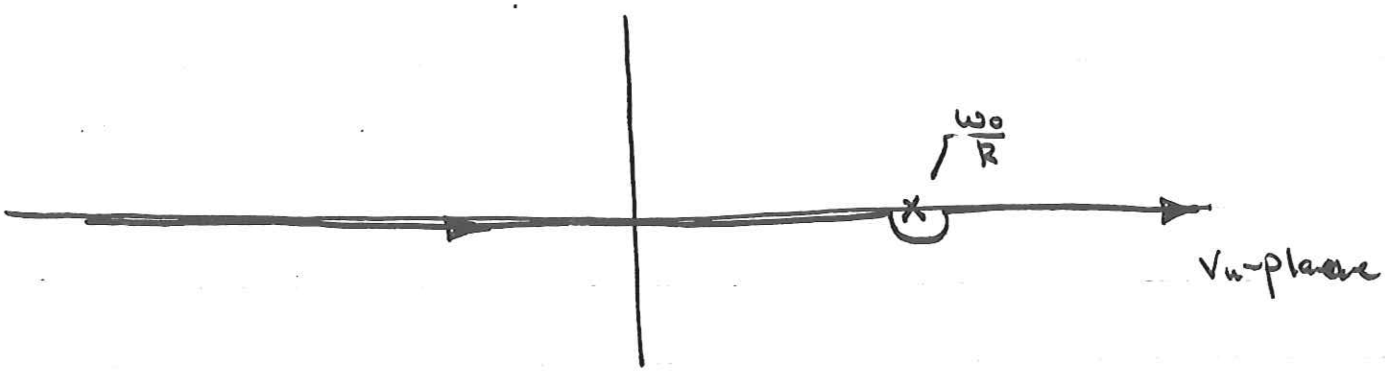
IF $Im(\omega_0) > 0$ THEN THE SINGULARITY LIES ABOVE THE INTEGRATION CONTOUR (IN V_{11}) AND THE INTEGRAL IS WELL DEFINED.

IF WE ARE REQUIRED TO DETERMINE $\epsilon(k, \omega_0)$ FOR $Im(\omega_0) < 0$ WE MUST BE SURE THAT WHAT WE DETERMINE IS THE ANALYTIC CONTINUATION OF $\epsilon(k, \omega_0)$ FOR $\omega_0 > 0$. (REMEMBER OUR METHOD OF EVALUATING $\phi_1(t)$ RELIED ANALYTICALLY CONTINUING THE INTEGRAND INTO THE LOWER HALF ω - PLANE.) THE PROPER ANALYTIC CONTINUATION OF $\epsilon(k, \omega_0)$ FOR $Im(\omega_0) < 0$

IS OBTAINED BY DEFORMING THE INTEGRATION PATH IN V_{11} SO THAT IT REMAINS BELOW THE SINGULARITY.



IF ω_0 IS REAL THE CONTOUR IS



THUS, FOR ω_0 - real

$$\epsilon(k,\omega_0) = 1 + \frac{4\pi q^2}{mk^2}\left\{ P\int d^3v \, \frac{\underset{\sim}{k}\cdot\frac{\partial f_0}{\partial v}}{\omega_0 - \underset{\sim}{k}\cdot\underset{\sim}{v}} - \frac{i\pi}{|k|} k \frac{\partial}{\partial v_{\parallel}} \int d^2v_{\perp} f_0 \Big|_{v_{\parallel}=\frac{\omega_0}{k}} \right\}$$

LET US ASSUME THAT A SOLUTION EXISTS FOR $\epsilon(\omega_0) = 0$ WITH $\omega_0/k \gg V_{th}$

THEN $$P\int d^3v \, \frac{\underset{\sim}{k}\cdot\frac{\partial f_0}{\partial \underset{\sim}{v}}}{\omega_0 - \underset{\sim}{k}\cdot\underset{\sim}{v}} \cong -\frac{k^2}{\omega_0^2}\int d^3v \, f_0$$

AND $$f(v_{\parallel})\Big|_{v_{\parallel}=\frac{\omega_0}{k}} = \int d^2v_{\perp} f_0 \quad : \text{ IS SMALL}$$

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2} - \frac{i\pi\omega_p^2}{k^2}\frac{k}{|k|}\frac{\partial f(v_{\parallel})}{\partial v_{\parallel}}\Big|_{v_{\parallel}=\frac{\omega_0}{k}} = \epsilon_R + i\epsilon_I = 0$$

$\epsilon_I << \epsilon_R$

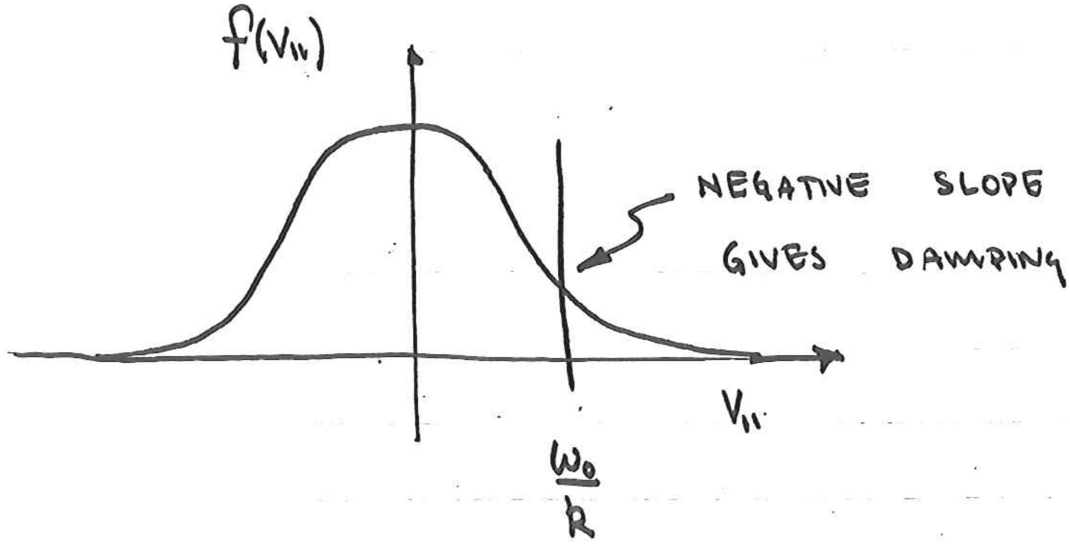
SO $\qquad \omega_0 = \bar{\omega}_0 + \delta\omega_0 \qquad$ WHERE $\quad \epsilon_R(\bar{\omega}_0) = 0$

WITH $\qquad \delta\omega_0 << \bar{\omega}_0 \qquad\qquad 1 - \frac{\omega_p^2}{\bar{\omega}_0^2} = 0 \qquad \bar{\omega}_0 = \pm\,\omega_p$

$$\frac{\partial \epsilon_R}{\partial \bar{\omega}_0}\,\delta\omega + i\,\epsilon_I \cong 0$$

$$\delta\omega = -i\,\epsilon_I \Big/ \frac{\partial \epsilon_R}{\partial \bar{\omega}_0} = +i\,\frac{\pi\,\omega_p^2}{k^2|k|}\,k\,\frac{\partial f}{\partial V_{\parallel}} \left[\frac{2\omega_p}{\bar{\omega}_0^3}\right]^{-1}$$

THUS IF $\left.\frac{k}{\bar{\omega}_0}\frac{\partial f}{\partial V_{\parallel}}\right|_{V_{\parallel}=\frac{\bar{\omega}_0}{k}} < 0$ THE MODE IS DAMPED



DAMPING OCCURS BECAUSE PARTICLES WITH VELOCITY $V_{\parallel}$ SUCH THAT $\omega_0 \equiv k V_{\parallel}$ SEE A NEARLY TIME INDEPENDENT FIELD. PARTICLES WHICH ARE SLIGHTLY SLOWER THAN THE WAVE WILL BE ACCELLERATED AND GAIN ENERGY FROM THE WAVE. PARTICLES WHICH ARE SLIGHTLY FASTER THAN THE WAVE WILL SLOW DOWN AND GIVE ENERGY TO THE WAVE. SINCE THERE ARE MORE SLOWER PARTICLES THAN FASTER ONES $(\partial f / \partial V_{\parallel} < 0)$ THE WAVE DAMPS.

Physical Interpretation of Landua damping

Total Energy of Plasma (particles + fields) is conserved

- NO collisions dissipation
- NO sources

In damping process

$$\underbrace{\text{Wave}}_{\substack{\text{Field Energy} \\ +\text{ Energy of Coherent} \\ \text{motion}}} \xrightarrow{\text{Energy}} \underbrace{\text{Resonant Particles}}_{\substack{\text{Kinetic Energy} \\ \text{of particles} \\ \text{with } \omega \sim k v_z}}$$

$$\boxed{U = \frac{1}{8\pi} E^2 \frac{\partial}{\partial \omega}(\omega \epsilon)}$$

Plasma Waves $\quad \epsilon \cong 1 - \frac{\omega_p^2}{\omega^2}$

$U = \frac{\epsilon |E|^2}{8\pi}$ but $\epsilon = 0$?

$$\frac{\partial}{\partial \omega}\omega\epsilon = \frac{\partial}{\partial \omega}\left(\omega - \frac{\omega_p^2}{\omega}\right) = 1 + \frac{\omega_p^2}{\omega^2}$$

Energy stored in E

Energy stored in u

Examine Resonant Particles

1D in z-direction

Electric field

$$E(z,t) = -\frac{\partial}{\partial z}\phi_0 \cos(k z - \omega_0 t)$$

Remember, damping is weak $\delta\omega \ll \omega_0$
treat ϕ_0 as constant

Transform to a frame moving with the wave

$$\left.\begin{aligned} z &= z' + \frac{\omega_0}{k} t' \\ V_z &= V_z' + \frac{\omega_0}{k} \end{aligned}\right\}$$

Galilean
neglect relativity

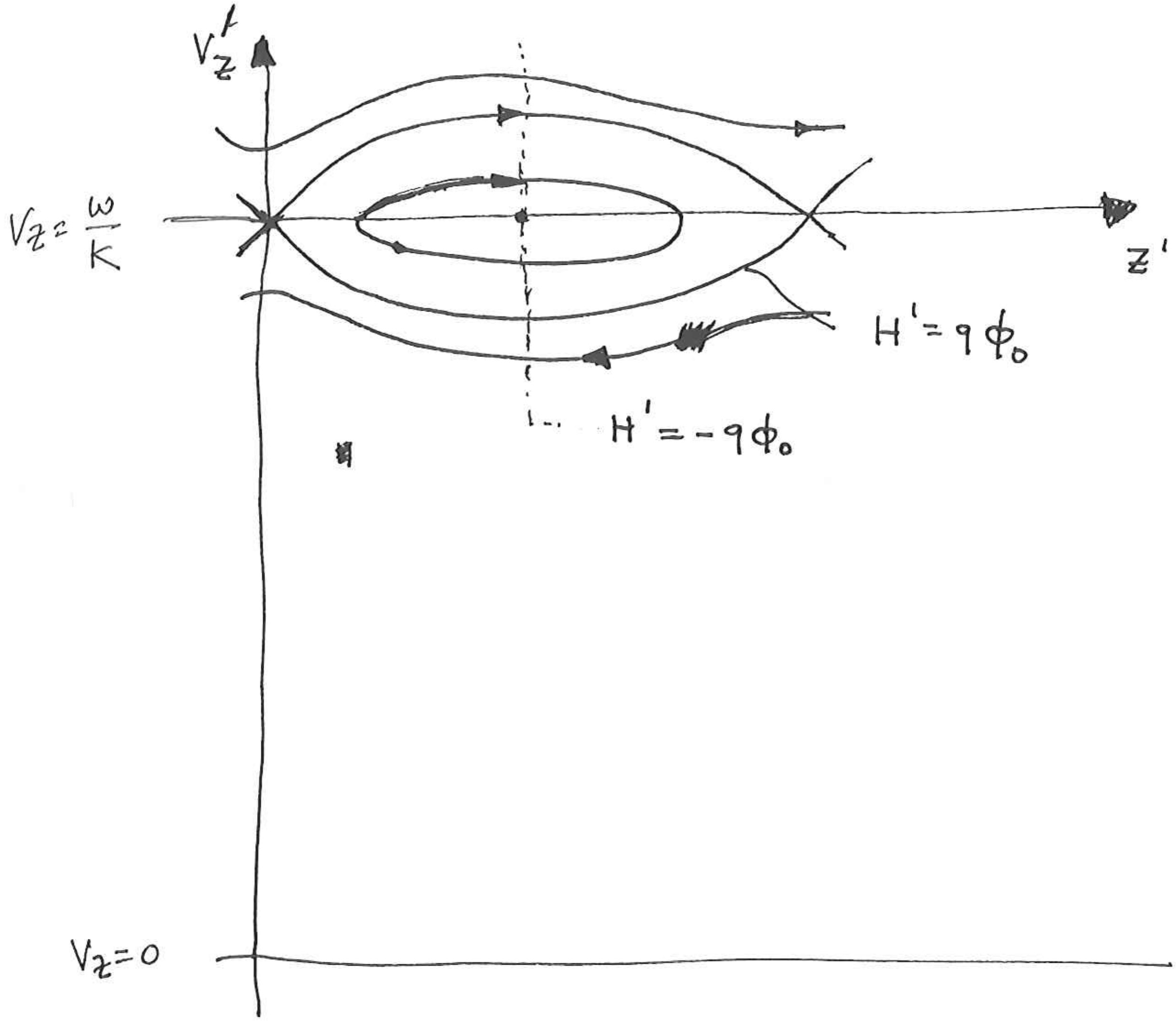
In the prime frame

$$E(z',t') = -\frac{\partial}{\partial z}\phi_0 \cos(k z')$$

no time dependence!

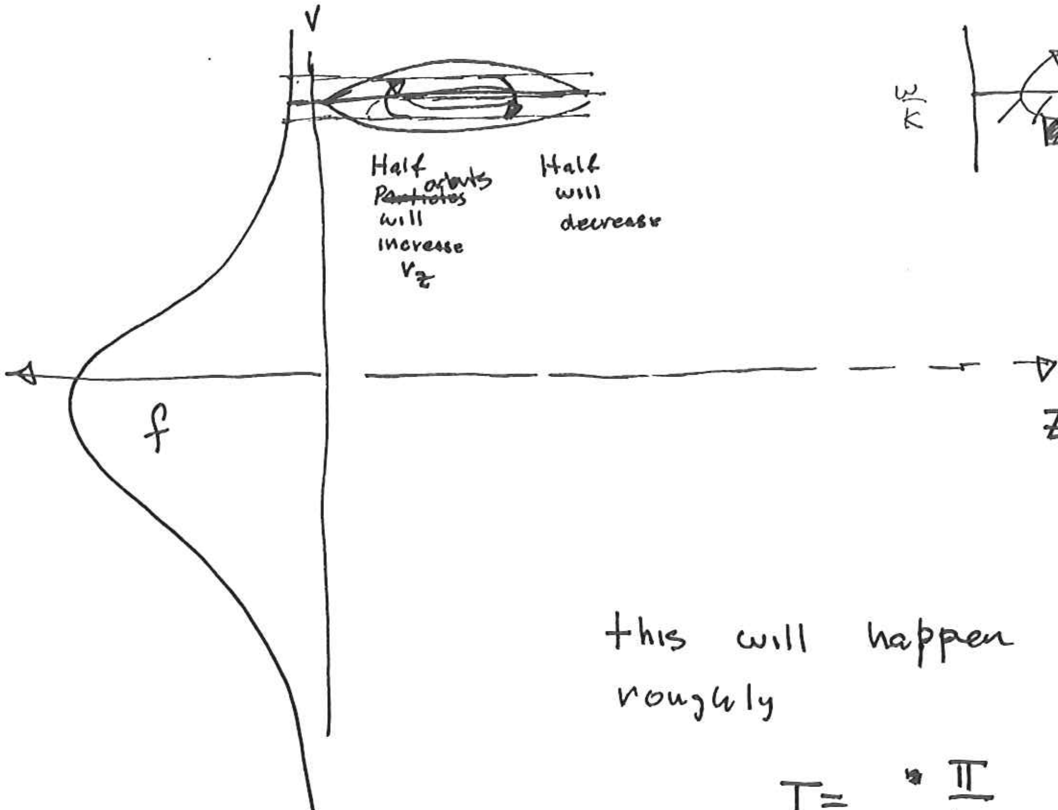
Hamiltonian in this frame is conserved

$$H' = \tfrac{1}{2} m v_z'^2 + q\phi_0 \cos k_z z' = \text{const} \quad \text{for particles}$$

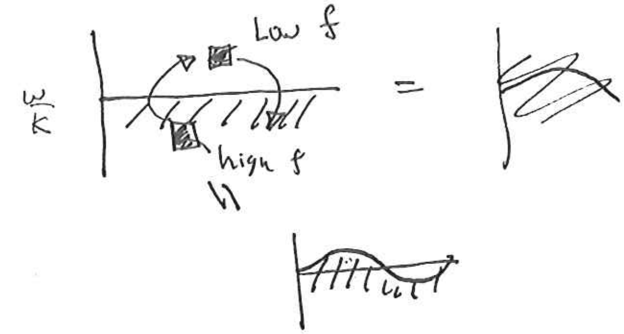


~~How~~

Initial distribution $f_0(v_z)$



Particle energy increased



this will happen until roughly

$$T = \frac{\pi}{\omega_B} \text{ — bounce time}$$

where T is the time ~~to~~ for a trapped particle to complete one circuit in the well

Estimate this time by expanding around "o" point

$$\cos(k_z z) \simeq -1 + \frac{1}{2}(k_z z' - \pi)^2$$

$$H' = \underbrace{\frac{1}{2} m v_z'^2 - q\phi_0 + q\phi_0 \frac{1}{2}(k_z z' - \pi)^2}_{\text{Harmonic Oscillator}}$$

$$\omega_B^2 = \frac{k_z^2 q \phi_0}{m}$$

$$T = \pi \Big/ \sqrt{\frac{k_z^2 q\phi_0}{m}}$$

for small amplitude wave $\phi_0 \to 0$, $T \to \infty$

When does linear theory apply?

1) if damping rate γ

$$\gamma = -\operatorname{Im}(\delta\omega)$$

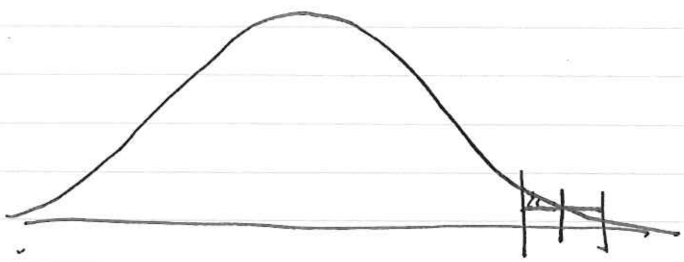
is large enough such that $\gamma T > 1$

wave damps before particles bounce

Linear theory o.k.

otherwise nonlinear theory applies

ENERGY REQUIRED TO flatten f



$$\delta f = -\left(V_z - \frac{\omega}{k_z}\right) \left.\frac{\partial f}{\partial V_z}\right|_{\frac{\omega}{k}}$$

$$\delta V \sim \sqrt{\frac{2\phi_0}{m}}$$

change in particle

$$\delta U = \int_{\frac{\omega}{k} - \delta V}^{\frac{\omega}{k} + \delta V} \frac{1}{2} m v^2 \, \delta f \, d^3v$$

$$\delta U = \left.\frac{\partial f}{\partial V_z}\right|_{\frac{\omega}{k}} \int_{\frac{\omega}{k} - \delta V}^{\frac{\omega}{k} + \delta V} dV_z \; \frac{1}{2} m V_z^2 \left(V_z - \frac{\omega}{k}\right)$$

$$\cong -\left(\frac{\partial f}{\partial V_z}\right)\bigg|_{\frac{\omega}{k}} m\left(\frac{\omega}{k}\right) \delta V^2$$

if $\delta V \gtrsim \left.\frac{E^2}{8\pi}\right|_{\text{initial}} \frac{\partial}{\partial \omega} \omega \epsilon$ linear damping occurs

The Plasma Dispersion Function

$$\epsilon(\underset{\sim}{k},\omega) = 1 + \frac{4\pi q^2}{mk^2}\int d^3v \, \frac{\underset{\sim}{k}\cdot \frac{\partial f_0}{\partial \underset{\sim}{v}}}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}$$

D

take $\underset{\sim}{k} = k\hat{z}$

$$f_0 = \frac{n_0}{(2\pi T/m)^{3/2}} \exp\left(-\frac{\frac{1}{2}m(v_\perp^2 + v_z^2)}{T}\right)$$

consider a Maxwellian Equilibrium Distribution function

$$\underset{\sim}{k}\cdot\frac{\partial f_0}{\partial \underset{\sim}{v}} = -\frac{k v_z m}{T} f_0$$

do $v_\perp$ integration

$$\epsilon(k,\omega) = 1 - \frac{4\pi q^2 n_0}{k^2 T}\int_{-\infty}^{\infty} \frac{dv_z}{(2\pi T/m)^{1/2}} \frac{k v_z}{\omega - k v_z} \exp(-\tfrac{1}{2}m v_z^2/T)$$

$$= 1 + \frac{4\pi q^2 n_0}{k^2 T}\int_{-\infty}^{\infty} \frac{dv_z}{\left(\frac{2\pi T}{m}\right)^{1/2}}\left[1 + \frac{\omega}{k v_z - \omega}\right] \exp(-\tfrac{1}{2}m v_z^2/T)$$

$$\epsilon(k,\omega) = 1 + \frac{4\pi q^2 n_0}{k^2 T}\left[1 + \int_{-\infty}^{\infty} \frac{dV_z}{\left(\frac{2\pi T}{m}\right)^{1/2}} \frac{\omega}{kV_z - \omega} \exp\left(-\tfrac{1}{2} m V_z^2 / T\right)\right]$$

NORMalize V_z to V_{th} $\qquad V_{th} = (2T/m)^{1/2}$

$\left(\frac{1}{2} m V_t^2 = T\right)$

call $x = V_z / V_t$

call $\omega / kV_t = \xi$

$$\epsilon(k,\omega) = 1 + \frac{4\pi q^2 n_0}{k^2 T}\left[1 + \xi \int_{-\infty}^{\infty} \frac{dx}{\pi^{1/2}} \frac{1}{x - \xi} e^{-x^2}\right]$$

the integral

$$Z(\xi) \equiv \int_{-\infty}^{\infty} \frac{dx}{\pi^{1/2}} \frac{e^{-x^2}}{x - \xi}$$

is called the plasma dispersion function

for ξ

Related to error function

A related function used by the Some others is

$$W(\xi) = \int_{-\infty}^{\infty} \frac{dx}{\pi^{1/2}} \frac{x\, e^{-x^2}}{\xi - x}$$

$$1 + \xi Z(\xi) = -W(\xi)$$

For large argument $\xi >> 1$

$$\frac{1}{x-\xi} = \frac{-1}{(\xi - x)} \simeq -\frac{1}{\xi}\left[1 + \frac{x}{\xi} + \frac{x^2}{\xi^2} + \frac{x^3}{\xi^3} \cdots\cdots\right]$$

$$Z(\xi) = -\int_{-\infty}^{\infty} \frac{dx}{\pi^{1/2}} e^{-x^2} \frac{1}{\xi}\left[1 + \frac{x}{\xi} + \frac{x^2}{\xi^2} + \frac{x^3}{\xi^3} + \cdots\right.$$

$$= -\frac{1}{\xi}\left[1 + \frac{0}{\xi} + \frac{1}{2\xi^2} + \frac{0}{\xi^3} + \frac{3}{4}\frac{1}{\xi^4}\right]$$

$$1+\xi Z(\xi) = 1-\left(1+\frac{1}{2\xi^2}+\frac{3}{4}\frac{1}{\xi^4}\right)$$

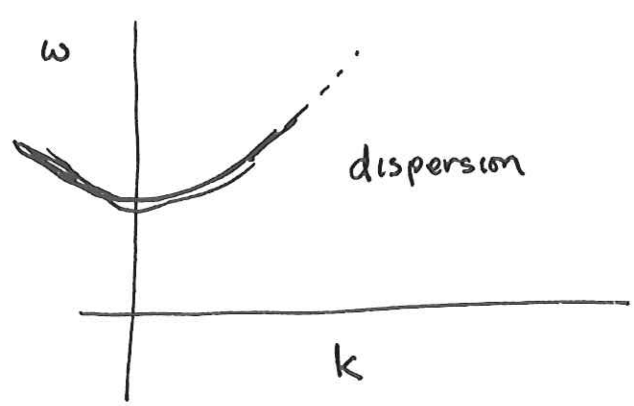
$$\epsilon = 1 - \frac{4\pi q^2 n_0}{k^2 T}\left[\frac{1}{2}\frac{k^2 v_t^2}{\omega^2}+\frac{3}{4}\frac{k^4 v_t^4}{\omega^4}\right]$$

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2}\left[1+\frac{3}{2}\frac{k^2 v_t^2}{\omega^2}\right] \qquad \frac{2T}{m} = v_t^2$$

thermal correct

$$\epsilon = 0$$

$$\omega^2 = \omega_p^2\left[1+3\frac{k^2 T_e}{m\omega^2}\right]$$



Tabulated by Fried and Conte

PLASMA DISPERSION FUNCTION

Definition[16] (first form valid only for $\operatorname{Im}\zeta > 0$):

$$Z(\zeta) = \pi^{-1/2} \int_{-\infty}^{+\infty} \frac{dt\ \exp\left(-t^2\right)}{t-\zeta} = 2i\exp\left(-\zeta^2\right) \int_{-\infty}^{i\zeta} dt\ \exp\left(-t^2\right).$$

Physically, $\zeta = x + iy$ is the ratio of wave phase velocity to thermal velocity.

Differential equation:

$$\frac{dZ}{d\zeta} = -2\left(1 + \zeta Z\right),\ Z(0) = i\pi^{1/2}; \quad \frac{d^2 Z}{d\zeta^2} + 2\zeta\frac{dZ}{d\zeta} + 2Z = 0.$$

Real argument ($y = 0$):

$$Z(x) = \exp\left(-x^2\right)\left(i\pi^{1/2} - 2\int_0^x dt\ \exp\left(t^2\right)\right).$$

Imaginary argument ($x = 0$):

$$Z(iy) = i\pi^{1/2}\exp\left(y^2\right)\left[1 - \operatorname{erf}(y)\right].$$

Power series (small argument):

$$Z(\zeta) = i\pi^{1/2}\exp\left(-\zeta^2\right) - 2\zeta\left(1 - 2\zeta^2/3 + 4\zeta^4/15 - 8\zeta^6/105 + \cdots\right).$$

Asymptotic series, $|\zeta| \gg 1$ (Ref. 17):

$$Z(\zeta) = i\pi^{1/2}\sigma\exp\left(-\zeta^2\right) - \zeta^{-1}\left(1 + 1/2\zeta^2 + 3/4\zeta^4 + 15/8\zeta^6 + \cdots\right),$$

where

$$\sigma = \begin{cases} 0 & y > |x|^{-1} \\ 1 & |y| < |x|^{-1} \\ 2 & y < -|x|^{-1} \end{cases}$$

Symmetry properties (the asterisk denotes complex conjugation):

$$Z(\zeta^*) = -\left[Z(-\zeta)\right]^*;$$

$$Z(\zeta^*) = \left[Z(\zeta)\right]^* + 2i\pi^{1/2}\exp[-(\zeta^*)^2] \quad (y > 0).$$

Two-pole approximations[18] (good for ζ in upper half plane except when $y < \pi^{1/2}x^2\exp(-x^2)$, $x \gg 1$):

$$Z(\zeta) \approx \frac{0.50 + 0.81i}{a - \zeta} - \frac{0.50 - 0.81i}{a^* + \zeta}, \quad a = 0.51 - 0.81i;$$

$$Z'(\zeta) \approx \frac{0.50 + 0.96i}{(b - \zeta)^2} + \frac{0.50 - 0.96i}{(b^* + \zeta)^2}, \quad b = 0.48 - 0.91i.$$

Imaginary Part of Z for real ξ

$$\mathrm{Im}\{Z(\xi)\} = \pi^{1/2} e^{-\xi^2}$$

For small argument

$$Z(\xi) \simeq i\pi^{1/2} e^{-\xi^2} - 2\xi(1 - 2\xi^2/3 + 4\xi^4/15 - $$

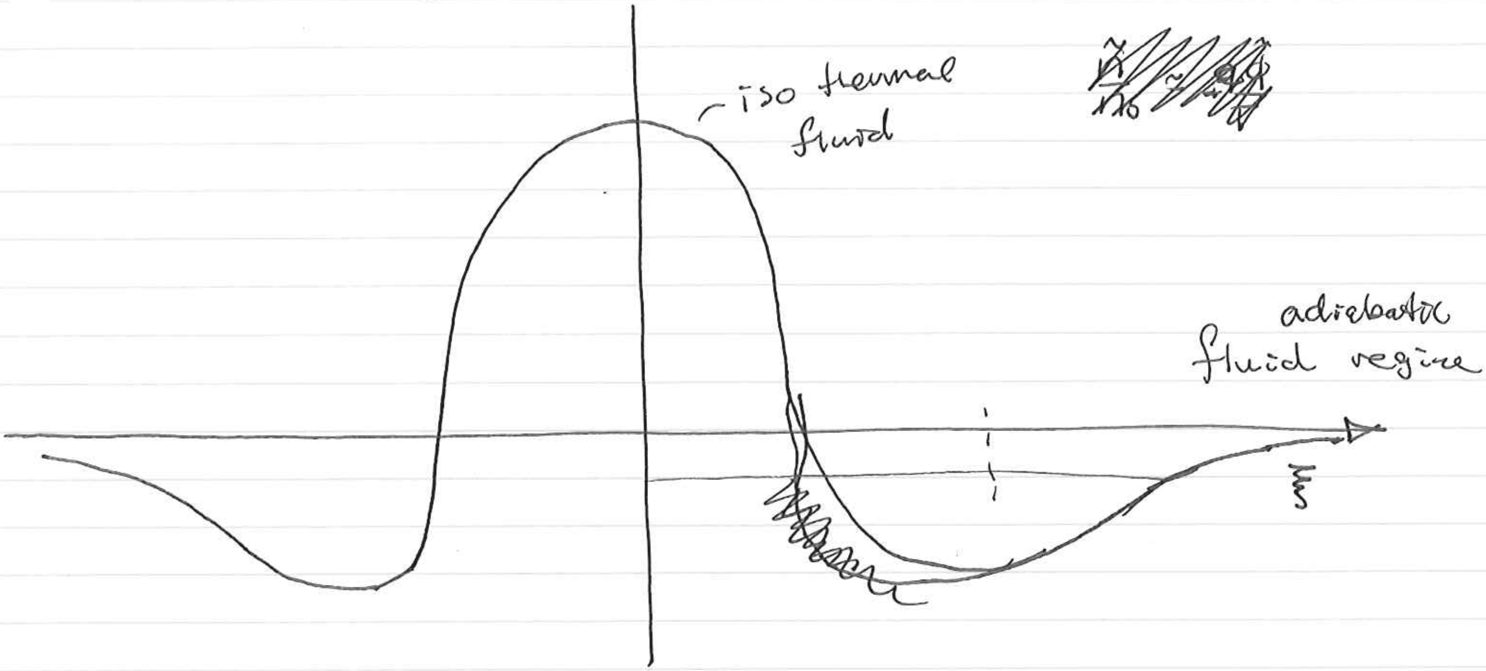
$$1 + \xi Z(\xi) \simeq \cancel{1 + i}$$

$$1 - 2\xi^2 + i\pi^{1/2}\xi e^{-\xi^2}$$

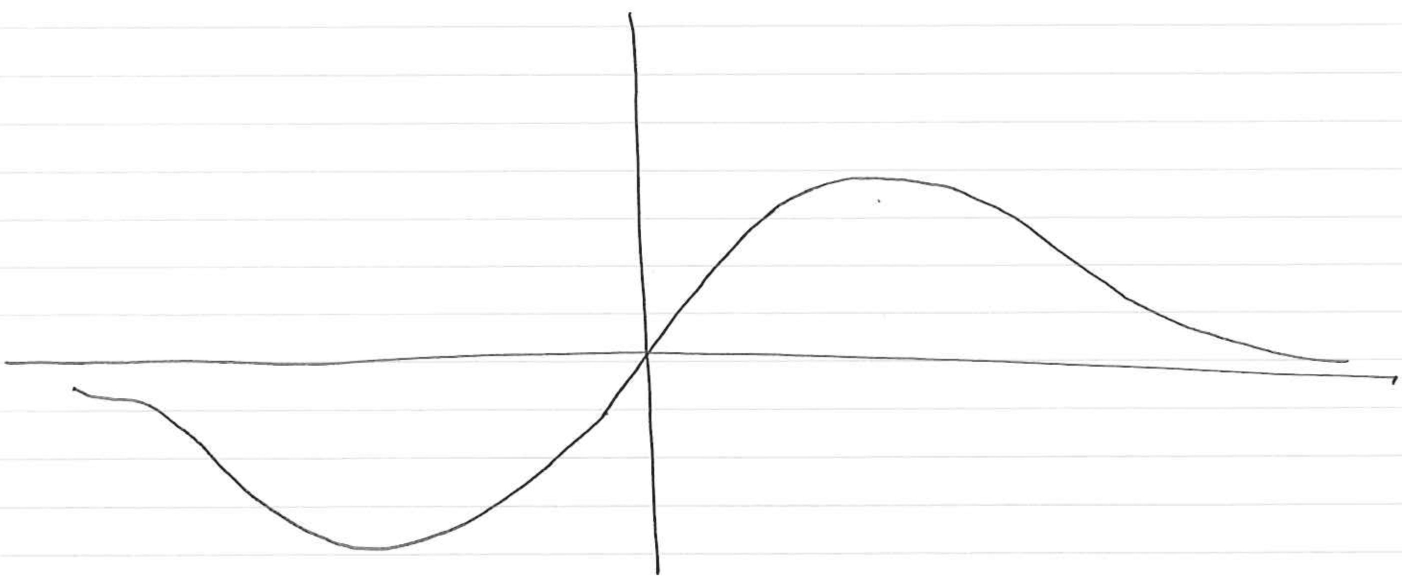
For large argumen

$$1 + \xi Z(\xi) \simeq -\left(\frac{1}{2\xi^2} + \frac{3}{4\xi^4}\right) + i\pi^{1/2}\xi e^{-\xi^2}$$

$Re\left\{1+\xi Z(\xi)\right\}$



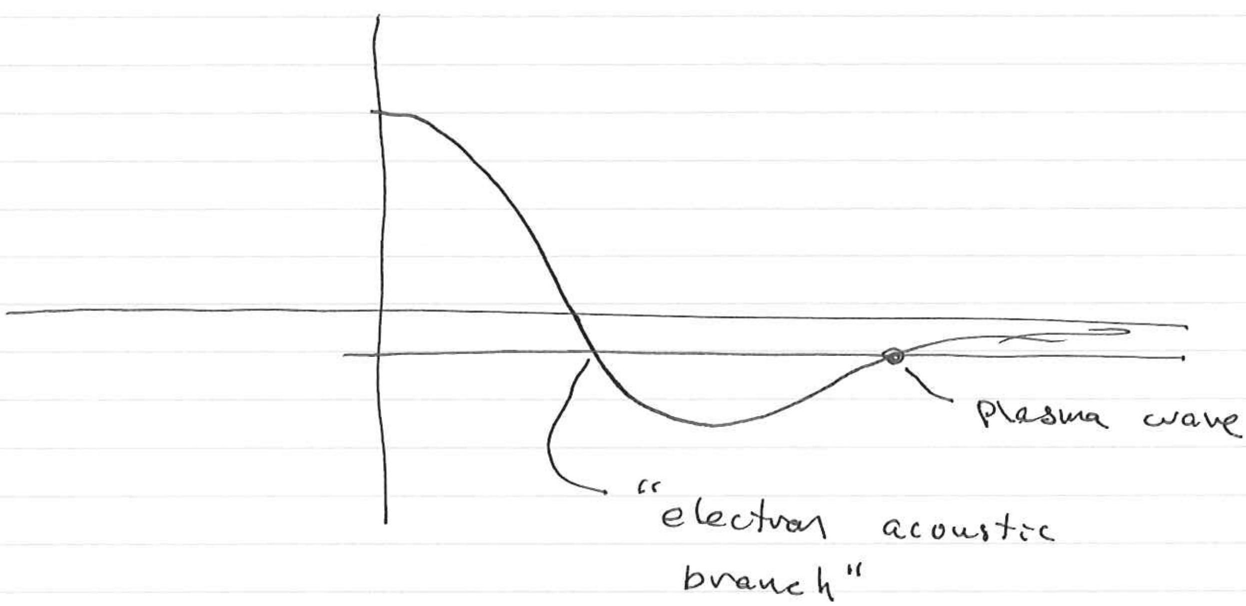
$Im\ \xi Z(\xi)$



$$\epsilon = 1 + \frac{4\pi q^2 n_0}{T}\left[\, 1 + \xi Z(\xi) \,\right]$$

suppose we didn't worry about $\mathrm{Im}\{\epsilon\}$

$\epsilon_r = 0$ has two solutions



(highly damped)

suppose $\omega \to 0$

$$\boxed{\delta n = -n_0 \frac{q\phi}{T}}$$

"adiabatic"

isothermal