Fitzpatrick kinetic.pdf

G&R Ch 23

Chouldhuri Ch 12.4

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Electrostatic Waves in a warm Plasma

Recall for a cold plasma we found

$$\epsilon = 1 - \sum_{e,i} \frac{\omega_{pe,i}^2}{\omega^2}$$

where $\omega_{pe,i}^2 = \frac{4\pi n_{e,i} q_{e,i}^2}{m_{e,i}}$

$$\omega_{pe}^2 \gg \omega_{pi}^2$$

normal electrostatic mode $\epsilon = 0$ $\omega \simeq \pm \omega_{pe}$

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* We will investigate the case of of nonzero Temperature
* How will we do this
* Solve Vlasov Equation

Step #1 find an equilibrium solution in which $f_{e,i}$ is independent of time

we will assume $\underset{\sim}{E} = 0$ in equilibrium

$\cancel{\&}$ $$\sum_{e,i} q_{e,i} n_{e,i0} = 0$$

CHARGE NEUTRALITY

STEP #2 We will assume a small electric field is present in the form of a wave

$$\underset{\sim}{E} = -\nabla\phi$$

$$\phi = Re\left\{\hat{\phi}\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)}\right\}$$

$\hat{\phi}$ = complex amplitude

$\underset{\sim}{k}$ = vector wave number

ω = frequency

STEP #3 We will solve the linearized VE to find the perturbed distribution function equilibrium

$$f(\underline{x},\underline{v}) = f_0 + Re\{\hat{f} e^{i(\underline{k}\cdot\underline{x}-\omega t)}\}$$

perturbation

assume $|\hat{f}| << |f_0|$

Step #4 We will compute the perturbed charge density

$$\rho = Re\{\hat{\rho} e^{i(\underline{k}\cdot\underline{x}-\omega t)}\}$$

Step #5 We will solve the Poisson equation

$$-\nabla^2 \cdot Re\{\hat{\phi} e^{i(\underline{k}\cdot\underline{x}-\omega t)}\} = 4\pi Re\{\hat{\rho} e^{i(\underline{k}\cdot\underline{x}-\omega t)}\}$$

From which we will extract $\underline{\underline{\epsilon(\omega,\underline{k})}}$

Step #1 — Find equilibrium solution corresponding to $\underset{\sim}{E} = 0$

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial f}{\partial \underset{\sim}{x}} + \frac{q}{m} \underset{\sim}{E} \cdot \frac{\partial f}{\partial \underset{\sim}{v}} = 0$$

$$f = f_0 + \mathrm{Re}\{\hat{f} e^{i(\underset{\sim}{k} \cdot \underset{\sim}{x} - \omega t)}\}$$

$$\underset{\sim}{E} = -\nabla \phi \qquad \phi = \mathrm{Re}\{\hat{\phi} e^{i(\underset{\sim}{k} \cdot \underset{\sim}{x} - \omega t)}\}$$

Equilibrium

$$\left.\begin{array}{l} \dfrac{\partial f_0}{\partial t} = 0 \\ \underset{\sim}{E} = 0 \end{array}\right\} \qquad \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}} f_0 = 0$$

$$f_0 = f_0(\underset{\sim}{v})$$

arbitrary function of velocity

independent of space

$$n_0 = \int d^3v\, f_0$$

we require $\sum_{e,i} q_{e,i} \int d^3v\, f_{e,i0} = 0$

Step #2

$$\underset{\sim}{E} = -\nabla\phi$$

$$\phi = Re\{\hat{\phi}\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x}-\omega t)}\}$$

$$\underset{\sim}{E} = Re\{-i\underset{\sim}{k}\hat{\phi}\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x}-\omega t)}\}$$

Step #3 solve the linearized VE

$$\frac{\partial f}{\partial t} + \underset{\sim}{v}\cdot\nabla f + \frac{q}{m} Re\{-i\underset{\sim}{k}\hat{\phi}\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x}-\omega t)}\}\cdot\frac{\partial}{\partial\underset{\sim}{v}} f = 0$$

$f = f_0 +$ perturbation perturbation $\propto \phi$

DROP SECond order IN ϕ TERM

$$\overrightarrow{\left(\frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\nabla\right) Re\{\hat{f}\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x}-\omega t)}\}} + Re\{-i\underset{\sim}{k}\,\cancel{\hat{\phi}}\,\frac{q\hat{\phi}}{m}\cdot\frac{\partial f_0}{\partial\underset{\sim}{v}}\} = 0$$

$$Re\{-i(\omega-\underset{\sim}{k}\cdot\underset{\sim}{v})\hat{f}\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x}-\omega t)} - i\underset{\sim}{k}\frac{q\hat{\phi}}{m}\cdot\frac{\partial f_0}{\partial\underset{\sim}{v}}\} = 0$$

$$-i(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})\hat{f} - i \frac{q\hat{\phi}}{m} \underset{\sim}{k} \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}} = 0$$

so

$$\boxed{\hat{f} = -\frac{q}{m} \frac{\hat{\phi}}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})} \underset{\sim}{k} \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}}}$$

$\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}$ = doppler shifted frequency for particles with velocity $\underset{\sim}{v}$

Step #4

$$\rho = \cancel{\rho} \, Re\{\hat{\rho} \, e^{i(\underset{\sim}{k} \cdot \underset{\sim}{x} - \omega t)}\}$$

$$\hat{\rho} = \sum_{e,i} \int d^3v \, q_{e,i} \hat{f}_{e,i}$$

$$\hat{\rho} = -\sum_{e,i} \int d^3v \, \frac{q_{e,i}^2}{m_{e,i}} \, \frac{\underset{\sim}{k} \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}}}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})} \, \hat{\phi}$$

note: $\hat{\rho}$ is proportional to $\hat{\phi}$

Step #5 Poisson Equation

$$-\nabla^2 \mathrm{Re}\{\hat{\phi} e^{i(\underset{\sim}{k} \cdot \underset{\sim}{x} - \omega t)}\} = 4\pi \, \mathrm{Re}\{\hat{\rho} e^{i(\underset{\sim}{k} \cdot \underset{\sim}{x} - \omega t)}\}$$

$$k^2 \hat{\phi} = 4\pi \hat{\rho} = -\sum_{e,i} \int d^3v \, \frac{4\pi q_{e,i}}{m_{e,i}} \, \frac{\underset{\sim}{k} \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}}}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})} \, \hat{\phi}$$

OR

$$k^2 \underbrace{\left(1 + \frac{1}{k^2} \sum_{e,i} \int d^3v \, \frac{4\pi q_{e,i}}{m_{e,i}} \, \frac{\underset{\sim}{k}}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})} \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}}\right)}_{\epsilon(\omega, k)} \hat{\phi} = 0$$

$$\boxed{\epsilon(\omega, \underset{\sim}{k}) = 1 + \frac{1}{k^2} \sum_{e,i} \int d^3v \, \frac{4\pi q_{e,i}}{m_{e,i}} \, \frac{\underset{\sim}{k} \cdot \partial f_0 / \partial \underset{\sim}{v}}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})}}$$

Reality check

FIRST DO velocity integral by parts

$$\epsilon(\omega, \underset{\sim}{k}) = 1 - \sum_{e,i} \frac{1}{k^2} \int d^3v \, \frac{4\pi q_{e,i}^2}{m_{e,i}} f_0 \, \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{v}} \frac{1}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})}$$

$$\underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{v}} \frac{1}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})} = -\frac{1}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})^2} \, \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{v}} (-\underset{\sim}{k} \cdot \underset{\sim}{v})$$

$$\sum_{ij} k_i \frac{\partial}{\partial v_i} k_j v_j = \sum_{ij} k_i k_j \frac{\partial v_j}{\partial v_i}$$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$= \sum_i k_i^2 = k^2$$

$$\epsilon(\omega, \underset{\sim}{k}) = 1 - \sum_{e,i} \int d^3v \, \frac{4\pi q_{e,i}^2}{m_{e,i}} \frac{f_0(\underset{\sim}{v})}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})^2}$$

Suppose plasma is cold,

$f_0(\underset{\sim}{v}) \to 0$ when $|\underset{\sim}{v}| > V_{typical} \sim \sqrt{\frac{2T}{m}}$

if $k V_{typical} \ll \omega$ than can neglect Doppler shift

$$\epsilon(\omega, \underset{\sim}{k}) = 1 - \sum_{e,i} \frac{4\pi q_{e,i}^2}{m_{e,i}\omega^2} \underbrace{\int d^3v \, f_0(\underset{\sim}{v})}_{n_{e,i0}}$$

$$= 1 - \sum_{e,i} \frac{\omega_{pe,i}^2}{\omega^2}$$ our old result

OLD Result applies when $V_{typ} \ll \frac{\omega}{|k|}$

Doppler shift is small

Interpretation each group of electrons acts as a density $dn = d^3v \, f_0(\underset{\sim}{v})$

has Doppler Shift $\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}$

$$\epsilon(\omega, k) = 1 - \sum_{e,i} \int \frac{4\pi q_{e,i}^2 \, dn_{e,i}}{m_{e,i}(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})^2}$$

$$d\,\omega_{pe,i}^2 = \frac{4\pi q_{e,i}^2 \, dn}{m_{e,i}}$$

Note, for electrostatic dielectric only velocities in direction of $\underset{\sim}{k}$ are important

$$\epsilon(\omega, \underset{\sim}{k}) = 1 - \sum_{e,i} \frac{4\pi q^2}{m} \int \frac{d^3v \, f(\underset{\sim}{v})}{(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v})^2}$$

$\angle$.

Take $\underset{\sim}{k}$ to be in x-direction

$$\epsilon(\omega, k) = 1 - \sum_{e,i} \frac{4\pi q^2}{m} \int \frac{dv_x dv_y dv_z \, f(v_x, v_y, v_z)}{(\omega - k v_x)^2}$$

Do v_y & v_z integrals

$$\epsilon(\omega, k) = 1 - \sum_{e,i} \frac{4\pi q^2}{m} \int \frac{dv_x \, f^{1D}(v_x)}{(\omega - k v_x)^2}$$

$$f_{1D}(v_x) = \int dv_y dv_z f(v_x, v_y, v_z)$$

ONE-DIMENSIONAL PHYSICS

NEXT STEP CONSIDER FIRST APPROXIMATION

$$\underset{\sim}{k}\cdot\underset{\sim}{V} \ll \omega \qquad \underset{\sim}{V} = \underset{\sim}{u} + \delta\underset{\sim}{V} \qquad |\underset{\sim}{k}\cdot\delta\underset{\sim}{V}| \ll \omega$$

average

assume

$$\frac{1}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{V}} = \frac{1}{\underbrace{\omega - \underset{\sim}{k}\cdot\underset{\sim}{u}}_{\text{doppler freq}} - \underset{\sim}{k}\cdot\delta\underset{\sim}{V}} = \frac{1}{\omega_d - \underset{\sim}{k}\cdot\delta\underset{\sim}{V}}$$

EXPAND DENOMINATOR

$$\frac{1}{(\omega_d - \underset{\sim}{k}\cdot\delta\underset{\sim}{V})^2} \simeq \frac{1}{\omega_d^2}\left[1 + 2\frac{\underset{\sim}{k}\cdot\delta\underset{\sim}{V}}{\omega_d} + \frac{3(\underset{\sim}{k}\cdot\delta\underset{\sim}{V})^2}{\omega_d^2} + \cdots\right]$$

$$\int \frac{d^3v\, f_0(\underset{\sim}{v})}{(\omega_d - \underset{\sim}{k}\cdot\delta\underset{\sim}{V})} \simeq \frac{1}{\omega_d^2}\left[\int d^3v\, f_0 \left(1 + \frac{2\underset{\sim}{k}\cdot\delta\underset{\sim}{V}}{\omega_d} + \frac{3(\underset{\sim}{k}\cdot\delta\underset{\sim}{V})^2}{\omega_d^2} + \cdots\right.\right.$$

$$\doteq \frac{1}{\omega_d^2}\left[n_0 + \frac{0}{\omega_d} + \frac{3}{\omega_d^2}\int d^3v\,(\underset{\sim}{k}\cdot\delta\underset{\sim}{v})^2 f_0\right]$$

$$\int d^3v\,(\underset{\sim}{k}\cdot\delta\underset{\sim}{v})^2 f_0 = \frac{1}{m}\,\underset{\sim}{k}\cdot\underset{\approx}{\pi}\cdot\underset{\sim}{k} = k^2\frac{p}{m}$$

for TE

USE $P = nT$

$$\epsilon \simeq 1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2 T}{m\omega^2}\right)$$

$\frac{T}{m}$ has units ~~#~~ of velocity^2

$\frac{T}{m} = V_{th}^2$ $\quad V_{th} =$ typical thermal velocity

small correction if $|kV_t| \ll \omega$

or $\quad V_t \ll \left|\frac{\omega}{k}\right|$ $\quad$ thermal velocity much less than phase velocity of wave.

Lets solve for ω for a given k

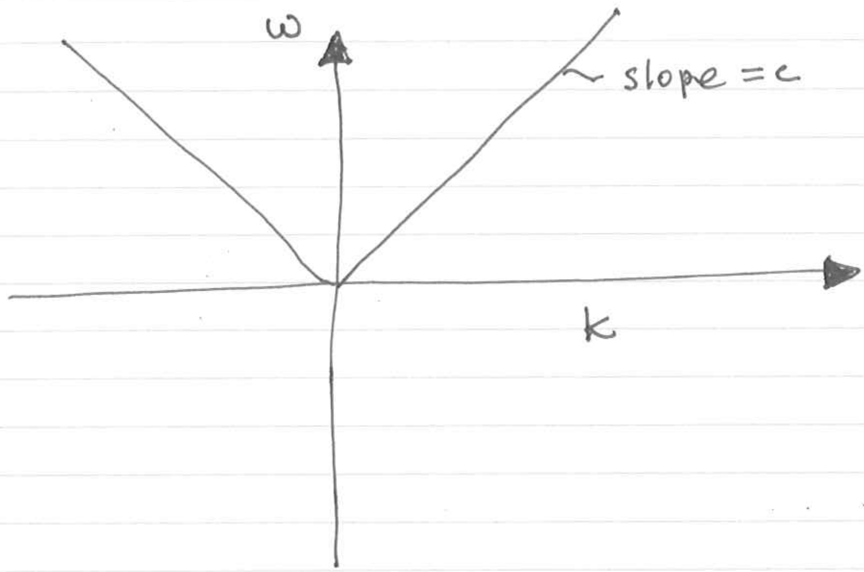
$$\epsilon(\omega, k) = 0 \Rightarrow \omega(k)$$

Normal mode frequency

This is known as a dispersion relation

For Light waves in ~~t~~ vacuum

$$\omega^2 = k^2 c^2$$



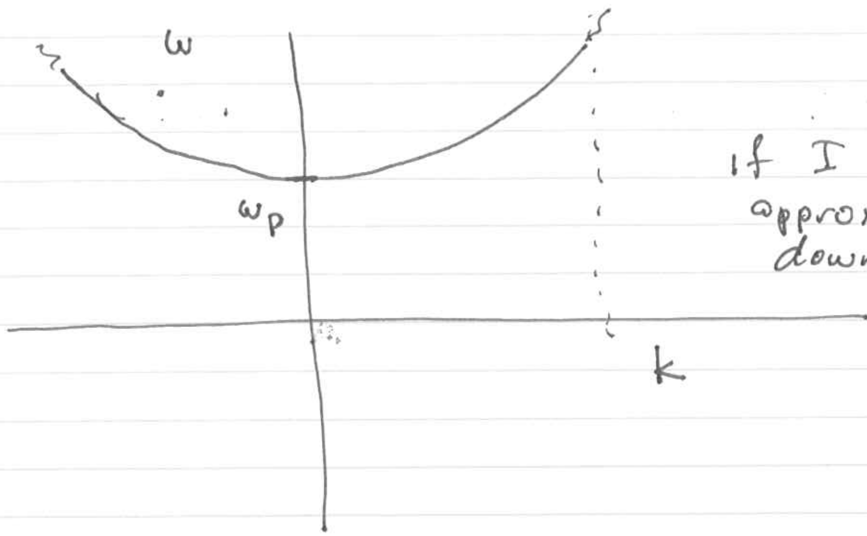
solve $\epsilon \simeq 0$

Appears to be a quadratic relation

$$\omega^2 \simeq \omega_p^2 \left(1 + 3\frac{k^2 T}{m\omega_p^2}\right) \simeq \omega_p^2 + 3\frac{k^2 T}{m}$$

$$\cancel{\omega \simeq \sqrt{\omega_p^2 + 3\frac{k^2 T}{m}}}$$

$$\omega = \sqrt{\omega_p^2 + 3\frac{k^2 T}{m}} \simeq \omega_p \left(1 + \frac{3}{2}\frac{k^2 T}{m\omega_p^2}\right)$$



if I make k too big approximation breaks down

Can we understand this based on fluid equations?

Ans: yes

$$\omega = \omega_p^2 \left(\cancel{2}\, 1 + \frac{3}{2} k^2 \lambda_D^2 \right)$$

$$\lambda_D^2 = \frac{T}{4\pi n e^2} = \frac{T}{m\,\omega_p^2}$$

good as long as $k\lambda_D \ll 1$

wave lengths long compared with Debye length

⊛ Apply same procedure to solution of fluid equations

continuity

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} n\underset{\sim}{u} = 0$$

momentum

$$\underset{\sim}{E} = -\nabla\phi$$

$$(mn)\left[\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{u}\right] = -\nabla p + q\,n\underset{\sim}{E}$$

internal energy

ratio of specific heats

$$\left(\frac{\partial}{\partial t} p + \underset{\sim}{u}\cdot\nabla p\right) + \gamma_s p \nabla\cdot\underset{\sim}{u} = 0$$

Equilibrium : $p = p_0$ $n = n_0$ — constants

$\underset{\sim}{u}_0,\ \phi_0 = 0$

no flow no potential

Perturbation $p = p_0 + \mathrm{Re}\{\hat{p}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega t}\}$

$n = n_0 + \mathrm{Re}\{\hat{n}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega t}\}$

$\underset{\sim}{u} = \mathrm{Re}\{\hat{\underset{\sim}{u}}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega t}\}$ $\phi = \mathrm{Re}\{\hat{\phi}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega t}\}$

continuity

drop $\hat{n}\hat{u}$ (non linear term)

$$-i\omega\hat{n} + i\underset{\sim}{k}\cdot n_0 \hat{\underset{\sim}{u}} = 0$$

$$m n_0 (-i\omega \hat{\underset{\sim}{u}}) = -i\underset{\sim}{k}\,\hat{p} - q n_0 i k \hat{\phi}$$

$$-i\omega\hat{p} + \gamma_s p_0 i \underset{\sim}{k}\cdot\hat{\underset{\sim}{u}} = 0$$

eliminate $\hat{p}$ & $\hat{\underset{\sim}{u}}$ to find $\hat{n}$ in terms of $\hat{\phi}$

$$\frac{\hat{p}}{p_0} = \gamma_s \frac{\hat{n}}{n_0}$$

$$-i\omega m n_0 \hat{\underset{\sim}{u}} = -i\underset{\sim}{k}\left(p_0 \gamma_s \frac{\hat{n}}{n_0} + q n_0 \hat{\phi} \right)$$

$$+i\omega\hat{n} = i n_0 \underset{\sim}{k}\cdot\hat{\underset{\sim}{u}} \qquad \frac{p_0}{n_0} = T$$

$$\omega^2 m \hat{n} = k^2 \left(\gamma_s \frac{p_0}{n_0} \frac{\hat{n}}{n_0} + q n_0 \hat{\phi} \right)$$

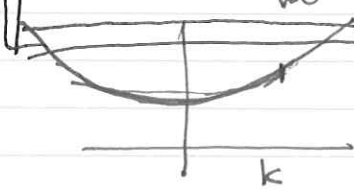
$$\hat{n} = \frac{k^2 \frac{q n_0}{m} \hat{\phi}}{\omega^2 - \frac{k^2 \gamma_s T}{m}}$$

charge density $\quad 4\pi \hat{\rho} = 4\pi q \hat{n}$

Poisson Equation

$$k^2 \hat{\phi} = 4\pi q \hat{n} = \frac{k^2 \omega_p^2}{\omega^2 - k^2 \gamma_s T/m}$$

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2 - k^2 \gamma_s T/m} \qquad \boxed{\omega^2 = k^2 \frac{\gamma_s T}{m} + \omega_p^2}$$



if we assume $k^2 \gamma_s T/m \ll \omega^2$ as befor

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2}\left[1 + \gamma_s \frac{k^2 T}{m\omega^2} + \cdots\right]$$

compare with  the same

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2}\left[1 + 3 \frac{k^2 T}{m\omega^2} + \cdots\right.$$

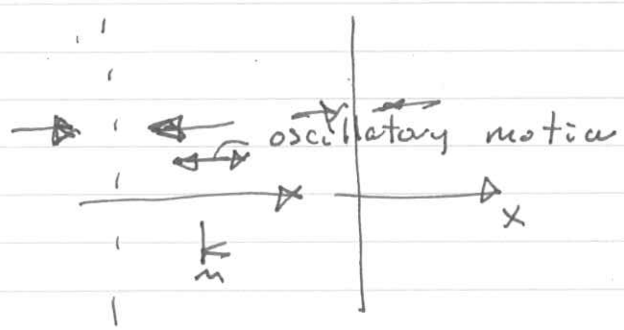
$$\gamma_s = \frac{n_f + 2}{n_f}$$

To get $\gamma_s = 3$ we must have $n_f = 1$ (not 3)

Explanation: ~~collisionless plasma~~

~~compression~~

Wave perturbation involves a compression in 1D



This increases internal energy associated with motion in x direction

In a collisional fluid ~~b collisi~~ raises $\frac{1}{2} m \delta v_x^2$

collisions quickly redistribute this energy to other degrees of freedom

$n_f = 3$ $\frac{1}{2} m \delta v_y^2 + \frac{1}{2} m \delta v_x^2$

In collisionless plasma

energy remains in $\frac{1}{2} m \delta v_x^2$

not shared

$n_f = 1$

Landau Damping

We have been avoiding ~~the~~ a big problem - the crux

$$\epsilon = 1 + \frac{4\pi q^2}{m k^2} \int d^3v \frac{\underset{\sim}{k} \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}}}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})}$$

OR

$$= 1 - \frac{4\pi q^2}{m} \int d^3v \frac{f_0}{(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v})^2}$$

what happens when $\underset{\sim}{k} \cdot \underset{\sim}{v} \simeq \omega$

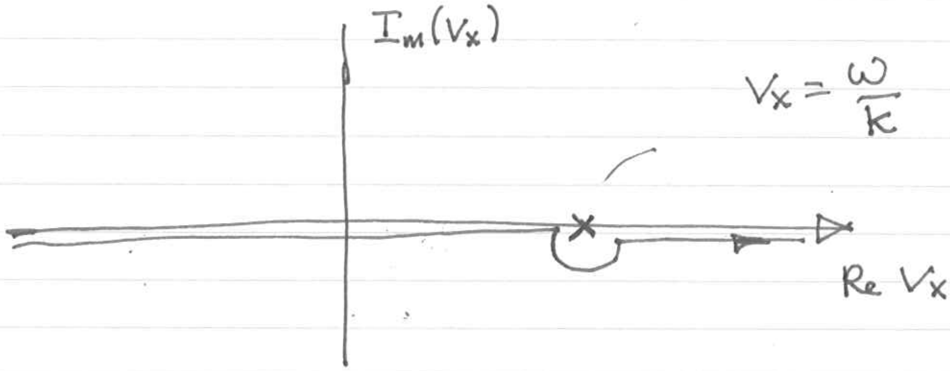
what happens when $\omega = \underset{\sim}{k} \cdot \underset{\sim}{v}$?

Integral is singular

Focus on 1D

$$\epsilon = 1 + \frac{4\pi q^2}{m k^2} \int d v_x \frac{k\, \partial f_0 / \partial v_x}{\omega - k v_x}$$

SHORT ANSWER MUST PEFORM
CONTOUR OF INTEGRATION



$V_x = \frac{\omega}{k}$ doppler ~~th~~ shift = 0

ϵ becomes complex

dissipation — waves are damped

Landau damping