Relation to Fluid Mechanics

Moments of the Vlasov Equation

Take VE & integrate over all velocity

$$\int d^3v \left[\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}} f + \frac{q}{m} \underset{\sim}{F} \cdot \frac{\partial f}{\partial \underset{\sim}{v}} \right] = \int d^3v\, C(f)$$

Because Flow in phase space is incompressible $\quad \parallel$

$$\int d^3v \left[\frac{\partial f}{\partial t} + \frac{\partial}{\partial \underset{\sim}{x}} \cdot (\underset{\sim}{v} f) + \frac{\partial}{\partial \underset{\sim}{v}} \frac{q}{m} (\underset{\sim}{F} f) \right] = 0$$

Collisions do not create or destroy particles

do by parts

$$\frac{\partial}{\partial t} \int d^3x\, f + \frac{\partial}{\partial \underset{\sim}{x}} \cdot \int d^3v (\underset{\sim}{v} f) + \underbrace{\int_{S_\infty} d^2v\, \underset{\sim}{n} \cdot \frac{q}{m} f}_{0} = 0$$

$f \to 0$ as $|\underset{\sim}{v}| \to \infty$

$$\frac{\partial}{\partial t} \int d^3v\, f + \frac{\partial}{\partial \underset{\sim}{x}} \cdot \int d^3v\, \underset{\sim}{v} f = 0$$

what is this?

ans: continuity equation

$n(\mathbf{x}, t) = \int d^3 \cancel{x}\, f$ particle density

$\mathbf{u}(\mathbf{x}, t) = \dfrac{\int d^3 v\, \mathbf{v} f}{\int d^3 v\, f}$ average velocity of particles at x, t

continuity

$$\boxed{\frac{\partial n}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot n\mathbf{u} = 0}$$

consider $m\mathbf{v}$ moment

$$\int d^3 v\, m\mathbf{v} \left[\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{v} f + \frac{\partial}{\partial \mathbf{v}} \cdot \left(\frac{\mathbf{F}}{m} f \right) \right]$$

$$= \int d^3 x\, m\mathbf{v}\, C(f)$$

$\parallel$

0

collisions conserve momentum

$$\frac{\partial}{\partial t}\int d^3v\, m\underset{\sim}{v} f + \frac{\partial}{\partial \underset{\approx}{x}} \cdot \int d^3v\, m\underset{\sim}{v}\underset{\sim}{v} f$$

$$- \int d^3v \left(\frac{\underset{\approx}{F}}{m} \cdot \frac{\partial}{\partial \underset{\approx}{v}} m\underset{\approx}{v} \right) = 0$$

$$\frac{\partial \underset{\sim}{v}}{\partial \underset{\sim}{v}} = \underline{I}$$

$$\frac{\partial}{\partial t} mn\underset{\approx}{u} + \frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\approx}{\Pi} - \int d^3v\, \underset{\approx}{F} f$$

$$\left(F_x \frac{\partial}{\partial v_x} + F_y \frac{\partial}{\partial v_y} + F_z \frac{\partial}{\partial v_z} \right) m\underset{\approx}{v}$$

unit vectors

$$\underset{\sim}{v} = (\hat{a}_x v_x + \hat{a}_y v_y + \hat{a}_z v_z)$$

$$\underset{\approx}{\Pi} = \int d^3v\, m\underset{\sim}{v}\underset{\sim}{v} f$$

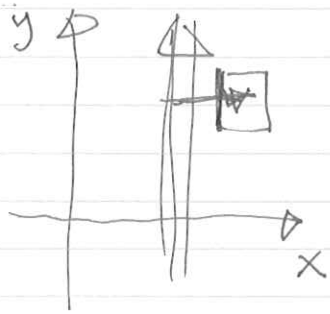
momentum flux tensor

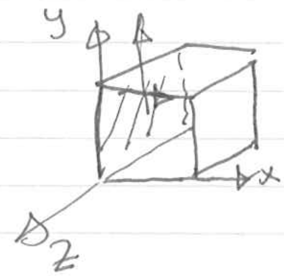
$$\begin{bmatrix} mv_xv_x & mv_xv_y & mv_xv_z \\ mv_yv_x & mv_yv_y & + mv_yv_z \\ & & \\ mv_zv_x & + mv_zv_y & + mv_{zz} \end{bmatrix}$$

symmetric tensor

Π_{xy} = the momentum density in y direction that is being "transported" in the x-direction

flow in y moving in x





box in x, y z s

momentum flux

$$\int d^3v \, \underset{\sim}{F} f = \text{Rate of input of momentum density due to } \underset{\sim}{F}$$

$$\underset{\sim}{F} = q\left(\underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c}\right)$$

Then $\displaystyle\int d^3v \, \underset{\sim}{F} f = \left(q \underset{\sim}{E} \int d^3v f + \int d^3v \, q \frac{\underset{\sim}{v} f \times \underset{\sim}{B}}{c} \right)$

$$= qn\left(\underset{\sim}{E} + \frac{\underset{\sim}{u} \times \underset{\sim}{B}}{c}\right)$$

conservation of momentum densit

$$\frac{\partial}{\partial t} n\underset{\sim}{u} + \frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\approx}{\Pi} - qn\left(\underset{\sim}{E} + \frac{\underset{\sim}{u} \times \underset{\sim}{B}}{c}\right) = 0$$

Lets look in more detail at $\underset{=}{\Pi}$

$$\underset{\approx}{\Pi} = \int d^3v \, m\underset{\sim}{v}\underset{\sim}{v} f$$

second moment of f

Pattern emerging

$n = \int d^3v \, f$ zero moment

continuit

$$\frac{\partial}{\partial t} n + \frac{\partial}{\partial x} n\underset{\sim}{u} = 0$$

$n\underset{\sim}{u} = \int d^3v \, \underset{\sim}{v} f$ first moment

$\underset{=}{\Pi} = \int d^3v \, m\underset{\sim}{v}\underset{\sim}{v} f$

second mom

$$\frac{\partial}{\partial t} mn\underset{\sim}{u} + \frac{\partial}{\partial x} \cdot \underset{=}{\Pi}$$

$$\frac{\partial}{\partial t} \underset{\approx}{\Pi} + \frac{\partial}{\partial \underset{\sim}{x}} \cdot \#$$ third moment

$\vdots$

for ever.

system does not close $\not\int$

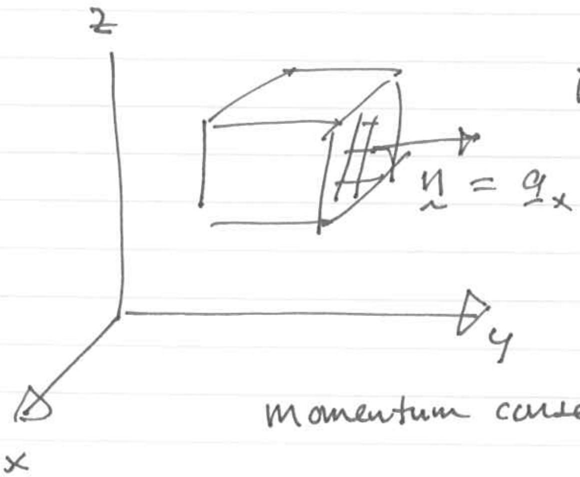
closure requires approximation — spec of f what does $\underline{\underline{\Pi}}$ momentum flux tensor

(stress tensor)

$$\frac{\partial}{\partial t} m n \underset{\sim}{u} + \nabla \cdot \underline{\underline{\Pi}} = \bar{\underset{\sim}{F}} n$$

$$= \underbrace{q n}_{\#\ \text{density}} \overbrace{\left(\underset{\sim}{E} + \frac{\dot{\underset{\sim}{u}} \times \underset{\sim}{B}}{c} \right)}^{\text{average force on local fluid parti.}}$$

units force density = Force / volume



integrate over a volume in physical space

$(\underset{\sim}{a}_x \cdot \underline{\underline{\Pi}})$

momentum conservation

$$\underbrace{\frac{d}{dt} \int d^3x \, (m n \underset{\sim}{u})}_{\substack{\text{Rate of change of} \\ \text{momentu in particles} \\ \text{in volume}}} + \underbrace{\int d^2x \, \overset{\text{outward normal}}{\underset{\sim}{n}} \cdot \underline{\underline{\Pi}}}_{\substack{\text{Rate at which momentum} \\ \text{is transported} \\ \text{(throug surface} \\ \text{out of volume}}} = \underbrace{\int d^3x \, \bar{\underset{\sim}{F}} n}_{\substack{\text{Rate of input of} \\ \text{momentum from} \\ \text{E\&M field}}}$$

$$\underset{\approx}{\Pi} = \int d^3v \, m(\underset{\sim}{v}\underset{\sim}{v}) f$$

average velocity

$$\underset{\sim}{v} = \underset{\sim}{u} + \delta \underset{\sim}{v}$$

deviation

$$\int d^3v \, \delta \underset{\sim}{v} f = 0$$

$\delta \underset{\sim}{v} = \underset{\sim}{v} - \underset{\sim}{u}$ (change of variables)

$$\underset{\approx}{\Pi} = \int d^3v \, m(\underset{\sim}{u} + \delta\underset{\sim}{v})(\underset{\sim}{u} + \delta\underset{\sim}{v}) f$$

$$= m\overset{n}{\underset{\sim}{u}\underset{\sim}{u}} + \underbrace{\int d^3v \, m \, \delta\underset{\sim}{v}\delta\underset{\sim}{v} f}_{\text{"small pi"}}$$

$\underset{\approx}{\pi}$ stress tensor

$$\nabla\cdot\underset{=}{\Pi} = \nabla\cdot(mn\underset{\sim}{u}\underset{\sim}{u}) + \nabla\cdot\underset{=}{\pi}$$

$$\frac{\partial}{\partial \underset{\sim}{x}} \cancel{\nabla}\cdot(m\underset{\sim}{u}\underset{\sim}{u}n) = \sum_i \frac{\partial}{\partial x_i}(nmu_i\underset{\sim}{u})$$

$$= \underset{\sim}{u} \frac{\partial}{\partial x_i}(nmu_i) + nmu_i \frac{\partial}{\partial x_i}\underset{\sim}{u}$$

$$\frac{\partial}{\partial t}(mn\underset{\sim}{u}) = mn\frac{\partial u}{\partial t} + m\underset{\sim}{u}\frac{\partial n}{\partial t}$$

ρ_m = mass density

$$m\underset{\sim}{u}\underbrace{\left(\frac{\partial n}{\partial t} + \frac{\partial}{\partial \underset{\sim}{x}}\cdot(n\underset{\sim}{u})\right)}_{0} + mn\left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\frac{\partial}{\partial \underset{\sim}{x}}\underset{\sim}{u}\right)$$

$$= -\frac{\partial}{\partial \underset{\sim}{x}}\cdot\underset{\approx}{\pi} + qn\left(\underset{\sim}{E} + \frac{\underset{\sim}{u}\times\underset{\sim}{B}}{c}\right)$$

$$\begin{bmatrix} \Pi_{xx} & \Pi_{xy} & \Pi_{xz} \\ \Pi_{yx} & \Pi_{yy} & \Pi_{yz} \\ \Pi_{zx} & \Pi_{zy} & \Pi_{zz} \end{bmatrix}$$

$(\underset{\sim}{n} \cdot \underline{\underline{\Pi}})_x = \cancel{(\Pi_{xx}, \Pi_{xy}, \Pi_{yz})}$ rate

Π_{xx} rate at which x-component of momentum density is flowing in $\underset{\sim}{n}$-direction.

$(\underset{\sim}{n} \cdot \underline{\underline{\Pi}})_y =$ rate at which y-component of momentum is flowing in x-direction

Closure

remember that collisions drive f to a maxwell Boltzmann distribution when collisions are strong enough

$\cancel{f}$.

$$f = \frac{n}{(2\pi T/m)^{3/2}} \exp\left[- \frac{\frac{1}{2} m (\underbrace{\underset{\sim}{v} - \underset{\sim}{u}}_{\delta \underset{\sim}{v}})^2}{T} \right] + \delta f$$

$\boxed{n, T, \underset{\sim}{u} \text{ still depend on space}}$ Local TE

small correction $\propto (\frac{1}{\nu})$

$$\underset{\approx}{\Pi} = \int d^3v \, m \begin{bmatrix} \delta v_x \delta v_x & \delta v_x \delta v_y & \delta v_x \delta v_z \\ \delta v_y \delta v_x & \delta v_y \delta v_y & \delta v_y \delta v_z \\ \delta v_z \delta v_x & \delta v_z \delta v_y & \delta v_z \delta v_z \end{bmatrix} \frac{n}{(2\pi T/m)^{3/2}} e^{-\frac{\frac{1}{2} m (\delta v_x^2 + \delta v_y^2 + \delta v_z^2)}{T}}$$

$+ \delta \underset{\approx}{\Pi}$ – small correction off diagonal terms are zero

diagonal terms are equal

$$2\int d^3v\, \underbrace{\tfrac{1}{2}m\, \delta v_x^2 \frac{n}{(2\pi T/m)^{3/2}}}_{Tn} \exp\left(-\frac{\frac{1}{2}m(\delta v_x^2+\delta v_y^2+\delta v_z^2)}{T}\right)$$

$= nT = p$ ideal gas presure

Navier-Stokes E

small stess due to dep. from TE

$$\rho_m\left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{u}\right) = -\nabla p - \nabla\cdot\delta\underset{\approx}{\Pi} + qn\left(\underset{\sim}{E} + \frac{\underset{\sim}{u}\times\underset{\sim}{B}}{c}\right)$$

for a simple fluid $\delta\underset{\approx}{\Pi} \propto \frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{x}} + \left(\frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{x}}\right)^{T}$

$\nabla\cdot\delta\underset{\approx}{\Pi}$ "viscous stress"

In a plasma "thermal force"

Energy Moment

We don't multiply by $m\underset{\sim}{v}\underset{\sim}{v}$ and integrate.

Why? collisions don't conserve this quantity

$$\int d^3v\, m\underset{\sim}{v}\underset{\sim}{v}\, C(f) \neq 0$$

However, for elastic collisions

$$\int d^3v\, \tfrac{1}{2} m v^2\, C(f) = 0$$

Multiply VE by $\frac{1}{2}mv^2$ & integrate

Result

$$\frac{\partial}{\partial t}\left(\rho_m \tfrac{1}{2}u^2 + \tfrac{3}{2}p\right) + \frac{\partial}{\partial \underset{\sim}{x}}\cdot\left(\rho_m \frac{u^2}{2}\underset{\sim}{u} + \tfrac{3}{2}p\underset{\sim}{u}\right)$$

$$= -\nabla\cdot\left(\overset{pi}{\tilde{\underset{\approx}{\Pi}}}\cdot\underset{\sim}{u}\right) + qn\underset{\sim}{u}\cdot\left(\underset{\sim}{E} + \frac{\underset{\sim}{u}\times\underset{\sim}{B}}{c}\right) - \frac{\partial}{\partial\underset{\sim}{x}}\cdot\underset{\sim}{q}$$

$$p = \frac{1}{3}\int d^3v\,(mv_x^2 + mv_y^2 + mv_z^2)\, f = \frac{1}{3}\,\mathrm{TR}(\delta\underset{\approx}{\Pi})$$

Result Energy Conservation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_m u^2 + \frac{3}{2}p\right) + \frac{\partial}{\partial \underset{\sim}{x}}\cdot\left[\frac{1}{2}\rho_m \underset{\sim}{u}\, u^2 + \frac{3}{2}\underset{\sim}{u}\, p + \underset{\sim}{u}\cdot\underset{\approx}{\Pi} + \underset{\sim}{q}\right]$$

$$= q n \underset{\sim}{u}\cdot \underset{\sim}{E}$$ ← rate at which ~~energy~~ fields do work on particles

where

$$= \underset{\sim q}{j}\cdot \underset{\sim}{E}$$

~~$\frac{1}{2}\rho_m u^2 + \frac{3}{2}p$~~

$$\frac{3}{2}p = \int d^3v\, \frac{1}{2} m\, \delta\underset{\sim}{v}\cdot\delta\underset{\sim}{v}\, f = \frac{1}{2}\mathrm{Tr}\{\underset{\approx}{\Pi}\} = \frac{3}{2}\ \text{pressure}$$

$$\underset{\sim}{q} = \int d^3v\, \frac{1}{2} m\, \delta\underset{\sim}{v}\, \delta v^2 f = \text{Heat Flux density}$$

$$\left(\frac{1}{2}\rho_m u^2 + \frac{3}{2}p\right) = \text{Fluid energy density}$$

$\frac{1}{2}\rho_m u^2 =$ Kinetic energy density associated with mean flow $\underset{\sim}{u}$

$\frac{3}{2}p =$ internal energy of ideal gas (energy associated w random motion)

Sign Errors in problem

Should be $X = x + V_y/\Omega \quad Y = y - V_x/\Omega$

Moments of VE - Boltzmann Eqn

$$\int d^3v \begin{pmatrix} 1 \\ m\underset{\sim}{v} \\ \frac{1}{2}mv^2 \end{pmatrix} \left(\frac{\partial f}{\partial t} + \underset{\sim}{v}\cdot\nabla f + \frac{\underset{\sim}{F}}{m}\cdot\frac{\partial f}{\partial \underset{\sim}{v}} \right) = \int d^3x \begin{pmatrix} 1 \\ m\underset{\sim}{v} \\ \frac{1}{2}mv^2 \end{pmatrix} C(f)$$

First mass - continuity

$$\frac{\partial n}{\partial t} + \nabla\cdot n\underset{\sim}{u} = 0 \qquad \text{continuity}$$

Second — momentum density balance

$$\frac{\partial}{\partial t} nm\underset{\sim}{u} + \nabla\cdot\underset{=}{\Pi} - \int \underset{\sim}{F} f\, d^3v = 0$$

$$\underset{=}{\Pi} = mn\underset{\sim}{u}\underset{\sim}{u} + \underset{=}{\mathrm{II}} \qquad \underset{=}{\mathrm{II}} = \int d^3\delta v\, m\,\delta\underset{\sim}{v}\,\delta\underset{\sim}{v}\, f$$

stress tensor

~~rewrite momentum be~~

Can use first two equations to rewrite

$$mn\left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{u}\right) = -\frac{\partial}{\partial \underset{\sim}{x}}\cdot\underline{\underline{\Pi}} + \underbrace{\int \underset{\sim}{F} f\, d^3v}_{qn\left(\underset{\sim}{E} + \frac{\underset{\sim}{u}\times\underset{\sim}{B}}{c}\right)}$$

---

if $\quad f = \dfrac{n}{(2\pi T/m)} \exp\left[-\dfrac{\frac{1}{2}m\,\delta v^2}{T}\right]$

Local maxwellian

identity tensor

$$\underline{\underline{\Pi}} = nT\,\underline{\underline{I}}$$

pressure $\quad p = nT \quad$ ideal gas

$$-\frac{\partial}{\partial \underset{\sim}{x}}\cdot\underline{\underline{\Pi}} = -\nabla(nT) = -\nabla p$$

Chapman - Enskog Expansion

Choudhury Sec, 3.4

~~Applies to gases and fluids~~

Applies when collisions dominate

$$\frac{\partial}{\partial t} f + \underset{\sim}{v} \cdot \nabla f + \frac{\underset{\sim}{F}}{m} \cdot \frac{\partial f}{\partial \underset{\sim}{v}} = C(f)$$

$$= -\nu (f - f_m)$$

maxwellian

collision frequency

In some cases collision rate is higher than rate for other processes

$$f = f_m + \underset{\delta f}{\text{small correction}}$$

Equation for small correcti.

$$\frac{\partial f_m}{\partial t} + \underset{\sim}{v} \cdot \nabla f_m + \frac{\underset{\sim}{F}}{m} \cdot \frac{\partial f_m}{\partial \underset{\sim}{v}} = -\nu \, \delta f$$

$\underset{\sim}{\delta f} \quad \propto \quad \nabla T \;\; \& \;\; \left(\frac{\partial u_i}{\partial x_j}\right)$ gradients in velocity

Thermal conduction

Viscous stress

$$\Lambda_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\delta\Pi_{ij} = -2\mu\left(\Lambda_{ij} - \frac{1}{3}\delta_{ij}\,\nabla\cdot\underline{u}\right)$$

$$\mu = \frac{Tn}{\nu} \quad \text{viscosity}$$

collision rate

Separate energy associated with mean flow

Momentum Equation

$$\underset{\sim}{u} \cdot \left[\frac{\partial}{\partial t} \left(\rho_m \underset{\sim}{u} \right) + \frac{\partial}{\partial \underset{\sim}{x}} \cdot \left(\rho_m \underset{\sim}{u}\underset{\sim}{u} + \underset{\approx}{\Pi} \right) \right] = qn \underset{\sim}{E} \cdot \underset{\sim}{u}$$

+ Some tricks

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_m u^2\right) + \frac{\partial}{\partial \underset{\sim}{x}} \cdot \left(\frac{1}{2}\rho_m u^2 \underset{\sim}{u}\right)$$

$$+ \left(\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\approx}{\Pi}\right) \cdot \underset{\sim}{u} = qn \underset{\sim}{E} \cdot \underset{\sim}{u}$$

SUBTRACT FROM Energy Conservation

$$\frac{\partial}{\partial t} \frac{3}{2} p + \frac{\partial}{\partial \underset{\sim}{x}} \cdot \left(\underset{\sim}{u} \frac{3}{2} p\right) + \frac{\partial}{\partial \underset{\sim}{x}} \cdot \left(\underset{\sim}{u} \cdot \underset{\approx}{\Pi} + q\right) - \left(\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\approx}{\Pi}\right) \cdot \underset{\sim}{u} = 0$$

isotropic pressu

VISCOUS STRESS

let $\underset{\approx}{\Pi} = p \underset{\approx}{I} + \delta \underset{\approx}{\Pi}$

70

$$\frac{\partial}{\partial \underset{\sim}{x}} \cdot (\underset{\approx}{\Pi} \cdot \underset{\sim}{u}) - \left(\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\approx}{\Pi}\right) \cdot \underset{\sim}{u} = \left(\underset{\approx}{\Pi} : \frac{\partial}{\partial \underset{\sim}{x}} \underset{\sim}{u}\right)$$

$$= \sum_{ij} \Pi_{ij} \frac{\partial}{\partial x_i} u_j$$

Internal energy

$$\frac{\partial}{\partial t} \frac{3}{2} p + \frac{\partial}{\partial \underset{\sim}{x}} \cdot \left(\underset{\sim}{u} \frac{3}{2} p\right) + \underbrace{\underset{\approx}{\Pi} : \frac{\partial}{\partial \underset{\sim}{x}} \underset{\sim}{u}}_{\text{viscous heatn}} + \underbrace{\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{q}}_{\text{heat flux}} = 0$$

viscous stress

if $\underset{\approx}{\Pi} = p \underset{\approx}{I} + \delta \underset{\approx}{\Pi}$ near Local TE

$$\underset{\approx}{\Pi} : \frac{\partial}{\partial \underset{\sim}{x}} \underset{\sim}{u} = \underbrace{p \frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{u}}_{pdV \sim \text{work done compressing gas}} + \underbrace{\delta \underset{\approx}{\Pi} : \frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{x}}}_{\text{heatin due to viscous stress}}$$

pdV ~ work

done compressing gas

heatin due to viscous stress

special case $\quad \delta\underline{\underline{\pi}} \to 0 \quad \underset{\sim}{q} \to 0$

no viscous stress, no heat flow (adiabatic)

$$\frac{\partial}{\partial t}\frac{3}{2}P + \frac{\partial}{\partial \underset{\sim}{x}}\frac{3}{2}\underset{\sim}{u}P + P\left(\frac{\partial}{\partial \underset{\sim}{x}}\cdot\underset{\sim}{u}\right) = 0$$

where did "3" come from

$3 = n_f$ number of degrees of freedom

group terms

$$\frac{n_f}{2}\left(\frac{\partial}{\partial t} + \underset{\sim}{u}\cdot\frac{\partial}{\partial x}\right)P + \left(\frac{n_f+2}{2}\right)P\frac{\partial}{\partial \underset{\sim}{x}}\cdot\underset{\sim}{u} = 0$$

$$\frac{dP}{dt}$$

$$\gamma_s = \frac{n_f+2}{n_f} = \frac{5}{3}$$

$$\frac{1}{P}\frac{dP}{dt} + \gamma_s\frac{\partial}{\partial \underset{\sim}{x}}\cdot\underset{\sim}{u} = 0$$

continuity

$$\frac{1}{\rho_m}\frac{d\rho_m}{dt} + \frac{\partial}{\partial \underset{\sim}{x}}\cdot\underset{\sim}{u} = 0$$

$$\frac{1}{P}\frac{dP}{dt} = -\gamma_s\frac{\partial}{\partial \underset{\sim}{x}}\cdot\underset{\sim}{u} = \gamma_s\frac{1}{\rho_m}\frac{d\rho_m}{dt}$$

Can integrate

$$\ln P = \gamma_s \ln \rho_m + \text{const}$$

$$\frac{P}{P_0} = \left(\frac{\rho_m}{\rho_{m0}}\right)^{\gamma_s}$$

P_0 = initial pressu

ρ_m = initial mass density

γ_s = ratio of specific heats $\quad \left.\frac{dT}{dQ}\right|_V \quad \left.\frac{dT}{dQ}\right|_P$

$$\boxed{\gamma_s = \frac{n_f+2}{n_f}}$$