Lecture 3 - Vlasov Theory

PHYS 761

In this lecture we will discuss the Vlasov equation, the equation that describes the evolution of a distribution function in phase space in the absense of collisions.

We will use the following references

- Chapters 4-6 of Nicholson *Introduction to Plasma Theory*
- Chapters 22-23 of Goldston and Rutherford *Introduction to Plasma Physics*

Recall that the distribution function describes the density of particles in phase space $(\boldsymbol{x}, \boldsymbol{v})$. The number of particles contained within the differential phase space volume d^3xd^3v is given by

$$dN = f(\boldsymbol{x}, \boldsymbol{v}, t)d^3xd^3v. \tag{1}$$

Our goal will be to obtain an evolution equation for the distribution function, $f(\boldsymbol{x}, \boldsymbol{v}, t)$. We will find that

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{x}} + \frac{q}{m}\left(\boldsymbol{E} + \frac{\boldsymbol{v} \times \boldsymbol{B}}{c}\right) \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0. \tag{2}$$

This is the Vlasov equation.

1. Distribution function picture

Note that when discussing the distribution function, $\boldsymbol{x}$, $\boldsymbol{v}$, and t are treated as independent variables. This is opposed to the classical description of a trajectory $\boldsymbol{x}(t)$, $\boldsymbol{v}(t)$ where t is an independent variable while $\boldsymbol{x}$ and $\boldsymbol{v}$ are dependent variables.

Question: If the distribution function is known, does this provide the equivalent information as the single particle picture? No, it does not. Suppose a system consists of N particles. The complete state of the system is given by 6N variables, the position and velocity of each particle. We will instead apply a statistical treatment. For most problems, we are not interested in distinguishing between particles that have the same position and velocity. Furthermore, we can average over distances of the order of the Debye length, ignoring binary collisions between particles.

We really aren't interested in tracking each individual particle, as they are indistinguishable. Suppose we were to adopt a statistical picture of the 6N particles,

$$dP = f_N\left(\boldsymbol{x}_1, \boldsymbol{v}_1, ..., \boldsymbol{x}_N, \boldsymbol{v}_N, t\right) d^3x_1d^3v_1....d^3x_Nd^3v_N, \tag{3}$$

here dP is the probability of finding particle 1 in phase space volume $d^3x_1d^3v_1$ up to particle N in phase space volume $d^3x_Nd^3v_N$. Here we assume that each particle is distinguishable, and f_N is the N-particle distribution function.

In practice this is much more information than we need. If we make the assumption that particles are weakly interacting, then we can reduce this picture to the familiar distribution function which provides the density of *all* particles in phase space. This is valid under the following assumptions.

- In a dilute gas, particles quickly become uncorrelated. Collisions are infrequent, and particles which collide are unlikely to collide again.
- Single inter-particle interactions have small impact. Rather, a given particle interacts with many particles at once.
- A particle must travel a long distance before its next collision, at which point its correlation due to the past collision is lost. Consider the ratio of the mean free path to the typical inter-particle spacing. The size of a typical atom (e.g. hydrogen) is the Bohr radius, a_0. We define the mean free path, $\lambda_{\rm mfp}$, as the distance that a particle travels before colliding. The number of particles within the cylindrical volume of length $\lambda_{\rm mfp}$ is 1,

  $$n a_0^2 \lambda_{\rm mfp} = 1,$$

  so the ratio of $\lambda_{\rm mfp}$ to the typical particle spacing, $L = n^{1/3}$, is

  $$\frac{\lambda_{\rm mfp}}{L} = \frac{L^3}{a_0^2 L} \gg 1.$$

  As discussed in Lecture 2, this is typically a very large number for plasma parameters of interest. A more quantitative calculation of the mean free path will be consider in our lectures on collisions.

Consider the single particle distribution function, $f_1(\boldsymbol{x}_1, \boldsymbol{v}_1, t)$, such that the probability that a given particle is in a region of phase space is given by

$$dP = f_1(\boldsymbol{x}_1, \boldsymbol{v}_1, t) d^3x_1 d^3v_1. \tag{4}$$

A two particle distribution function, $f_2(\boldsymbol{x}_1, \boldsymbol{v}_1, \boldsymbol{x}_2, \boldsymbol{v}_2, t)$, can be similarly defined such that

$$dP = f_2(\boldsymbol{x}_1, \boldsymbol{v}_1, \boldsymbol{x}_2, \boldsymbol{v}_2, t) d^3x_1 d^3v_1 d^3x_2 d^3v_2 \tag{5}$$

describes the joint probability that particle 1 is in phase space volume $d^3x_1 d^3v_1$ and particle 2 is in phase space volume $d^3x_2 d^3v_2$.

Under the assumption of statistical independence, the two-particle distribution function can be written as

$$f_2(\boldsymbol{x}_1, \boldsymbol{v}_1, \boldsymbol{x}_2, \boldsymbol{v}_2, t) = f_1(\boldsymbol{x}_1, \boldsymbol{v}_1, t) f_1(\boldsymbol{x}_2, \boldsymbol{v}_2, t). \tag{6}$$

This can similarly be done for the N-particle distribution function,

$$f_N(\boldsymbol{x}_1, \boldsymbol{v}_1, ..., \boldsymbol{x}_N, \boldsymbol{v}_N, t) = f_1(\boldsymbol{x}_1, \boldsymbol{v}_1, t) ... f_1(\boldsymbol{x}_N, \boldsymbol{v}_N, t). \tag{7}$$

The distribution function which describes density in phase space (1) is related via

$$f(\boldsymbol{x}, \boldsymbol{v}, t) = N f_1(\boldsymbol{x}, \boldsymbol{v}, t). \tag{8}$$

We therefore recover the expected number of paticles

$$N = \int d^3v\, f = N \int d^3v\, f_1 \tag{9}$$

The Vlasov equation can be obtained by considering the equations satisfied by the single particle distribution function (see Nicholson chapter 4). In this lecture we will instead use a more heuristic argument.

2. Hueristic derivation of the Vlasov equation

The distribution function satisfies a conservation equation, the Vlasov equation.

For comparison, we will consider number conservation in 3D. Consider the total number of particles in a fluid bounded within a volume, N_V, in 3D space. As particles cannot be created or destroyed (neglecting any atomic processes), N_V can only change if there is a flux of particles into the volume,

$$\frac{\partial N_V}{\partial t} = -\int_{\partial V} d^2x\, n\boldsymbol{u}\cdot\hat{\boldsymbol{n}}, \tag{10}$$

where $\hat{\boldsymbol{n}}$ is the outward unit normal on V,

$$N_V = \int_V d^3x\, n, \tag{11}$$

and $\boldsymbol{u}$ is the fluid velocity. The condition (10) can now be expressed as

$$\frac{\partial}{\partial t}\left(\int_V d^3x\, n\right) + \int_V d^3x\, \nabla\cdot(\boldsymbol{u}n) = 0 \tag{12}$$

using the divergence theorem. As this is true for any volume, we have the following conservation equation for the density,

$$\frac{\partial n}{\partial t} + \nabla\cdot(\boldsymbol{u}n) = 0. \tag{13}$$

We would like to write a similar expression for the conservation of the distribution function. The total number of particles in a phase space volume, $\mathcal{V}$,

$$N_{\mathcal{V}} = \int_{\mathcal{V}} d^3x d^3v\, f(\boldsymbol{x},\boldsymbol{v},t), \tag{14}$$

should only change due to a flux of particles through $\partial\mathcal{V}$,

$$\frac{\partial N_{\mathcal{V}}}{\partial t} = -\int_{\partial\mathcal{V}} dS\, \hat{\boldsymbol{n}}\cdot\boldsymbol{U}, \tag{15}$$

where $\boldsymbol{U}$ is a 6D velocity in phase space and dS is a 5D surface area element bounding $\mathcal{V}$. We can apply the divergence theorem to write

$$\int_{\mathcal{V}} d^3x d^3v \left(\frac{\partial f}{\partial t} + \frac{\partial}{\partial \boldsymbol{x}}\cdot(\boldsymbol{U}f) + \frac{\partial}{\partial \boldsymbol{v}}\cdot(\boldsymbol{U}f)\right) = 0. \tag{16}$$

As this must be true for any volume $\mathcal{V}$, we find that

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \boldsymbol{x}}\cdot(\boldsymbol{U}f) + \frac{\partial}{\partial \boldsymbol{v}}\cdot(\boldsymbol{U}f) = 0, \tag{17}$$

must be satisfied. The velocity in phase space is

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{F}/m \end{bmatrix}, \tag{18}$$

as

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{v} \end{bmatrix} = \boldsymbol{U}. \tag{19}$$

So we find

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + \frac{\boldsymbol{F}}{m}\cdot\frac{\partial f}{\partial \boldsymbol{v}} + f\frac{\partial}{\partial \boldsymbol{v}}\cdot\frac{\boldsymbol{F}}{m} = 0. \tag{20}$$

Here $\boldsymbol{F} = q\,(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}/c)$. We can note that the velocity-space divergence of $\boldsymbol{F}$ vanishes, using the vector identity $\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot\nabla\times\boldsymbol{A} - \boldsymbol{A}\cdot\nabla\times\boldsymbol{B}$,

$$\frac{\partial}{\partial \boldsymbol{v}}\cdot\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right) = \frac{\boldsymbol{B}}{c}\cdot\nabla\times\boldsymbol{v} = 0. \tag{21}$$

So we arrive at the Vlasov equation:

$$\boxed{\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + \frac{\boldsymbol{F}}{m}\cdot\frac{\partial f}{\partial \boldsymbol{v}} = 0} \tag{22}$$

The left hand side represents to the total time derivative of f along a test particles orbit with position $\boldsymbol{x}(t)$ and velocity $\boldsymbol{v}(t)$,

$$\left.\frac{df}{dt}\right|_{\text{orbit}} = \frac{\partial f}{\partial t} + \frac{d\boldsymbol{x}(t)}{dt}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + \frac{d\boldsymbol{v}(t)}{dt}\cdot\frac{\partial f}{\partial \boldsymbol{v}} = 0. \tag{23}$$

Patches of phase space density can be moved around and carry their values of f with them.

Note that the electro-magnetic fields appear in the Vlasov equation through $\boldsymbol{F}$. In principle, the time evolution of f must be coupled to the time-evolution of the fields through Maxwell's equations.

$$\nabla\cdot\boldsymbol{E} = 4\pi\sum_s q_s n_s = 4\pi\sum_s q_s\int d^3v\, f_s(\boldsymbol{x},\boldsymbol{v},t) \tag{24}$$

$$\nabla\times\boldsymbol{B} = \frac{4\pi}{c}\sum_s q_s n_s \boldsymbol{u}_s + \frac{\partial \boldsymbol{E}}{\partial t} = \frac{4\pi}{c}\sum_s q_s\int d^3v\, f_s(\boldsymbol{x},\boldsymbol{v},t)\boldsymbol{v} + \frac{\partial \boldsymbol{E}}{\partial t} \tag{25}$$

The Vlasov equation must be solved for each species, s, coupled with solutions for $\boldsymbol{E}$ and $\boldsymbol{B}$.

What assumptions have we made along the way?

- We have smoothed over any discrete particle effects. Thus we have neglected collisions between particles that occur on length scales shorter than λ_D. This averaging is valid under the assumption that the number of particles in a Debye sphere is very large, $n\lambda_D^3 >> 1$, or the plasma parameter $\Lambda > 1$.
- We have ignored any radiation, such as Bremstrahlung or cyclotron.

Note that we have found that the 6D divergence of the phase space velocity vanishes,

$$\frac{\partial}{\partial \boldsymbol{x}}\cdot\boldsymbol{U} + \frac{\partial}{\partial \boldsymbol{v}}\cdot\boldsymbol{U} = 0. \tag{26}$$

This will have important conservation consequences. For example, we can show that the total number of particles is conserved,

$$\begin{aligned}\frac{d}{dt}\left(\int_{\mathcal{V}} d^3x d^3v\, \boldsymbol{v} f(\boldsymbol{x},\boldsymbol{v},t)\right) &= \int_{\mathcal{V}} d^3x d^3v\left(\boldsymbol{v}\cdot\frac{\partial(\boldsymbol{v}f)}{\partial \boldsymbol{x}} + \frac{\boldsymbol{F}}{m}\cdot\frac{\partial(\boldsymbol{v}f)}{\partial \boldsymbol{v}}\right)\\ &= \int_{\mathcal{V}} d^3x d^3v\, \nabla_{\boldsymbol{x},\boldsymbol{v}}\cdot(\boldsymbol{U}\boldsymbol{v}f)\\ &= \int_{\partial\mathcal{V}} dS\,\hat{\boldsymbol{n}}\cdot\boldsymbol{U}\boldsymbol{v}f, \end{aligned} \tag{27}$$

if the flux through the boundary, $\hat{\boldsymbol{n}}\cdot\boldsymbol{U}\boldsymbol{v}f$, vanishes. Here $\nabla_{\boldsymbol{x},\boldsymbol{v}}$ is the 6D gradient.

3. Properties of Vlasov solutions

Suppose have knew the solutions for all trajectories.

$$\boldsymbol{x}_T(t;\boldsymbol{x}_0,\boldsymbol{v}_0) \tag{28}$$

$$\boldsymbol{v}_T(t;\boldsymbol{x}_0,\boldsymbol{v}_0), \tag{29}$$

where $\boldsymbol{x}_0$, $\boldsymbol{v}_0$ are the initial conditions, which satisfy

$$\frac{d\boldsymbol{x}_T(t)}{dt} = \boldsymbol{v}_T(t) \tag{30}$$

$$\frac{d\boldsymbol{v}_T(t)}{dt} = \frac{\boldsymbol{F}(\boldsymbol{x}_T(t),\boldsymbol{v}_T(t))}{m} \tag{31}$$

with $\boldsymbol{x}_T(t=0)=\boldsymbol{x}_0$ and $\boldsymbol{v}_T(t=0)=\boldsymbol{v}_0$.

Let $f_0(\boldsymbol{x}_0,\boldsymbol{v}_0)$ be the initial distribution function. If we know know $\boldsymbol{x}_T$, $\boldsymbol{v}_T$ as a function of $\boldsymbol{x}_0$, $\boldsymbol{v}_0$,

$$\boldsymbol{x}_0(t;\boldsymbol{x}_T,\boldsymbol{v}_T) \tag{32}$$

$$\boldsymbol{v}_0(t;\boldsymbol{x}_T,\boldsymbol{v}_T), \tag{33}$$

then

$$f = f_0(\boldsymbol{x}_0(t;\boldsymbol{x},\boldsymbol{v}),\boldsymbol{v}_0(t;\boldsymbol{x},\boldsymbol{v})) \tag{34}$$

is a solution of the Vlasov equation.

Suppose there exists a constant of motion for the classical particles. Consider, for example, the Hamiltonian for a charge particle in the presence of an electrostatic field $\boldsymbol{E}=-\nabla\Phi$,

$$H = \frac{mv^2}{2} + q\Phi(\boldsymbol{x}). \tag{35}$$

As $\partial\Phi/\partial t = 0$, H is a constant of the motion,

$$\frac{dH}{dt} = m\boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{dt} + q\boldsymbol{v}\cdot\nabla\Phi = q\boldsymbol{v}\cdot\nabla\Phi - q\boldsymbol{v}\cdot\nabla\Phi = 0. \tag{36}$$

This implies that $f(\boldsymbol{x},\boldsymbol{v},t) = f(H(\boldsymbol{x},\boldsymbol{v}))$ is a solution of the Vlasov equation, where $f(H)$ is *any* function of the energy. We can directly apply the Vlasov operator to show that this is true,

$$\left(\boldsymbol{v}\cdot\frac{\partial H}{\partial\boldsymbol{x}} + \frac{\boldsymbol{F}}{m}\cdot\frac{\partial H}{\partial\boldsymbol{v}}\right)\frac{\partial f}{\partial H} \stackrel{?}{=} 0$$

$$(q\boldsymbol{v}\cdot\nabla\Phi + -q\nabla\Phi\cdot\boldsymbol{v})\frac{\partial f}{\partial H} = 0. \tag{37}$$

As another example, consider the canonical momentum. If we have electro-magnetic fields

$$\boldsymbol{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t} \tag{38}$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}, \tag{39}$$

the Hamiltonian is

$$H(\boldsymbol{x},\boldsymbol{p},t) = \frac{(\boldsymbol{p}-q\boldsymbol{A}(\boldsymbol{x},t))}{2m} + q\Phi(\boldsymbol{x},t). \tag{40}$$

The Hamiltonian now has explicit time-dependence, so the energy is no longer a constant of the motion. However, if each of the vector components of $\boldsymbol{A}$ and Φ are independent of some coordinate x, then $\partial H/\partial x = 0$, implying that

$$p_x = mv_x + \frac{q}{c}A_x \tag{41}$$

is a constant of the motion from Hamilton's equations. We can again directly apply the Vlasov operator,

$$\left(\boldsymbol{v}\cdot\frac{\partial p_x}{\partial \boldsymbol{x}}+\frac{\boldsymbol{F}}{m}\cdot\frac{\partial p_x}{\partial \boldsymbol{v}}\right)\frac{\partial f}{\partial p_x}\overset{?}{=}0$$

$$\left(\frac{q}{c}\boldsymbol{v}\cdot\frac{\partial A_x}{\partial \boldsymbol{x}}+\frac{q}{c}\left(\boldsymbol{v}\times\boldsymbol{B}\cdot\hat{\boldsymbol{x}}\right)\right)\overset{?}{=}0$$

$$\frac{q}{c}\boldsymbol{v}\cdot\frac{\partial \boldsymbol{A}}{\partial x}=0 \tag{42}$$

where we have used the vector identity $\boldsymbol{A}\times(\nabla\times\boldsymbol{B})=\nabla\boldsymbol{B}\cdot\boldsymbol{A}-\boldsymbol{A}\cdot\nabla\boldsymbol{B}$.

4. Consideration of collisions

Question: How do particles influence each other? Through E&M forces due to the average charge and current distributions.

The collisions between particles occurs on the scale of the Debye length, λ_D, discussed in Lecture 1, as the electric fields are shielded out on longer length scales. If collisions are included, another term is added to the Vlasov equation,

$$\frac{\partial f_s}{\partial t}+\boldsymbol{v}\cdot\frac{\partial f_s}{\partial \boldsymbol{x}}+\frac{\boldsymbol{F}_s}{m_s}\cdot\frac{\partial f_s}{\partial \boldsymbol{v}}=\sum_{s'}C(f_s,f_{s'}), \tag{43}$$

where f_s is the distribution function of species s and $C(f_s,f_{s'})$ is the collision operator which describes the change in distribution function f_s due to its interaction with $f_{s'}$. Including the effects of collisions, (43) is referred to as the Boltzmann equation.

The collision operator is nonlinear integral operator, describing binary collisions between two distribution functions. The Landau form of the collision operator is written as

$$C(f_s,f_{s'})=-\frac{2\pi q_s^2 q_{s'}^2\ln\Lambda}{m_a}\frac{\partial}{\partial \boldsymbol{v}}\cdot\left(\int d^3v'\left(\frac{u^3\overleftrightarrow{\boldsymbol{I}}-\boldsymbol{u}\boldsymbol{u}}{u^3}\right)\cdot\left(\frac{f_a(\boldsymbol{v})}{m_b}\frac{\partial f_b(\boldsymbol{v}')}{\partial \boldsymbol{v}'}-\frac{f_b(\boldsymbol{v}')}{m_a}\frac{\partial f_a(\boldsymbol{v})}{\partial \boldsymbol{v}}\right)\right), \tag{44}$$

where $\boldsymbol{u}:=\boldsymbol{v}-\boldsymbol{v}'$ is the relative velocity of the species and $\overleftrightarrow{\boldsymbol{I}}$ is the identity tensor. We will discuss collisions is much more detail in upcoming lectures. This operator can be found by considering the classical scattering cross-section between charged particles and summing over all possible interactions. The Landau form of the collision operator assumes that small-angle collision dominate, which is typically appropriate for the Coulomb potential.

The collision operator has several important properties

- Conservation of particles, energy, and momentum
- If f is a thermal equilibrium (Maxwell-Boltzmann) distribution

$$f=\frac{n_0}{(2\pi T/m)^{3/2}}\exp\left(-\frac{mv^2/2+q\Phi}{T}\right),$$

  then $C(f)=0$ ($\partial f/\partial t=0$ due to collisions).
- Collisions drive f to a thermodynamic equilibrium. Collisions increase the entropy until TE is reached. This statement is known as Boltzmann's H theorem.
- Collisions act locally in physical space and non-locally in velocity space. In order of particles to collide, they must be physically near each other (within λ_D of each other). When we write down the Vlasov equation, we have considered any length scales smaller than λ_D to have been averaged over. In this sense, collisions always occur at the same physical position. However, particles with very different velocities can collide with each other.