The Vlasov Equation (collisionless Boltzmann Equation)

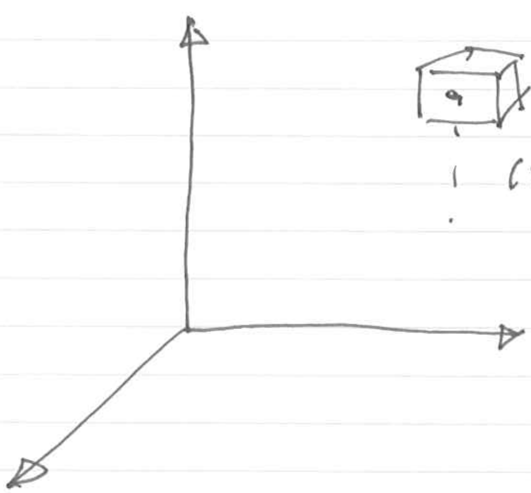
Goal: obtain an evolution equation for the distribution function

$$f(\underset{\sim}{x}, \underline{v}, t).$$

[redacted]$+1$ args

Recall the definition of $f(\underset{\sim}{x}, \underset{\sim}{v}, t)$

$$dN = f(\underset{\sim}{x}, \underset{\sim}{v}, t)\, d^3x\, d^3v$$



$dN = \#$ of particles in BOX

Assumes no collision

Answer:

force/m

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial f}{\partial \underset{\sim}{x}} + \frac{q}{m}\left(\underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c}\right) \cdot \frac{\partial f}{\partial \underset{\sim}{v}} = 0$$

~~comments~~

Comments:

Newton's Laws & Classical Mechanics

$\underset{\sim}{x}, \underset{\sim}{v}$ are dependent variables

problem is solved when we know

$\underset{\sim}{x}(t), \underset{\sim}{v}(t)$ particle description

independent variable

Vlasov description

$\underset{\sim}{x}, \underset{\sim}{v}$, & t are all independent variables

problem is solved when we know

$f(\underset{\sim}{x}, \underset{\sim}{v}, t)$

Fluid exqutions

$\underset{\sim}{x}$ & t are indedendent variables

Problem is solved when we know

$\left.\begin{array}{ll} \underset{\sim}{u}(\underset{\sim}{x}, t) & \text{velocity} \\ n(\underset{\sim}{x}, t) & \text{density} \\ T(\underset{\sim}{x}, t) & \text{Temperature} \end{array}\right\}$ dependent variables

Before we start we should ask the basic question. Does knowledge of $f(\underset{\sim}{x}, \underset{\sim}{v}, t)$ completely describe the system?

<u>Answer: No</u>

Lets say our system consists of N particles. Then the complete state of the system is given by 6N variables, Three components of position and velocity for each particle. Certainly a lot more information than we can handle.

Even if we adopt a statistical description our pdf would have many ~~[illegible]~~ arguments

pdf

6N+1 args

$$dP = F_N(\underset{\sim}{x}_1, \underset{\sim}{v}_1, \underset{\sim}{x}_2, \underset{\sim}{v}_2, \ldots \underset{\sim}{x}_N, \underset{\sim}{v}_N, t)\, d^3x_1 d^3v_1 \cdots d^3x_N d^3v_N$$

dP = probability of finding particle #1 in box 1
and " #2
and " ⋮ N in box N

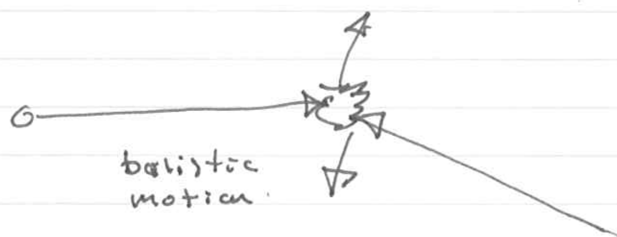
Here we are assuming particles are distinguishable.

Evolution equation for F is known as the Liouville Equation.

So how do we get away with characterizing the system in terms of ~~a function of~~ only 6 arguments?

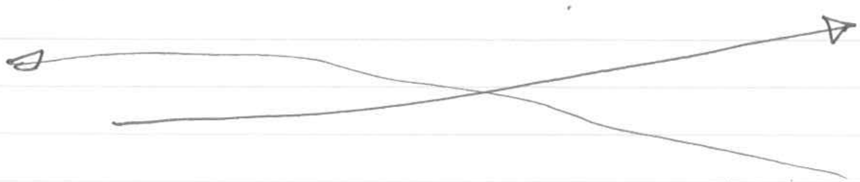
Answer, particles are "weakly" interacting. When does this apply?

<u>dilute gas</u>



infrequent collisions by particles not likely encounter each other again soon

Plasma



Single inter particle reactions have small effect given particle interacts with many at once

Mainly, particles are uncorrelated.

IF particles are uncorrellated

$$F_N(\underset{\sim}{x}_1, \underset{\sim}{v}_1, \underset{\sim}{x}_2, \underset{\sim}{v}_2 \cdots \underset{\sim}{x}_N, \underset{\sim}{v}_N) = F_1(\underset{\sim}{x}_1, \underset{\sim}{v}_1) F_1(\underset{\sim}{x}_2, \underset{\sim}{v}_2) \cdots F(\underset{\sim}{x}_N, \underset{\sim}{v}_N)$$

joint pdf is product of single particle pdfs

our distribut

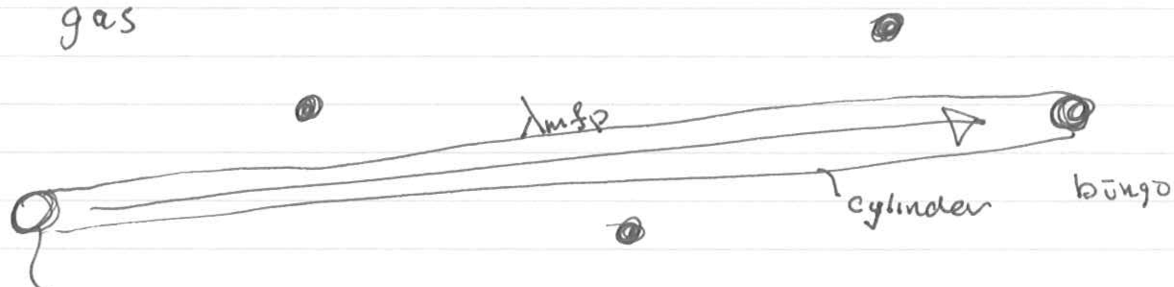
total # of particle

$$f(\underset{\sim}{x}, \underset{\sim}{v}) = N \, F_1(\underset{\sim}{x}, \underset{\sim}{v})$$

why $\int d^3x\, d^3v\, F_1 = 1$

$\int d^3x\, d^3v\, f = N$

Justification for a dilute gas



a_0 = size of atom/molecule

λ_{mfp} = how far it goes before collision

cross-section for collision

volume of cylinder $= a_0^2 \lambda_{mfp}$

$$n \, a_0^2 \lambda_{mfp} = 1$$

$$\lambda_{mfp} = \frac{1}{n a_0^2}$$

normalize λ_{mfp} to typical particle spacing $r_s = n^{-1/3}$

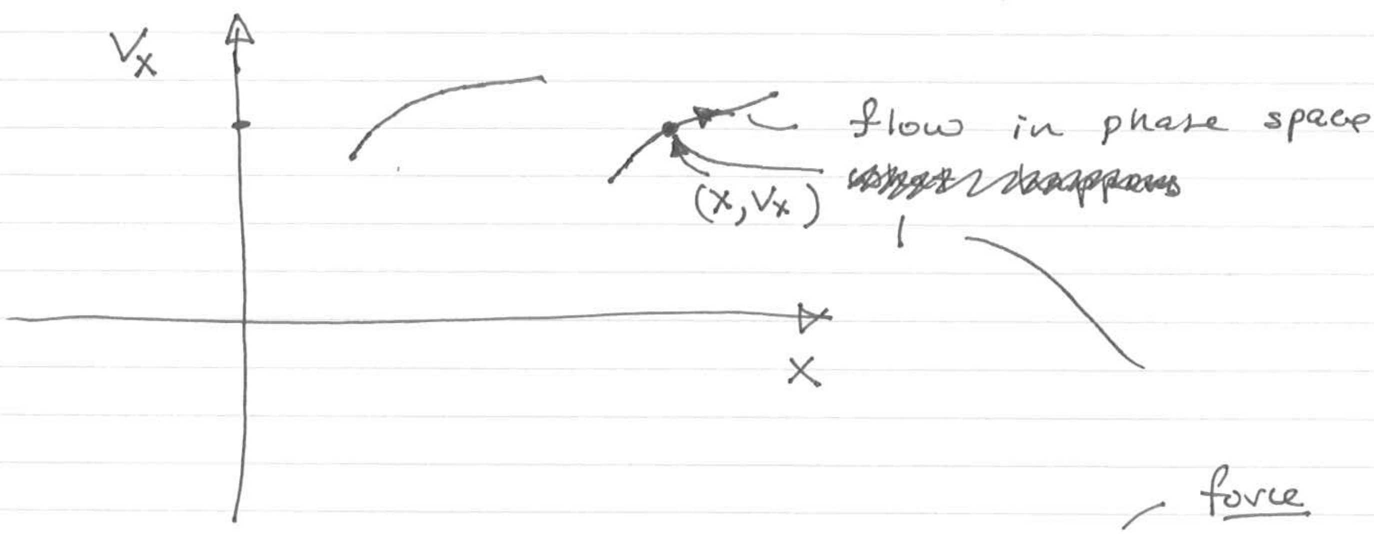
$$\frac{\lambda_{mfp}}{r_s} = \frac{r_s^2}{a_0^2} \underbrace{\frac{1}{r_s^3 n}}_{\sim 1} = \frac{r_s^2}{a_0^2} \gg 1$$

Particle has traveled "long distance" before next collision at which point correlation with past collision is lost.

~~simplify~~

What is $\underset{\sim}{U}$ 6D velocity in phase space

Focus on $x - V_x$ phase space



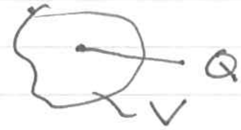
$$\left.\frac{dx}{dt}\right|_{\text{particle}} = V_x \qquad \frac{dV_x}{dt} = \frac{\overbrace{F_x}^{\text{force}}}{m}$$

V_x & $\frac{F_x}{m}$ are x components of $\underset{\sim}{U}$

$$\left.\begin{matrix} V_y & \frac{F_y}{m} \\ V_z & \frac{F_z}{m} \end{matrix}\right\} \text{ y \& z components}$$

Anologous situation, charge conservation in 3D

$$\frac{dQ_V}{dt} = -\int_S d\underset{\sim}{S} \cdot \underset{\sim}{J}$$



number density

$$Q_V = \int q\, n \, d^3x$$

"fluid velocity"

$$\underset{\sim}{J} = q\, n\, \underset{\sim}{u}(\underset{\sim}{x}, t)$$

$$\frac{d}{dt}\int_V d^3x \, n = -\int_S d\underset{\sim}{S} \cdot \underset{\sim}{u}\, n$$

Use Divergence theorem

$$\int_S d\underset{\sim}{S} \cdot \underset{\sim}{u}\, n = \int_V d^3x \, \nabla \cdot (\underset{\sim}{u}\, n)$$

~~TRUE FOR any static~~

$$\int_V d^3x \left(\frac{\partial n}{\partial t} + \nabla \cdot \underset{\sim}{u}\, n \right) = 0$$

TRUE FOR ANY V

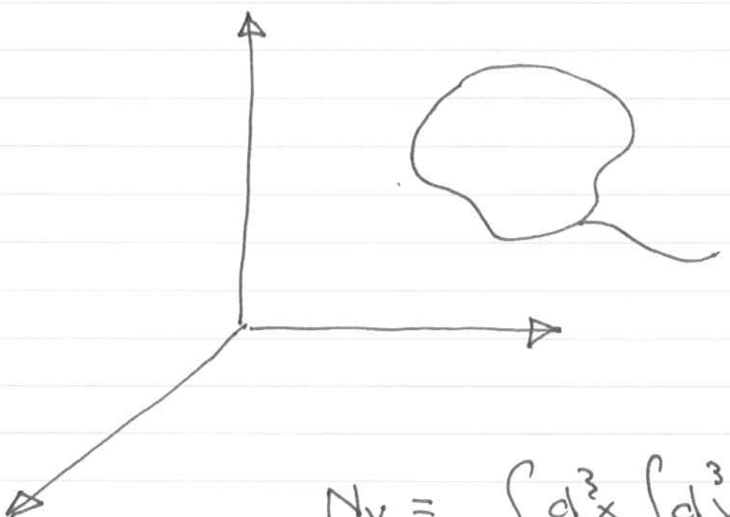
$$\therefore \quad \frac{\partial n}{\partial t} + \nabla \cdot (\underset{\sim}{u}\, n) = 0$$

Now lets derive the Vlasov Equation

~~Outline of derivation by analogy~~

Conservation of # of particles

Phase space



6-dimensional volume
V (stationary)

$$N_V = \int_V d^3x \int d^3v \, f(\underset{\sim}{x}, \underset{\sim}{v}, t)$$

N_V = # of particles in V

$\frac{dN}{dt}$ = − Rate at which particles leave volume by passing through suface

$$= -\int d\underset{\sim}{S} \cdot \underset{\sim}{U} f$$

$\underset{\sim}{U}$ — 6D velocity in phase space

$d\underset{\sim}{S}$ = 5D surface area
direction out of V

Also by divergence theorem

$$\int_S d\underset{\sim}{S} \cdot \underset{\sim}{U} f = \int_V d^3x\, d^3v \; \nabla_6 \cdot (\underset{\sim}{U} f)$$

six dimensional divergence

$$\nabla_6 \cdot (\underset{\sim}{U} f) = \frac{\partial}{\partial x}(V_x f) + \frac{\partial}{\partial V_x}\left(\frac{F_x}{m} f\right) + \frac{\partial}{\partial y} + \cdots$$

$$= \frac{\partial}{\partial \underset{\sim}{x}} \cdot (\underset{\sim}{V} f) + \frac{\partial}{\partial \underset{\sim}{V}} \cdot \left(\frac{\underset{=}{F}}{m} f\right)$$

Continuity exquation in phase space

$$\frac{d}{dt}\int_V d^3x\, d^3v\, f = \int_V d^3x\, d^3v \frac{\partial f}{\partial t}$$

$$\int d^3x\, d^3x \left[\frac{\partial f}{\partial t} + \nabla_6 \cdot (\underset{\sim}{U} f)\right] = 0$$

$$\frac{\partial f}{\partial t} + \nabla_6 \cdot (\underset{\sim}{U} f) = 0$$

OR

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \underset{\sim}{x}} \cdot (\underset{\sim}{V} f) + \frac{\partial}{\partial \underset{\sim}{V}} \cdot \left(\frac{\underset{=}{F}}{m} f\right) = 0$$

NOT THERE YET

$$\frac{\partial}{\partial \underset{\sim}{x}} \cdot (\underset{\sim}{V} f) = \underset{\sim}{V} \cdot \frac{\partial f}{\partial \underset{\sim}{x}} + f \frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{V}$$

What is $\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{V}$ Ans 0

$\underset{\sim}{V}$ and $\underset{\sim}{x}$ are both independent variables

fluid velocity

$\left(\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{u}(x) \neq 0 \right)$

$$\frac{\partial}{\partial \underset{\sim}{V}} \cdot \left(\frac{\underset{\sim}{F}}{m} f \right) \quad \text{where} \quad \underset{\sim}{F} = q \left(\underset{\sim}{E}(\underset{\sim}{x}, t) + \frac{\underset{\sim}{V} \times \underset{\sim}{B}(\underset{\sim}{x}, t)}{c} \right)$$

$$= \frac{\underset{\sim}{F}}{m} \cdot \frac{\partial f}{\partial \underset{\sim}{V}} + \frac{f}{m} \frac{\partial}{\partial \underset{\sim}{V}} \cdot \underset{\sim}{F}$$

$$\frac{\partial}{\partial \underset{\sim}{V}} \cdot \underset{\sim}{F} = \frac{\partial}{\partial \underset{\sim}{V}} \cdot q \frac{\underset{\sim}{V} \times \underset{\sim}{B}}{c}$$

$$= \frac{\partial}{\partial V_x} \frac{q}{c} \left(V_y B_z - V_z B_y \right) + \frac{\partial}{\partial V_y} \frac{q}{c} \left(V_z B_x - V_x B_z \right)$$

$$+ \frac{\partial}{\partial V_z} \frac{q}{c} \left(V_x B_y - V_y B_x \right) = 0$$

Interaction with "smoothed density and currents" effects of particle discreteness absent

Examples of missing effects

1) collisions
2) spontaneous radiation
   Bremstrahlung
   cyclotron radiation

~~what is includes~~

1) stimulated radiation

Flow is incompressible in phase space!

$$\nabla_6 \cdot \underset{\sim}{U} = \frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{v} + \frac{\partial}{\partial \underset{\sim}{v}} \cdot \frac{q}{m} \underset{\sim}{F} = 0$$

Important consequences

Q: How do particles influence each other

A: Through E&M Forces due to average charge and current distributions

$$\underset{\sim}{F} = q\left(\underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c} \right)$$

$$\nabla \cdot \underset{\sim}{E} = 4\pi\rho = 4\pi \sum_q q\, n_q(\underset{\sim}{x}, t)$$

$$n_q(\underset{\sim}{x}, t) = \int d^3v\, f_q(\underset{\sim}{x}, \underset{\sim}{v}, t)$$

$$\nabla \times \underset{\sim}{B} = \frac{4\pi}{c} \underset{\sim}{J} + \frac{1}{c} \frac{\partial \underset{\sim}{E}}{\partial t}$$

$$\underset{\sim}{J} = \sum_q q\, n_q \bar{u}_q = \sum_q q \int d^3v\, \underset{\sim}{v} f_q$$