PHYS761: An alternate introduction to Landau damping

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1 Introduction

The purpose of these notes is to clarify and elucidate the idea of Landau damping. I may not go through the whole derivation but I will try to capture the essential aspects and physical implications of this phenomenon.

2 Electrostatic linear analysis of the Vlasov-Poisson system

2.1 Vlasov's analysis

The linearized Vlasov-Poisson system of equations can be written as

$$\begin{aligned} \frac{\partial f_{1s}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} f_{1s} - \frac{q_s}{m_s}\boldsymbol{\nabla}\varphi_1 \cdot \frac{\partial f_0(\boldsymbol{v})}{\partial \boldsymbol{v}} = 0 \\ -\boldsymbol{\nabla}^2\varphi_1 = \sum_s 4\pi q_s \int d^3\boldsymbol{v}\, f_{1s} \end{aligned} \tag{1}$$

where $f_{1s} \ll f_0$. To obtain the dielectric constant $\epsilon(\boldsymbol{k}, \omega_k)$, Vlasov assumed

$$\begin{aligned} f_{1s} &= \sum_{k=-\infty}^{\infty} \hat{f}_{1s} \exp(i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)) \\ \varphi_1 &= \sum_{k=-\infty}^{\infty} \hat{\varphi}_1 \exp(i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)) \end{aligned} \tag{2}$$

which means that the perturbations can be written as a linear combination of independent Fourier modes, i.e., normal modes. Another way to describe normal modes would be to recall the identity

$$\int d^3\boldsymbol{r} \exp(i\boldsymbol{k}\cdot\boldsymbol{r}) = \delta(\boldsymbol{k}), \quad \int dt \exp(i\omega t) = \delta(\omega) \tag{3}$$

where δ is the Dirac-delta distribution. For the Fourier(normal) modes in equation (2), we can write a similar identity

$$\int d^3\boldsymbol{r}\, \exp(i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_k t)) \exp(i(\boldsymbol{k}^{'}\cdot\boldsymbol{r} - \omega^{'} t)) = \delta(\boldsymbol{k} - \boldsymbol{k}^{'}) \exp(-i(\omega - \omega^{'})t) = \delta(\boldsymbol{k} - \boldsymbol{k}^{'}) \tag{4}$$

Similarly, we can write the identity

$$\int dt\, \exp(i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)) \exp(i(\boldsymbol{k}^{'}\cdot\boldsymbol{r} - \omega^{'} t)) = \delta(\omega - \omega^{'}) \exp(i(\boldsymbol{k} - \boldsymbol{k}^{'})\cdot\boldsymbol{r}) = \delta(\omega - \omega^{'}) \tag{5}$$

These identities imply that summing over two modes can only yield a non-zero value if the modes are identical. In other words, the modes are linearly independent of each other. Note that normal mode frequencies don't have to be purely real. Using the ansatz in equation (2), we can write equation (1) as

$$\begin{aligned} -i\omega f_{1s} + i\boldsymbol{k}\cdot\boldsymbol{v} f_{1s} - \frac{q_s}{m_s}\varphi_1 i\boldsymbol{k}\cdot\frac{\partial f_0(\boldsymbol{v})}{\partial \boldsymbol{v}} = 0 \\ k^2\varphi_1 = \sum_s 4\pi q_s \int d^3\boldsymbol{v}\, f_{1s} \end{aligned} \tag{6}$$

Eliminating f_{1s} from the linearized Poisson's equation, we get a dispersion relation

$$k^2\varphi_1 = \sum_s \frac{4\pi q_s^2}{m_s}\int d^3\boldsymbol{v}\,\frac{\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}}{(-\omega+\boldsymbol{k}\cdot\boldsymbol{v})}\varphi_i \tag{7}$$

And the dielectric response function

$$\epsilon(\boldsymbol{k},\omega_k) = 1-\sum_s \frac{\omega_{ps}^2}{k^2\,n_s}\int d^3\boldsymbol{v}\,\frac{\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}}{(-\omega+\boldsymbol{k}\cdot\boldsymbol{v})},\quad \omega_s^2 = \frac{4\pi q_s^2 n_s}{m_s} \tag{8}$$

where n_s is the equilibrium plasma density. In class, we obtained the dispersion relation(not shown here) for the electron plasma waves by solving $\epsilon(\boldsymbol{k},\omega)=0$ in the limit $\omega \ll \boldsymbol{k}\cdot\boldsymbol{v}$ and assuming a local thermal equilibrium, i.e.,

$$f_0(\boldsymbol{v}) = \frac{n_0}{\pi^{3/2}v_{th}^3}\exp\left(-\frac{v^2}{v_{th}^2}\right) \tag{9}$$

We also assumed that $\omega_{pi}^2/\omega_{pe}^2 \sim m_e/m_i \ll 1$. Using the $\omega \ll \boldsymbol{k}\cdot\boldsymbol{v}$, we can avoid dealing with the singularity at $\omega = \boldsymbol{k}\cdot\boldsymbol{v}$. This is exactly what Vlasov did in 1940. But what happens to the particles in the distribution function f_0 that have velocities close to the electron plasma waves? Vlasov doesn't make it entirely clear in his treatment. He suggests that the integral over the singularity can be written as a principal value integral. This is not always true as we'll find out in the next section.

2.2 Landau's initial-value treatment

Before going into the details, we define Laplace and inverse Laplace transform, respectively as

$$\begin{aligned}\hat{f}_1 &= \int_0^\infty dt f_1(t)\exp(i\omega t),\\ f_1 &= \int_{i\omega_0-\infty}^{i\omega_0+\infty}\frac{d\omega}{2\pi}\hat{f}_1(\omega)\exp(-i\omega t),\end{aligned} \tag{10}$$

where ω is a complex number chosen with an imaginary part ω_0 such that $f_i(t)\exp(-\omega_0 t)\to\infty$ as $t\to 0$ and the Laplace transform is bounded and well-defined. The inverse Laplace transform is defined in the complex ω plane as shown in illustration below.

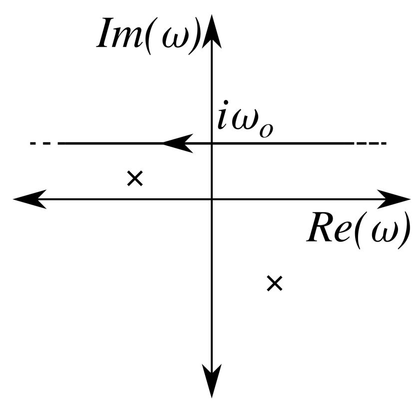

Figure 1: shows the contour of integration(Bromwich contour) for the inverse Laplace transform. All the singularities are below the line $\omega = i\omega_0$. The integral is a straight-line parallel to the $Re(\omega)$ axis and closes in a semi-circle as $Re(\omega)\to\infty$ in the $Im(\omega)<0$ direction enclosing all the finite singularities within the semi-circle contour.

Landau treated the electrostatic dielectric response calculation as an initial-value problem. That way one doesn't have to find a basis in which the perturbation can be decomposed as a sum of normal modes or even assume that the perturbation always grows as a sum of normal modes. This gives one more freedom in the way one expresses the perturbation and makes the procedure more general. Instead of using the ansatz in equation (2), we use the following ansatz

$$\begin{aligned} f_{1s} &= \sum_{k=-\infty}^{\infty} \exp(i\boldsymbol{k}\cdot\boldsymbol{r})\hat{f}_{1s}(t) \\ \varphi_1 &= \sum_{k=-\infty}^{\infty} \exp(i\boldsymbol{k}\cdot\boldsymbol{r})\hat{\varphi}_{1s}(t). \end{aligned} \tag{11}$$

Note that the relation (5) does not hold for the above ansatz anymore. Next, we substitute (11) into equation (1) and take the Laplace transform of the equation to get

$$\begin{aligned} \int_0^\infty dt\, \exp(i\omega t)\left(\frac{\partial f_{1s}}{\partial t} + i\boldsymbol{k}\cdot\boldsymbol{v} f_{1s} - \frac{q_s}{m_s}\varphi_1 i\boldsymbol{k}\cdot\frac{\partial f_0(\boldsymbol{v})}{\partial \boldsymbol{v}}\right) = 0 \\ \int_0^\infty dt\, \exp(i\omega t)\left(k^2\varphi_1 - \sum_s 4\pi q_s \int d^3\boldsymbol{v}\, f_{1s}\right) = 0 \end{aligned} \tag{12}$$

Using integration by parts for the first term in the Vlasov equation like so

$$\int_0^\infty dt\, \exp(i\omega t)\frac{\partial f_{1s}}{\partial t} = \exp(i\omega t) f_{1s}\Big|_0^\infty - i\omega\int_0^\infty dt\, \exp(i\omega t) f_{1s} = f_{1s}(\boldsymbol{v}, t=0) - i\omega\hat{f}_{1s} \tag{13}$$

we get the Laplace-transformed, Vlasov-Poisson system as

$$\begin{aligned} f_{1s}(\boldsymbol{v},0) - i\omega\hat{f}_{1s} + i\boldsymbol{k}\cdot\boldsymbol{v}\hat{f}_{1s} - \frac{q_s}{m_s}\hat{\varphi}_1 i\boldsymbol{k}\cdot\frac{\partial f_0(\boldsymbol{v})}{\partial\boldsymbol{v}} = 0 \\ k^2\hat{\varphi}_1 - \sum_s 4\pi q_s\int d^3\boldsymbol{v}\,\hat{f}_{1s} = 0 \end{aligned} \tag{14}$$

Eliminating $\hat{f}_{1s}$, we get a self-consistent, dielectric response equation

$$\begin{aligned} \left(k^2 - \sum_s \frac{\omega_{ps}^2}{n_s}\int d^3\boldsymbol{v}\frac{\boldsymbol{k}\cdot\frac{\partial f_0}{\partial\boldsymbol{v}}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\right)\hat{\varphi}_{1s} - \sum_s\frac{\omega_{ps}^2}{n_s q_s}\int d^3\boldsymbol{v}\frac{f_{1s}(\boldsymbol{v},0)}{i(\omega-\boldsymbol{k}\cdot\boldsymbol{v})} = 0 \\ \hat{\varphi}_{1s} = \frac{1}{\epsilon(\boldsymbol{k},\omega)}\sum_s\frac{\omega_{ps}^2}{n_s q_s}\int d^3\boldsymbol{v}\frac{f_{1s}(\boldsymbol{v},0)}{i(\omega-\boldsymbol{k}\cdot\boldsymbol{v})} \end{aligned} \tag{15}$$

This $\hat{\varphi}$ can then be inverted to obtain $\varphi(t)$

$$\varphi_1(t) = \int_{i\omega_0-\infty}^{i\omega_0+\infty}\frac{d\omega}{2\pi}\,\exp(-i\omega t)\hat{\varphi}_{1s} \tag{16}$$

First let simplify the integrand inside the velocity integral. Without loss of generality, we can assume that $\boldsymbol{k}\hat{z}$ where $\hat{z}$. This means that the velocity integral in the x and y-directions doesn't have a singularity. Let's define

$$F(v_z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dv_x\, dv_y f_0(\boldsymbol{v}) \tag{17}$$

Using these simplifications, the velocity integral inside $\epsilon(\boldsymbol{k},\omega)$ will become

$$\mathcal{I}_0 = \sum_s\frac{\omega_{ps}^2}{n_s}\int_{-\infty}^{\infty} dv_z\frac{F'(v_z)}{\omega/k_z - v_z} \tag{18}$$

Let's digress for the next few paragraphs and talk about how these integrals are calculated. First, we have to solve the velocity integrals inside equation (15). The contour for these integrals will look like this

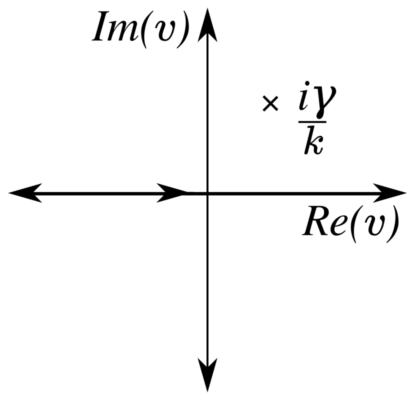

Figure 2: shows the contour of integration for velocity integral in the complex $\boldsymbol{v}$ plane. Once the value of the imaginary pole $i\gamma/k \leq |\delta|, 0 < \delta \ll 1$ we must deform the contour to preserve analytical continuity

Depending on the value of imaginary part of the pole, the integral $\mathcal{I}_0$ can have different values. The three different scenarios and their corresponding integrals are:

1. The pole is above the line $Re(\boldsymbol{v})$. This is the case of an unstable wave. In this case there is no singularity on the real line and hence an integral $\mathcal{I}_0$ is unchanged.

2. The pole in on the line. This is the case of a weakly damped or undamped pole. In that case, we use the idea of a principal value integral and write

$$\mathcal{I}_0 = \sum_s \frac{\omega_{ps}^2}{n_s} \left[\mathcal{P} \int \frac{F^{'}(v_z)}{\omega/k_z - v_z} + i\pi F^{'}\left(\frac{\omega}{k_z}\right)\right] \tag{19}$$

3. The pole is below the line $Re(\boldsymbol{v}) = 0$. This corresponds to the strongly damped case and the velocity integral can be written as

$$\mathcal{I}_0 = \sum_s \frac{\omega_{ps}^2}{n_s} \left[\int \frac{F^{'}(v_z)}{\omega - k_z v_z} + 2i\pi F^{'}\left(\frac{\omega}{k_z}\right)\right] \tag{20}$$

After we complete the velocity integral we need to invert $\hat{\varphi}_{1s}$ by using equation (16). For almost all the practical cases, the function $\hat{\varphi}_{1s}$ will only have a finite number of poles. Using the idea of analytical continuation, we can continue the contour in ω-space as long as we don't break and rejoin the contour over a pole. The illustration given below explains the process of analytical continuation

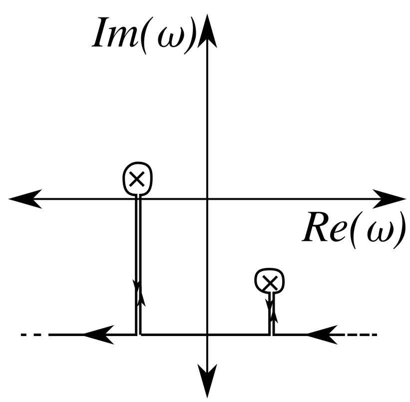

Figure 3: shows the analytically continued contour used for inverting the Laplace transformed variables to real space. One can deform the contour in any possible shape as long as it does not cross the poles. The contributions from the thin vertical lines will cancel each other and the only contribution will arise from the circles around the poles

Assuming the initial condition $f_{1s}(\boldsymbol{v},0)$ to be analytic in the complex $\boldsymbol{v}$-space, the numerator in the dielectric response equation (15) will be analytic — any singularities coming from denominator of the integral

$$\mathcal{I}_1 = \int d^3\boldsymbol{v}\ \frac{f_{1s}(\boldsymbol{v},0)}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v})} \tag{21}$$

will be integrable. Hence, there cannot be any singularities in $\hat{\varphi}_{1s}$ from the numerator $\mathcal{I}_1$. So the only source of singularities are the zeros of the dielectric function $\epsilon(\boldsymbol{k},\omega)$. The most general form of $\hat{\varphi}$ will be

$$\hat{\varphi} = \sum_i \frac{c_i}{(\omega - \omega_i)^{r_i}} + A(\omega) \tag{22}$$

where $A(\omega)$ is an analytical function and c_i is the residue from Cauchy's integral in $\epsilon(\boldsymbol{k},\omega)$. For simplicity,we assume that all the poles of $\hat{\varphi}_{1s}$ are multiplicity one, i.e., $r_i = 1\,\forall\, i$. Inverting the general form of ϕ we get a general solution of the form $\phi(t) \propto \exp(\gamma_{il}t)$ where γ_{il} is the largest imaginary part of the ω_is. When $\gamma_{il} < 0$, the electrostatic potential of the wave decays exponentially via Landau damping.

Let's look at what the perturbation to the distribution function looks like. To be continued...