ENERGY PRINCIPLE

Lets write down the Linear Equations for an arbitrary equilibrium.

$$\rho_m \frac{\partial^2 \xi}{\partial t^2} = -\nabla p_1 + \frac{J_1 \times B_0}{c} + \frac{J_0 \times B_1}{c} \equiv F(\xi)$$

force density

$$J_1 = \frac{c}{4\pi} \nabla \times B_1 \qquad B_1 = \nabla \times \xi \times B_0$$

$$p_1 = -\xi \cdot \nabla p_0 - \gamma_s p_0 \nabla \cdot \xi$$

multiply by $\frac{\partial \xi}{\partial t}$ and integrate over all plasma

$$\int d^3x \, \frac{\partial \xi}{\partial t} \rho \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial}{\partial t} \underbrace{\int d^3x \, \rho \, \frac{1}{2}\left(\frac{\partial \xi}{\partial t}\right)^2}_{K} = \frac{\partial}{\partial t} K$$

K = kinetic energy

$$\underline{\int d^3x \, \frac{\partial \xi}{\partial t} \cdot F(\xi)}$$

can show F is self adjoint with appropriate boundary conditions what does self adjoint mean

$$\int d^3x \; \boldsymbol{\xi}_1 \cdot F(\boldsymbol{\xi}_2) = \int d^3x \; \boldsymbol{\xi}_2 \cdot F(\boldsymbol{\xi}_1) \qquad \text{For any } \boldsymbol{\xi}_1 \; \& \; \boldsymbol{\xi}_2$$

thus,
$$\int d^3x \; \frac{\partial \boldsymbol{\xi}}{\partial t} \cdot F(\boldsymbol{\xi}) = \int d^3x \; \boldsymbol{\xi} \cdot F\left(\frac{\partial \boldsymbol{\xi}}{\partial t}\right)$$

$$\int d^3x \; \frac{\partial \boldsymbol{\xi}}{\partial t} \cdot F(\boldsymbol{\xi}) = \frac{1}{2} \int d^3x \left[\frac{\partial \boldsymbol{\xi}}{\partial t} \cdot F(\boldsymbol{\xi}) + \boldsymbol{\xi} \cdot F\left(\frac{\partial \boldsymbol{\xi}}{\partial t}\right)\right]$$

$$= \frac{\partial}{\partial t} \frac{1}{2} \int d^3x \; \boldsymbol{\xi} \cdot F(\boldsymbol{\xi})$$

$$= \frac{\partial}{\partial t}(-\Delta W)$$

$$\Delta W = -\frac{1}{2} \int d^3x \; \boldsymbol{\xi} \cdot F(\boldsymbol{\xi})$$

$$\frac{\partial}{\partial t}(K + \Delta W) = 0$$

$K + \Delta W = \text{constant}$ $\qquad \boldsymbol{\xi} = 0 \qquad$ constant $= 0$

$K \geq 0$

Variational Approach to growth rates

$$\underset{\sim}{\xi}(\underset{\sim}{x},t) = \hat{\underset{\sim}{\xi}}(x)\exp(\gamma t)$$

$$K = \gamma^2 \int \tfrac{1}{2} d^3x\, \rho\, |\hat{\underset{\sim}{\xi}}|^2 \quad \exp(2\gamma t)$$

$$\Delta W = \exp(2\gamma t)\, \tfrac{1}{2} \int d^3x\, \hat{\underset{\sim}{\xi}}(x)\cdot F(\hat{\underset{\sim}{\xi}})$$

<u>$K + \Delta W = 0$</u>

$$\gamma^2 = + \frac{\tfrac{1}{2}\int d^3x\, \hat{\underset{\sim}{\xi}}(x)\cdot F(\hat{\underset{\sim}{\xi}})}{\tfrac{1}{2}\int d^3x\, \rho\, |\hat{\underset{\sim}{\xi}}|^2} = -\frac{\Delta W(\hat{\xi},\hat{\xi})}{K(\hat{\xi},\hat{\xi})}$$

Quadratic EXPRESSION FOR γ^2

Find the FUNCTION, $\hat{\xi}$, which maximizes γ^2

<u>Calculus of variations</u> (EULER EQUATIONS)

$$\gamma^2 \rho \hat{\underset{\sim}{\xi}} = + F(\hat{\xi})$$

ORIGINAL

Equations

THUS, in order for there to be an instability ~~ΔW~~ ~~ΔW <~~

ΔW must be negative

$$\Delta W = -\frac{1}{2}\int d^3x \; \underset{\sim}{\xi} \cdot \underset{\sim}{F}(\underset{\sim}{\xi})$$

= WORK you must expend ~~EXPENDED~~ in MAKING PERTURBATION $\underset{\sim}{\xi}$

1) IF $\Delta W > 0$ FOR all ~~any~~ possible $\underset{\sim}{\xi}$ then configuration is stable

2) IT CAN ALSO be shown IF There exists any displacement $\underset{\sim}{\xi}$ which makes $\Delta W < 0$ then the configuration is unstable

$\Delta W > 0$ for all ξ is necessary and <u>sufficient</u> for stability

if there exists a $\underset{\sim}{\xi}$ such that $\Delta W < 0$ necessary & sufficient for instability

~~THUS~~ ~~MAXIMIZI~~

THUS, MAXIMIZING γ^2 IS EQUIVALENT to solving the Equations of motion

- IF ANY TEST FUNCTION GIVES a positive γ^2 then the actual ~~test~~ ~~fu~~ solution will have a larger γ^2

- IF ANY TEST FUNCTION MAKES $\Delta W < 0$ THEN THE SYSTEM IS UNSTABLE

What is $\Delta W(\xi, \xi)$

$$\Delta W = -\frac{1}{2}\int d^3x\ \underset{\sim}{\xi} \cdot \left[-\nabla p_1 + \frac{(\nabla \times \underset{\sim}{B}_1) \times B_0}{4\pi} + \frac{\underset{\sim}{J}_0 \times \underset{\sim}{B}_1}{c} \right]$$

$$\underset{\sim}{B}_1 = \nabla \times (\underset{\sim}{\xi} \times \underset{\sim}{B}_0)$$

$$p_1 = -\xi \cdot \nabla p_0 - \gamma_s p_0 \nabla \cdot \xi$$

FIRST TERM

$$\frac{1}{2}\int d^3x\ \xi \cdot \nabla p_1 = \frac{1}{2}\int_S d\vec{A} \cdot \underset{\sim}{\xi}\, p_1 - \frac{1}{2}\int d^3x\ p_1 \nabla \cdot \xi$$

LETS DROP THE SURFACE TERM FOR THE MOMENT

$$1^{st} \text{ Term} = +\frac{1}{2}\int d^3x \left[\underbrace{\gamma_s p_0 (\nabla \cdot \xi)^2}_{\substack{\text{energy invested} \\ \text{in compressing} \\ \text{plasma}}} + (\xi \cdot \nabla p_0)\nabla \cdot \underset{\sim}{\xi} \right]$$

SECOND TERM

$$-\frac{1}{2}\int d^3x \; \underset{\sim}{\xi} \cdot \frac{\nabla \times \underset{\sim}{B}_1 \times \underset{\sim}{B}_0}{4\pi} = \frac{1}{2}\int d^3x \; \frac{\underset{\sim}{\xi} \times \underset{\sim}{B}_0}{4\pi} \cdot \nabla \times \underset{\sim}{B}_1$$

$$= \frac{1}{2}\int_S d\vec{A} \cdot \frac{\underset{\sim}{B}_1 \times (\underset{\sim}{\xi} \times \underset{\sim}{B}_0)}{4\pi} + \frac{1}{2}\int d^3x \; \frac{\underset{\sim}{B}_1}{4\pi} \cdot \underbrace{\nabla \times (\underset{\sim}{\xi} \times \underset{\sim}{B}_0)}_{\underset{\sim}{B}_1}$$

DROP SURFACE TERM

$$= \underbrace{\frac{1}{2}\int d^3x \; \frac{|\underset{\sim}{B}_1|^2}{4\pi}}_{\text{energy in perturbed magnetic field}}$$

THIRD TERM

$$\underset{\sim}{J}_0 = J_{0\parallel} \underset{\sim}{b} + \underset{\sim}{J}_{0\perp}$$

$$\underset{\sim}{J}_{0\perp} = c\, \frac{\underset{\sim}{b} \times \nabla P_0}{B_0} \qquad \text{equilibrim}$$

$$-\frac{1}{2}\int d^3x \; \underset{\sim}{\xi} \cdot \frac{\underset{\sim}{J}_0 \times \underset{\sim}{B}_1}{c} = \frac{1}{2}\int d^3x \; \frac{J_{0\parallel}}{c} \, \underset{\sim}{b} \cdot \underset{\sim}{\xi} \times \underset{\sim}{B}^1$$

$$-\frac{1}{2}\int d^3x \; \underset{\sim}{\xi} \cdot \frac{\underset{\sim}{b} \times \nabla P_0}{B_0} \times \underset{\sim}{B}_1$$

AFTER SOME MANIPULATION

THIRD TERM BECOMES $= \frac{1}{2} \int d^3x \left\{ \frac{J_{0\parallel}}{c} \, \underset{\sim}{b} \cdot \underset{\sim}{\xi} \times \underset{\sim}{B}^1 - (\xi \cdot \nabla p_0) \left[\frac{\underset{\sim}{b} \cdot \underset{\sim}{B}_1}{B_0} + \nabla \cdot (\underset{\sim}{b} \xi_1 \right. \right.$

WHAT IS $\frac{\underset{\sim}{b} \cdot \underset{\sim}{B}_1}{B_0} = \frac{\underset{\sim}{B}_0 \cdot \underset{\sim}{B}_1}{B_0^2} = \frac{\underset{\sim}{B}_0 \cdot \nabla \times (\underset{\sim}{\xi} \times \underset{\sim}{B}_0)}{B_0^2}$

$$= \frac{1}{B_0^2} \left\{ \nabla \cdot ((\xi \times \underset{\sim}{B}_0) \times \underset{\sim}{B}_0) + \underset{\sim}{\xi} \times \underset{\sim}{B}_0 \cdot \nabla \times \underset{\sim}{B}_0 \right\}$$

$$= \frac{1}{B_0^2} \left\{ \underbrace{\nabla \cdot \underset{\sim}{B}_0 \underset{\sim}{\xi} \cdot \underset{\sim}{B}_0 - \nabla \cdot \underset{\sim}{\xi} B_0^2}_{-\nabla \cdot \underset{\sim}{\xi}_\perp B_0^2} + \xi_\perp \cdot \underset{\sim}{B}_0 \times \nabla \times \underset{\sim}{B}_0 \right\}$$

AFTER ALGEBRA

$$\frac{\underset{\sim}{B}_0 \cdot \underset{\sim}{B}_1}{B_0^2} = -\nabla \cdot \underset{\sim}{\xi}_\perp - \frac{\underset{\sim}{\xi}_\perp}{B_0^2} \cdot \nabla \frac{B_0^2}{2} - \underset{\sim}{\xi}_\perp \cdot \underset{\sim}{K}$$

$$\boxed{\frac{\underset{\sim}{B}_0 \cdot \underset{\sim}{B}_1}{B_0} = -B_0 \nabla \cdot \underset{\sim}{\xi}_\perp - \underset{\sim}{\xi}_\perp \cdot \nabla B_0 - \underset{\sim}{\xi}_\perp \cdot \underset{\sim}{K} \, B_0}$$

COMBINE TERMS

$$\frac{1}{2}\int d^3x \ \underset{\sim}{\xi}\cdot\nabla p_0 \ \nabla\cdot\underset{\sim}{\xi} - (\underset{\sim}{\xi}\cdot\nabla p_0)\left[\frac{\underset{\sim}{b}\cdot\underset{\sim}{B}_1}{B_0} + \nabla\cdot\underset{\sim}{b}\,\xi_\parallel\right]$$

$$= \frac{1}{2}\int d^3x \ \underset{\sim}{\xi}\cdot\nabla p_0\left[\nabla\cdot\underset{\sim}{\xi}_\perp - \frac{\underset{\sim}{b}\cdot\underset{\sim}{B}_1}{B_0}\right]$$

$$\nabla\cdot\underset{\sim}{\xi}_\perp = -\frac{\underset{\sim}{b}\cdot\underset{\sim}{B}_1}{B_0} - \frac{1}{B_0}\underset{\sim}{\xi}_\perp\cdot\nabla B_0 - \underset{\sim}{\xi}_\perp\cdot\underset{\sim}{K}$$

$$= \frac{1}{2}\int d^3x \ \underset{\sim}{\xi}\cdot\nabla p_0\left[-2\frac{\underset{\sim}{b}\cdot\underset{\sim}{B}_1}{B_0} - \frac{1}{B_0}\underset{\sim}{\xi}_\perp\cdot\nabla B_0 - \underset{\sim}{\xi}_\perp\cdot\underset{\sim}{K}\right]$$

complete square

$$\frac{1}{2}\int d^3x\left[\frac{(\underset{\sim}{b}\cdot\underset{\sim}{B}_1)^2}{4\pi} - \frac{2\underset{\sim}{\xi}\cdot\nabla p_0}{B_0}(\underset{\sim}{b}\cdot\underset{\sim}{B}_1)\right]$$

$$\frac{1}{2}\int d^3x\left[\frac{1}{4\pi}\left(\underset{\sim}{b}\cdot\underset{\sim}{B}_1 - \frac{4\pi\,\underset{\sim}{\xi}\cdot\nabla p_0}{B_0}\right)^2 - \frac{4\pi(\underset{\sim}{\xi}\cdot\nabla p_0)^2}{B_0^2}\right]$$

3 positive Terms

$$\Delta W = \frac{1}{2}\int d^3x \left\{ \frac{(\underset{\sim}{B}_{1\perp})^2}{4\pi} + \frac{\left(\underset{\sim}{b}\cdot\underset{\sim}{B}_1 - \frac{4\pi\, \xi\cdot\nabla p_0}{B_0}\right)^2}{4\pi} + \gamma p (\nabla\cdot\xi)^2 \right.$$

$$\frac{J_{0\parallel}}{c}\, \underset{\sim}{b}\cdot \underset{\sim}{\xi}\times\underset{\sim}{B}^{(1)} \quad - \underbrace{\xi\cdot\nabla p_0 \left[+ \frac{1}{B_0}\, \underset{\sim}{\xi}_\perp \cdot \nabla B_0 + \underset{\sim}{\xi}_\perp\cdot\underset{\sim}{\kappa} + \frac{4\pi(\xi\cdot\nabla p_0)}{B_0^2} \right.}$$

parallel current (kinks)

$$- 2\, \xi\cdot\nabla p_0\; \underset{\sim}{\xi}\cdot\underset{\sim}{\kappa} \Big]$$

pressure gradient

Two possibly negative terms

INTERCHANGE & ~~[illegible]~~

$$\underset{\sim}{B}_1 = \nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}_0) \qquad \xi_\perp$$

$$\underset{\sim}{b}\cdot\nabla p_0 = 0$$

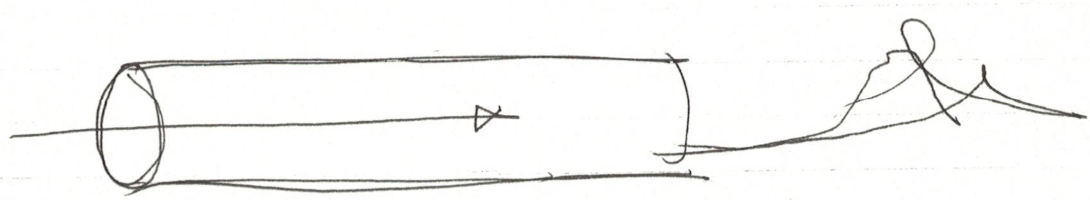
$$\underset{\sim}{b}\cdot(\cancel{\underset{\sim}{b}\cdot\nabla\underset{\sim}{b}}\ \underset{\sim}{\kappa}) = 0$$

<u>SPECIAL</u> <u>NOTES</u> $\xi_\parallel$ only appears in one place $(\nabla\cdot\underset{\sim}{\xi})^2$

can frequently pick $\xi_\parallel$ s.t. $(\nabla\cdot\xi)^2 = 0$

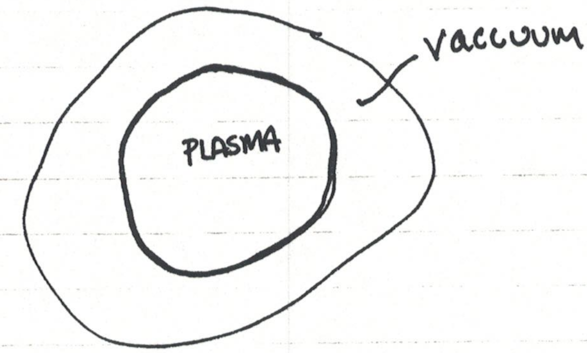
eliminate one term automatically

$$\underset{\sim}{b}\cdot\underset{\sim}{B}_1 =$$



wants to cork screw

what if we have a vacum region



* smooth pressure profile

$p_0 = 0$ on surface

$p_0' = 0$ on "

no surface currents

~~δW_f~~ ·

$$P_1 = - \xi \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \xi$$

$P_1 = 0$ on S

IN vacuum $\nabla \times \underset{\sim}{B}^1 = 0$

$$\delta W_v = \frac{1}{2} \int d^3x \, \frac{|\underset{\sim}{B}'|^2}{4\pi}$$

$$\left.\begin{matrix} \underset{\sim}{B}'_t \\ \underset{\sim}{E}^1_t \end{matrix}\right\} \text{continuous on } S$$

$$\Delta W = \Delta W_f + \Delta W_v$$

SURFACE TERM

$$\frac{1}{2}\int_S d\vec{A} \cdot \Big[\underbrace{\underset{\sim}{\xi}\, p'}_{\nabla p'} + \frac{\underset{\sim}{B}_1 \times (\xi \times B_0)}{4\pi} \Big]$$

POYNTING FLUX

$$\underset{\sim}{E}^{(1)} = -c\, \frac{\partial \xi}{\partial t} \times \underset{\sim}{B}_0$$

IF THE SURFACE IS A conducting, RIGID BOUNDARY

BOUNDARY

$\xi \cdot d\vec{A} = 0$ RIGID BOUNDARY

$E_t = 0$ CONDUCTING BOUNDARY