Introduce concept for plasma waves

$$\boxed{U = \frac{\overline{E}^2}{8\pi} + \underbrace{\tfrac{1}{2} m n v_{os}^2}_{K.E.}}$$

WAVE ENERGY & MOMENTUM

MAXWELLS EQUATIONS

$$\nabla \times \underset{\sim}{E} = -\frac{1}{c}\frac{\partial \underset{\sim}{B}}{\partial t} \qquad \nabla \cdot \underset{\sim}{B} = 0$$

$$\nabla \cdot \underset{\sim}{E} = 4\pi\rho \qquad \nabla \times \underset{\sim}{B} = \frac{4\pi}{c}\underset{\sim}{J} + \frac{1}{c}\frac{\partial \underset{\sim}{E}}{\partial t}$$

$$\underset{\sim}{E} \cdot \nabla \times \underset{\sim}{B} - \underset{\sim}{B} \cdot \nabla \times \underset{\sim}{E} = -\nabla \cdot \underset{\sim}{E} \times \underset{\sim}{B}$$

$$= \frac{4\pi}{c}\underset{\sim}{E} \cdot \underset{\sim}{J} + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{E^2 + B^2}{2}\right)$$

$$\frac{\partial}{\partial t}\frac{E^2 + B^2}{8\pi} + \nabla \cdot \frac{c}{4\pi}\underset{\sim}{E} \times \underset{\sim}{B} = -E \cdot j$$

POYNTING'S THEOREM

$$\frac{\partial}{\partial t} U + \nabla \cdot \underset{\sim}{S} = -\underset{\sim}{E} \cdot \underset{\sim}{j}$$

ELECTRO-MAGNETIC DENSITY — POWER FLUX — WORK DONE BY FIELDS ON CURRENTS

SIMILARLY FOR MOMENTUM

$$\frac{d}{dt}\left(P_{MECH} + P_{FIELD}\right) = +\nabla \cdot \overleftrightarrow{T}$$

WHERE $P_{FIELD} = \dfrac{1}{4\pi c}\,\mathbf{E} \times \mathbf{B}$ $\qquad P_{mech} = \sum_{\sigma} m n \mathbf{v}_{\sigma}$

$$\overleftrightarrow{T} = \frac{1}{4\pi}\left[\mathbf{E}\mathbf{E} + \mathbf{B}\mathbf{B} - \frac{1}{2}\overleftrightarrow{I}\,(E^2 + B^2)\right]$$

WHY IS THIS DISCRIPTION LESS THAN IDEAL FOR WAVES IN DISPERSIVE MEDIA?

ANS: BECAUSE IT KEEPS SEPERATE ACCOUNTS OF PARTICLE ENERGY AND ELECTROMAGNETIC FIELD ENERGY.

THE ENERGY DENSITY (AND MOMENTUM DENSITY) OF A WAVE ~~[illegible]~~ ~~[illegible]~~ HAS TWO COMPONENTS: THE

ELECTROMAGNETIC ENERGY DENSITY $(E^2+B^2)/8\pi$ AND THE ENERGY DENSITY OF THE COHERENT MOTION OF THE PLASMA PARTICLES. WHY IS IT IMPORTANT TO COMBINE THESE TWO? ANS: SUPPOSE I WANT TO MAKE A WAVE WITH GIVEN ELECTRIC FIELD AMPLITUDE E, THE AMOUNT OF ENERGY THAT I MUST INVEST IS THE SUM OF THE ELECTROMAGNETIC AND MECHANICAL ENERGIES. SIMILARLY, IF A WAVE WITH GIVEN AMPLITUDE DAMPS THE AMOUNT OF ENERGY THAT MUST BE DISSIPATED IS AGAIN THE SUM OF THE TWO COMPONENTS. (SIMILAR ARGUMENTS APPLY FOR MOMENTUM) FOR THESE REASONS WE WOULD LIKE AN EXPRESSION FOR THE ENERGY DENSITY, MOMENTUM DENSITY AND POWER FLUX OF A WAVE EXPRESSED IN TERMS OF THE AMPLITUDE OF THE WAVE ELECTRIC FIELD. IN PRINCIPLE THESE CAN BE OBTAINED FROM THE DISTRIBUTION FUNCTION HOWEVER THERE IS AN EASIER WAY.

WAVE PROPAGATION IN AN INFINITE HOMOGENEOUS PLASMA

ELECTRIC FIELD

$$\mathbf{E} = \mathrm{Re}\left(\hat{\mathbf{E}} \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)\right) + \text{c.c.}$$

$$\mathbf{B} = \hat{\mathbf{B}} \cdots$$

MAXWELLS EQUATIONS

$$i\mathbf{k}\times\hat{\mathbf{E}} = \frac{\omega}{c}\hat{\mathbf{B}} \qquad i\mathbf{k}\cdot\hat{\mathbf{E}} = 4\pi\rho$$

$$i\mathbf{k}\times\hat{\mathbf{B}} = \frac{4\pi}{c}\hat{\mathbf{J}} - \frac{i\omega}{c}\hat{\mathbf{E}} \qquad i\mathbf{k}\cdot\hat{\mathbf{B}} = 0$$

DIELECTRIC TENSOR

$$\frac{4\pi}{c}\hat{\mathbf{J}} - \frac{i\omega}{c}\hat{\mathbf{E}} = -\frac{i\omega}{c}\overset{\epsilon}{\underline{\underline{K}}}\cdot\hat{\mathbf{E}}$$

IN GENERAL $\qquad \overset{\epsilon}{\underline{\underline{K}}} = \overset{\epsilon}{\underline{\underline{K}}}(\mathbf{k},\omega)$

$$\underset{\sim}{k} \times \frac{c}{\omega} \underset{\sim}{k} \times \hat{E} = -\frac{\omega}{c} \overset{\epsilon}{\underset{\approx}{K}} \cdot \hat{E}$$

$$\frac{c^2}{\omega^2} \quad \underset{\sim}{k} \times (\underset{\sim}{k} \times \hat{E}) + \overset{\epsilon}{\underset{\approx}{K}} \cdot \hat{E} = 0$$

DISPERSION RELATION

$$\underset{\approx}{G} \cdot \hat{E} = 0 \qquad \mathrm{DET}\,| G | = 0$$

<u>FOR</u> <u>COLD</u> <u>PLASMA</u>

$$\overset{\epsilon}{\underset{\approx}{K}} = \underset{\approx}{I} - \sum_{s} \begin{bmatrix} \frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2} & \frac{i\,\omega_{ps}^2 \Omega_s}{\omega(\omega^2 - \Omega_s^2)} & 0 \\ -\frac{i\,\omega_{ps}^2 \Omega_s}{\omega(\omega^2 - \Omega_s^2)} & \frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2} & 0 \\ 0 & 0 & \frac{\omega_{ps}^2}{\omega^2} \end{bmatrix}$$

$$\omega_{ps}^2 = \frac{4\pi n_s e_s^2}{m_s} \qquad \Omega_s = \frac{e_s B}{m_s c}$$

NOTE: FOR COLD PLASMA $\underset{\approx}{K}$ IS INDEPENDENT OF $\underset{\sim}{k}$ HERMITIAN $\underset{\approx}{K}^{+} = \underset{\approx}{K}^{*}$

METHOD OF DETERMINING ENERGY DENSITY OF WAVE

INSTEAD OF COMPUTING SEPARATELY THE ELECTROMAGNETIC & MECHANICAL PORTIONS OF THE WAVE ENERGY DENSITY LETS ASK "HOW MUCH WORK DO I HAVE TO DO TO CREATE A WAVE OF GIVEN AMPLITUDE". TO DO THIS LETS INTRODUCE A TEST CURRENT AND ASSOCIATED CHARGE DENSITY WHICH IS SEPARATE FROM THE PLASMA CURRENTS AND CHARGES

THAT IS

$$\nabla \cdot \underset{\sim}{E} = \underbrace{4\pi\rho}_{\text{PLASMA}} + \underbrace{4\pi\rho_t}_{\text{TEST CHARGE}}$$

$$\nabla \times \underset{\sim}{B} = \underbrace{\frac{4\pi}{c}\underset{\sim}{J}}_{\text{PLASMA}} + \frac{1}{c}\frac{\partial E}{\partial t} + \underbrace{\frac{4\pi}{c}\underset{\sim}{J}_T}_{\text{TEST CURRENT}}$$

AND LET

$$\underset{\sim}{J}_T = \mathrm{Re}\left\{\hat{\underset{\sim}{J}}_T \exp\left(i\underset{\sim}{k}\cdot\underset{\sim}{x} - i(\omega_r + i\gamma)t\right) \cancel{+\text{c.c.}}\right\}$$

$$\boxed{-i(\omega_r + i\gamma)\hat{\rho}_T + i\underset{\sim}{k}\cdot\hat{J}_T = 0}$$

CONTINUITY OF TEST CHARGE

WHAT IS THE WORK THAT THE TEST CURRENT MUST DO ~~ON THE PLASMA?~~ to build up a wave of a given amplitude?

work done by field on J_T

$$W = -\int_{-\infty}^{t} dt' \, \underset{\sim}{j}_T \cdot \underset{\sim}{E}$$

$$= -\frac{1}{4}\int_{-\infty}^{0} dt' \left[\hat{\underset{\sim}{J}}_T \cdot \hat{\underset{\sim}{E}}^* + \hat{\underset{\sim}{J}}_T^* \cdot \hat{\underset{\sim}{E}} \right] \exp(2\gamma t')$$

$$= -\frac{1}{8\gamma}\left[\hat{\underset{\sim}{J}}_T \cdot \hat{\underset{\sim}{E}}^* + \hat{\underset{\sim}{J}}_T^* \cdot \hat{\underset{\sim}{E}} \right]$$

FROM MAXWELL'S EQUATIONS

$$i\underset{\sim}{k} \times \frac{c\underset{\sim}{k}}{\omega} \times \hat{\underset{\sim}{E}} + \frac{i\omega}{c} \overset{\epsilon}{\underset{\approx}{K}} \cdot \hat{\underset{\sim}{E}} = \frac{4\pi}{c} \hat{\underset{\sim}{J}}_T$$

$$= \frac{i\omega}{c} \underset{\approx}{G} \cdot \hat{\underset{\sim}{E}} = \frac{4\pi}{c} \hat{J}_T$$

$$\boxed{\underset{=}{G} = \underset{=}{\epsilon} - \underset{=}{I}\frac{k^2c^2}{\omega^2} + \frac{\underset{-}{k}\underset{-}{k}c^2}{\omega^2} \quad -\underset{=}{M}}$$

$$W = -\frac{1}{8\gamma}\left[\frac{i\omega}{4\pi} \hat{E}^* \cdot \underset{\approx}{G} \cdot \hat{E} - \frac{i\omega^*}{4\pi} \hat{E} \cdot \underset{\approx}{G}^* \cdot \hat{E}^* \right]$$

$\omega = \omega_r + i\gamma$

transpose

$$W = -\frac{1}{8\gamma}\,\frac{\hat{E}^*}{4\pi}\cdot\left[\, i(\omega\quad)\,\underset{\approx}{G}^{(\omega)} - i(\omega^*\quad)\left(\tilde{\underset{\approx}{G}}^{*}\right)^{\cancel{*}}\,\right]\cdot\hat{E}$$

TAYLOR EXPAND $\quad \omega = \omega_r + i\gamma \quad \gamma \ll \omega_r$

$$\omega\,\underset{\approx}{G} \cong \omega_r\,\underset{\approx}{G}(\omega_r) + i\gamma\,\frac{\partial}{\partial\omega_r}\,\omega_r\,\underset{\approx}{G}(\omega_r)$$

FOR LOSS FREE MEDIUM: $\left(\tilde{\underset{\approx}{G}}^{*}(\omega_r)\right) = \underset{\approx}{G}(\omega_r)$

$$W = \frac{\hat{E}^*}{\cancel{4\pi}\;16\pi}\cdot\frac{\partial}{\partial\omega_r}\,\omega_r\,\underset{\approx}{G}(\omega_r)\cdot\hat{E}$$

W = ENERGY DENSITY OF WAVE

FOR DISSIPATING MEDIUM $\left(\underset{\approx}{G}^{*}(\omega_r)\right)^{\dagger} \neq \underset{\approx}{G}(\omega_r)$

BUT IF $\quad \underline{\underline{G}}^{*\dagger} - \underline{\underline{G}} \ll \underline{\underline{G}}^{*\dagger} + \underline{\underline{G}}$

(WEAKLY DISSIPATIVE THE EXPRESSION FOR W IS STILL VALID)

COMPONENTS OF W

$$W = \frac{1}{16\pi}\, \hat{E}^* \cdot \frac{\partial}{\partial \omega_r} \omega_r \left[\overbrace{\frac{c^2}{\omega_r^2}\, \underline{k} \times \underline{k} \times \hat{E} + \underbrace{\underline{\underline{K}}}_{\underline{\underline{\epsilon}}} \cdot \hat{E}}^{G \cdot \hat{E}} \right]$$

PLASMA CONTRIBUTION

$$\underline{\underline{\epsilon}} \quad \cancel{K} = \underline{\underline{I}} + \underline{\underline{\Sigma}} \quad \frac{4\pi i \underline{\underline{\sigma}}}{\omega}$$

DISPLACEMENT CURRENT

$$W = \frac{1}{4\pi}\left[- \hat{E}^* \frac{c^2}{\omega_r^2} \cdot \underline{k} \times \underline{k} \times \hat{E} + \hat{E}^* \cdot \hat{E} + \underline{\hat{E}}^* \cdot \frac{\partial}{\partial \omega_r} \omega_r \underline{\underline{\Sigma}} \cdot \hat{E} \right]$$

$$\frac{c}{\omega_r}\, \underline{k} \times \hat{E} = \hat{B}, \quad -\frac{c}{\omega_r}\, \hat{E}^* \cdot \underline{k} \times \hat{B} = \hat{B} \cdot \frac{c}{\omega_r}\, \underline{k} \times \hat{E}^*$$

$$= \hat{B} \cdot \hat{B}^*$$

$$W = \frac{1}{\cancel{4\pi}\,16\pi}\left[\underbrace{\hat{E}^* \cdot \hat{E} + \hat{B}^* \cdot \hat{B}}_{\text{FIELD CONTRIBUTION}} + \underbrace{\hat{E}^* \cdot \frac{\partial}{\partial \omega_r}\, \omega_r \overset{(\underline{\underline{\epsilon}} - \underline{\underline{1}})}{\underline{\underline{\epsilon}}} \cdot \hat{E}}_{\text{PLASMA CONTRIBUTION}}\right]$$

Momentum Density of Wave

IN A SIMILAR MANNER WE MAY CALCULATE THE MOMENTUM DENSITY OF THE WAVE.

WHAT IS THE RATE AT WHICH MOMENTUM IS TRANSFERED FROM THE TEST CURRENT TO THE WAVE.

$$\cancel{\rho_t \underline{E} + \frac{j}{c}}$$

$$\frac{d}{dt}\underline{P}_t = e_t n_t \underline{E} + \frac{e_t n_t \underline{V}_t \times \underline{B}}{c}$$

$$\underline{P}_t = +\int_{-\infty}^{t} dt' \left[\rho_t \underline{E} + \frac{1}{c}\underline{j}_t \times \underline{B}\right]$$

$$\underset{\sim}{P}_t = \int_{-\infty}^{t} dt' \left[\hat{\rho}_t \, \hat{\underset{\sim}{E}}^* + \rho_t^* \hat{\underset{\sim}{E}} + \frac{1}{c} \hat{\underset{\sim}{J}}_t^* \times \hat{\underset{\sim}{B}} + \frac{1}{c} \hat{J}_t \times \hat{B}^* \right] e^{2\gamma}$$

$$i \underset{\sim}{k} \cdot \hat{E} = \overset{4\pi}{\hat{\rho}_t} \qquad \hat{B} = \frac{c}{\omega} \underset{\sim}{k} \times \hat{E}$$

$$\frac{1}{c} \hat{J}_t \times \hat{B}^* = \hat{J}_t \frac{1}{\omega^*} \times \underset{\sim}{k} \times \hat{E}^*$$

$$= \frac{\underset{\sim}{k}}{\omega^*} \hat{J}_t \cdot \hat{E}^* - \hat{E}^* \frac{k \cdot \hat{J}_t}{\omega^*}$$

$\omega_r \gg \gamma$

$$\cong \frac{\underset{\sim}{k}}{\omega} \hat{J}_t \cdot \underset{\sim}{E}^* - \hat{\rho}_t \underset{\sim}{E}^*$$

$$\underset{\sim}{P}_t = \int_{-\infty}^{t} dt' \exp(2\gamma t') \frac{\underset{\sim}{k}}{\omega} \left[\hat{\underset{\sim}{J}}_t \cdot \hat{\underset{\sim}{E}}^* + \hat{\underset{\sim}{J}}^* \cdot \hat{\underset{\sim}{E}} \right]$$

$$\underset{\sim}{P}_w = -\underset{\sim}{P}_t = + \frac{\underset{\sim}{k}}{\omega} \quad W$$

$$\underset{m}{P}_{\omega} = \frac{\underset{m}{k}}{\omega} W$$

QUANTUM MECHANICAL ANALOGY

$$\boxed{\begin{array}{l} \underset{m}{p} = \hbar \underset{m}{k} \qquad \mathcal{E} = \hbar\omega \\ \underset{m}{p} = \dfrac{\underset{m}{k}}{\omega} \mathcal{E} \end{array}}$$

THUS, EVEN FOR A DISPERSIVE ANISOTROPIC MEDIUM THE MOMENTUM DENSITY AND ENERGY DENSITY ARE SIMPLY RELATED.

~~Skip~~

Introducting wave action

$N_k \equiv$ the wave „action" $= \frac{1}{4}\epsilon_0 |\underline{E}_k|^2 \frac{\partial \epsilon}{\partial \omega}\Big|_{\omega=\omega_k}$

$$\boxed{\begin{aligned} U_k &= \omega_k N_k \\ P_k &= k N_k \end{aligned}}$$

analogy with quantum mechanics

$\hbar\omega$ photon energy

$\hbar k$ photon momentum

$N_k \to \hbar$ number of plasmons

The amount of energy to set up the wave. There are two states of the plasma before the wave ~~the~~ set up.

The energy in plasma ~~on~~ should consist from two part: energy of electrostatic field and kinetic energy of particles moving due to this field

Electron Plasma Oscillation

$$\epsilon = 1 - \frac{\omega_{pe}^2}{\omega^2} \qquad \omega_k = \omega_p$$

$$\frac{\partial \epsilon}{\partial \omega} = \frac{2\omega_{pe}^2}{\omega^3} \qquad \omega \frac{\partial \epsilon}{\partial \omega}\Big|_{\omega=\omega_k} = 2$$

$$U_k = \underbrace{\frac{1}{4}\epsilon_0 \, |E_k|^2}_{\text{time avearage}} \cdot 2$$

time avearage of electrostatic energy density (one $\frac{1}{2}$ - average over time, another $\frac{1}{2}$ - avaray ~~over~~ of E)

factor 2 shows, that avaraged kinetic energy density of the oscillating particles (electrons) $= \frac{1}{4}\epsilon_0 |E_k|^2$

Group Velocity & Power Flux

Group Velocity

SUPPOSE WE SOLVE

$$\mathrm{Det}\,\left|\underline{\underline{G}}(\omega, \underline{k})\right| = 0$$

AND OBTAIN THE SOLUTION   Talk about

$\omega = \omega(\underline{k})$ THE GROUP VELOCITY

IS DEFINED AS $\partial\omega / \partial \underline{k} \equiv \underline{V}_g$

THIS IS THE [redacted] VELOCITY OF A WAVE PACKET AND ALSO THE RATE AT WHICH THE ENERGY DENSITY IS TRANSPORTED

$$\underline{\Gamma} = \underline{V}_g W$$

$\underline{\Gamma}$ = POWER FLUX $\left(\dfrac{\text{ergs}}{\text{SEC} \cdot \text{CM}^2}\right)$

INORDER TO DETERMINE $\underset{\sim}{\Gamma}$ & $\underset{\sim}{V}_g$

$$\underset{\approx}{G} \cdot \underset{\sim}{E} = 0$$

LET $\omega = \omega_0 + \delta\omega$ $\qquad$ $\underset{\sim}{k} = \underset{\sim}{k}_0 + \delta \underset{\sim}{k}$

$$\underset{\sim}{E} = \underset{\sim}{E}_0 + \delta \underset{\sim}{E}$$

LINEARIZE:

$$\overset{0}{\underset{\approx}{G}(\omega_0, \underset{\sim}{k}_0) \cdot \underset{\sim}{E}_0} + \underset{\approx}{G}(\omega_0, \underset{\sim}{k}_0) \cdot \delta \underset{\sim}{E}$$

$$+ \delta\omega \frac{\partial}{\partial \omega_0} \underset{\approx}{G}(\omega_0, \underset{\sim}{k}_0) \cdot \underset{\sim}{E}_0 + \delta \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{k}_0} \underset{\approx}{G}(\omega_0, \underset{\sim}{k}_0) \cdot \underset{\sim}{E}_0 = 0$$

DOT ON LEFT WITH E_0^* & ASSUME $\underset{\approx}{G}$ IS HERMITIAN

$$\underset{\sim}{E}^0 \cdot \underset{\approx}{G} = 0$$

$$\delta\omega \qquad \underset{\sim}{E}_0^* \cdot \frac{\partial}{\partial \omega_0} \omega_0 \underset{\approx}{G} \cdot \underset{\sim}{E}_0 = - \delta \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{k}_0} \omega_0 \underset{\sim}{E}_0^* \cdot \underset{\approx}{G} \cdot \underset{\sim}{E}$$

$$\delta\omega \; W = +\,\delta \underset{\sim}{k} \cdot \left\{ -\frac{\omega_0}{4\pi} \frac{\partial}{\partial \underset{\sim}{k}_0} \underset{\sim}{E}_0^* \cdot \underset{\approx}{G} \cdot \underset{\sim}{E}_0 \right\}$$

IN OTHER WORDS

$$\delta\omega \, W = \delta \underset{\sim}{k} \cdot \underset{\sim}{V}_g W$$

$$\underset{\sim}{\Gamma} = -\frac{\omega_0}{4\pi} \frac{\partial}{\partial \underset{\sim}{k}_0} \hat{\underset{\sim}{E}}_0^* \cdot \underset{\approx}{G} \cdot \hat{\underset{\sim}{E}}_0$$

$$\underset{\sim}{\Gamma} = \underset{\sim}{S} + \underset{\sim}{T}$$

$$\underset{\sim}{S} = \frac{c}{4\pi} \left[\hat{\underset{\sim}{E}}_0^* \times \hat{\underset{\sim}{B}}_0 + \hat{\underset{\sim}{E}}_0 \times \hat{\underset{\sim}{B}}^* \right] / 4 \qquad \text{POYNTING}$$

$$\underset{\sim}{T} = -\frac{\omega_0}{4\pi} \frac{\partial}{\partial \underset{\sim}{k}_0} \hat{\underset{\sim}{E}}_0^* \cdot \underset{\approx}{\Sigma} \cdot \hat{E}_0 / 4 \qquad \text{PARTICLE CONT.}$$

* FOR COLD PLASMA $\underset{\approx}{\Sigma}$ IS INDEPENDENT OF $\underset{\sim}{k}$ ALL ENERGY FLUX IS POYNTING.

DAMPING: SUPPOSE

$$\underline{\underline{G}} = \underline{\underline{G}}^h + i\,\underline{\underline{G}}^a$$

WHERE $\left(\underline{\underline{G}}^h\right)^* = \underline{\underline{G}}^+$

$$\underline{\underline{G}}^h \gg \underline{\underline{G}}^a$$

$$\underline{E} = \underline{E}_0 + \delta\underline{E} \qquad \omega = \omega_0 + \delta\omega$$

$$\delta\omega \, \frac{\partial}{\partial\omega_0}\, \omega_0 \underline{E}^{0*} \cdot \underline{\underline{G}}^h \cdot \underline{E}^0 + \underline{E}^{0*} \cdot \omega \underline{\underline{G}}^a \cdot \underline{E}^0 = 0$$

$$\delta\omega \cdot W = -\frac{i}{4\pi}\overbrace{\omega_0\, E^{0*} \cdot \underline{\underline{G}}^a \cdot \underline{E}^0}^{\text{RATE AT WHICH ENERGY IS DISSIPATED}}$$

$\therefore$

$$\text{DAMPING RATE} = \frac{\omega_0}{4\pi}\,\frac{E^{0*} \cdot \underline{\underline{G}}^a \cdot E^0}{W}$$