Dynamics of Plasmas in Strong Magnetic Fields

So far we have considered dynamics of unmagnetized plasmas and motion of charged particles in strong, but prescribed fields. Now we will begin the study of plasma motion ~~with~~ in strong magnetic fields.

Problem is simple, just solve VE

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \nabla f + \frac{q}{m}\left(\underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c}\right) \cdot \frac{\partial f}{\partial \underset{\sim}{v}} = C(f)$$

+ ~~THIS IS NOT SO easy~~ Maxwell's

$$\nabla \cdot \underset{\sim}{E} = \sum_{q_i} 4\pi \int d^3v\, q f \qquad \nabla \cdot \underset{\sim}{B} = 0$$

$$\nabla \times \underset{\sim}{B} = \frac{4\pi}{c} \sum_{q_i} \int d^3v\, q \underset{\sim}{v} f + \frac{1}{c}\frac{\partial E}{\partial t}$$

$$-\frac{1}{c}\frac{\partial}{\partial t} \underset{\sim}{E} = \nabla \times \underset{\sim}{B}$$

This is not easy. Even the simplest case of linear waves in straight magnetic fields quickly degenerates into a Bessel fest. Lot's of insight is obtained from fluid description

Fluid equations

$$\frac{\partial n_q}{\partial t} + \nabla \cdot n_q \underset{\sim}{u}_q = 0$$ — continuity

$$m_q n_q \left(\frac{\partial \underset{\sim}{u}_q}{\partial t} + \underset{\sim}{u}_q \cdot \nabla \underset{\sim}{u}_q \right) = - \nabla \cdot \underset{\approx}{\Pi}_q + q n_q \left(\underset{\sim}{E} + \frac{\underset{\sim}{u}_q \times \underset{\sim}{B}}{c} \right) + \underset{\sim}{R}_q$$

← momentum

+ equation giving $\underline{\underline{\Pi}}$

$\underset{\sim}{R}_e = - \underset{\sim}{R}_i$ momentum conserved by electron-ion collisions

* To use these we need some scheme for coming up with $\underset{\approx}{\Pi}_q$ and $\underset{\sim}{R}_q$

* I will defer this problem until later since it really opens up a can of worms

* See Bragunskii Equations and discussion in Fitzpatrick 3.8. It's probably more than you want to know.

S.I. Braginskii, Transport Processes in a Plasma Reviews of Plasma Physics

Consultants Bureau NY (1965) V. 9 p. 205

Let's approach ~~the~~ things assuming the simplest $\underset{\approx}{\Pi}$ and $\underset{\sim}{R}$

$$\underset{\approx}{\Pi} = \underset{\approx}{I}\, p$$

p given by adiabatic equation of state

$$\underset{\sim}{R}_e = -m_e \nu_{ei} (\underset{\sim}{u}_e - \underset{\sim}{u}_i) = -\underset{\sim}{R}_i$$

collisions between electrons and ions produce a "drag" causing relaxation of velocity difference.

Slight impovement

$$\underset{\approx}{\Pi} = \begin{bmatrix} P_\perp & 0 & 0 \\ 0 & P_\perp & 0 \\ 0 & 0 & P_\parallel \end{bmatrix} = P_\perp \underset{\approx}{I} + (P_\parallel - P_\perp)\underset{\sim}{b}\underset{\sim}{b}$$

$\underset{\sim}{b}$ = direction of magnetic field

$P_\perp$ & $P_\parallel$ satisfy separate adiabatic laws. $\boxed{\frac{\partial}{\partial t} \gg \nu_c \text{ collision rate}}$

Even this system is complicated, large ~~terms~~ parameters, further simplifications are possible.

Simplest Case:

Cold plasma $T_e = T_i \to 0$ $\underline{\underline{P}} = 0$

This is a good approximation if

$|\underset{\sim}{u}| >> V_{th}$ thermal velocity

or if $\omega/K >> V_{th}$

actually $\frac{\partial}{\partial t} \gg V_{th} \nabla$

We have already investigated electrostatic waves in a cold unmagnetized plasma

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This model is also appropriate for high frequency E&M waves

As one lowers frequency $\frac{\omega}{K} \sim V_{th}$

and pressure effects must be considered.

If species α is a weak beam (number and energy density small compared with background) streaming through a Maxwellian plasma, then

$$\mathbf{J}^{\alpha\backslash\beta} = -\frac{m_\alpha}{m_\alpha + m_\beta}\nu_s^{\alpha\backslash\beta}\mathbf{v}f^\alpha - \frac{1}{2}\nu_\parallel^{\alpha\backslash\beta}\mathbf{vv}\cdot\nabla_{\mathbf{v}}f^\alpha - \frac{1}{4}\nu_\perp^{\alpha\backslash\beta}\left(v^2 \mathsf{I} - \mathbf{vv}\right)\cdot\nabla_{\mathbf{v}}f^\alpha.$$

B-G-K Collision Operator

For distribution functions with no large gradients in velocity space, the Fokker-Planck collision terms can be approximated according to

$$\frac{Df_e}{Dt} = \nu_{ee}(F_e - f_e) + \nu_{ei}(\bar{F}_e - f_e);$$

$$\frac{Df_i}{Dt} = \nu_{ie}(\bar{F}_i - f_i) + \nu_{ii}(F_i - f_i).$$

The respective slowing-down rates $\nu_s^{\alpha\backslash\beta}$ given in the Relaxation Rate section above can be used for $\nu_{\alpha\beta}$, assuming slow ions and fast electrons, with ϵ replaced by T_α. (For ν_{ee} and ν_{ii}, one can equally well use $\nu_\perp$, and the result is insensitive to whether the slow- or fast-test-particle limit is employed.) The Maxwellians F_α and $\bar{F}_\alpha$ are given by

$$F_\alpha = n_\alpha\left(\frac{m_\alpha}{2\pi kT_\alpha}\right)^{3/2}\exp\left\{-\left[\frac{m_\alpha(\mathbf{v}-\mathbf{v}_\alpha)^2}{2kT_\alpha}\right]\right\};$$

$$\bar{F}_\alpha = n_\alpha\left(\frac{m_\alpha}{2\pi k\bar{T}_\alpha}\right)^{3/2}\exp\left\{-\left[\frac{m_\alpha(\mathbf{v}-\bar{\mathbf{v}}_\alpha)^2}{2k\bar{T}_\alpha}\right]\right\},$$

where n_α, $\mathbf{v}_\alpha$ and T_α are the number density, mean drift velocity, and effective temperature obtained by taking moments of f_α. Some latitude in the definition of $\bar{T}_\alpha$ and $\bar{\mathbf{v}}_\alpha$ is possible;[20] one choice is $\bar{T}_e = T_i$, $\bar{T}_i = T_e$, $\bar{\mathbf{v}}_e = \mathbf{v}_i$, $\bar{\mathbf{v}}_i = \mathbf{v}_e$.

Transport Coefficients

Transport equations for a multispecies plasma:

$$\frac{d^\alpha n_\alpha}{dt} + n_\alpha\nabla\cdot\mathbf{v}_\alpha = 0;$$

$$m_\alpha n_\alpha\frac{d^\alpha\mathbf{v}_\alpha}{dt} = -\nabla p_\alpha - \nabla\cdot \mathsf{P}_\alpha + Z_\alpha e n_\alpha\left[\mathbf{E} + \frac{1}{c}\mathbf{v}_\alpha\times\mathbf{B}\right] + \mathbf{R}_\alpha;$$

$$\frac{3}{2} n_\alpha \frac{d^\alpha kT_\alpha}{dt} + p_\alpha \nabla \cdot \mathbf{v}_\alpha = -\nabla \cdot \mathbf{q}_\alpha - P_\alpha : \nabla \mathbf{v}_\alpha + Q_\alpha .$$

Here $d^\alpha/dt \equiv \partial/\partial t + \mathbf{v}_\alpha \cdot \nabla$; $p_\alpha = n_\alpha kT_\alpha$, where k is Boltzmann's constant; $\mathbf{R}_\alpha = \sum_\beta \mathbf{R}_{\alpha\beta}$ and $Q_\alpha = \sum_\beta Q_{\alpha\beta}$, where $\mathbf{R}_{\alpha\beta}$ and $Q_{\alpha\beta}$ are respectively the momentum and energy gained by the αth species through collisions with the βth; P_α is the stress tensor; and $\mathbf{q}_\alpha$ is the heat flow.

The transport coefficients in a simple two-component plasma (electrons and singly charged ions) are tabulated below. Here $\parallel$ and $\perp$ refer to the direction of the magnetic field $\mathbf{B} = \mathbf{b}B$; $\mathbf{u} = \mathbf{v}_e - \mathbf{v}_i$ is the relative streaming velocity; $n_e = n_i \equiv n$; $\mathbf{j} = -ne\mathbf{u}$ is the current; $\omega_{ce} = 1.76 \times 10^7 B \sec^{-1}$ and $\omega_{ci} = (m_e/m_i)\omega_{ce}$ are the electron and ion gyrofrequencies, respectively; and the basic collisional times are taken to be

$$\tau_e = \frac{3\sqrt{m_e}(kT_e)^{3/2}}{4\sqrt{2\pi}\, n\lambda e^4} = 3.44 \times 10^5 \frac{{T_e}^{3/2}}{n\lambda} \sec,$$

where λ is the Coulomb logarithm, and

$$\tau_i = \frac{3\sqrt{m_i}(kT_i)^{3/2}}{4\sqrt{\pi} n\, \lambda e^4} = 2.09 \times 10^7 \frac{{T_i}^{3/2}}{n\lambda} \mu^{1/2} \sec.$$

In the limit of large fields ($\omega_{c\alpha}\tau_\alpha \gg 1$, $\alpha = i, e$) the transport processes may be summarized as follows:[21]

momentum transfer	$\mathbf{R}_{ei} = -\mathbf{R}_{ie} \equiv \mathbf{R} = \mathbf{R}_\mathbf{u} + \mathbf{R}_T$;
frictional force	$\mathbf{R}_\mathbf{u} = ne(\mathbf{j}_\parallel/\sigma_\parallel + \mathbf{j}_\perp/\sigma_\perp)$;
electrical conductivities	$\sigma_\parallel = 1.96\sigma_\perp$; $\sigma_\perp = ne^2\tau_e/m_e$;
thermal force	$\mathbf{R}_T = -0.71n\nabla_\parallel(kT_e) - \frac{3n}{2\omega_{ce}\tau_e}\mathbf{b} \times \nabla_\perp(kT_e)$;
ion heating	$Q_i = \frac{3m_e}{m_i}\frac{nk}{\tau_e}(T_e - T_i)$;
electron heating	$Q_e = -Q_i - \mathbf{R} \cdot \mathbf{u}$;
ion heat flux	$\mathbf{q}_i = -\kappa_\parallel^i \nabla_\parallel(kT_i) - \kappa_\perp^i \nabla_\perp(kT_i) + \kappa_\wedge^i \mathbf{b} \times \nabla_\perp(kT_i)$;
ion thermal conductivities	$\kappa_\parallel^i = 3.9\frac{nkT_i\tau_i}{m_i}$; $\kappa_\perp^i = \frac{2nkT_i}{m_i{\omega_{ci}}^2\tau_i}$; $\kappa_\wedge^i = \frac{5nkT_i}{2m_i\omega_{ci}}$;
electron heat flux	$\mathbf{q}_e = \mathbf{q}_\mathbf{u}^e + \mathbf{q}_T^e$;
frictional heat flux	$\mathbf{q}_\mathbf{u}^e = 0.71nkT_e\mathbf{u}_\parallel + \frac{3nkT_e}{2\omega_{ce}\tau_e}\mathbf{b} \times \mathbf{u}_\perp$;

thermal gradient heat flux	$\mathbf{q}_T^e = -\kappa_\parallel^e \nabla_\parallel (kT_e) - \kappa_\perp^e \nabla_\perp (kT_e) - \kappa_\wedge^e \mathbf{b} \times \nabla_\perp (kT_e);$
electron thermal conductivities	$\kappa_\parallel^e = 3.2 \dfrac{nkT_e\tau_e}{m_e}; \quad \kappa_\perp^e = 4.7 \dfrac{nkT_e}{m_e \omega_{ce}^2 \tau_e}; \quad \kappa_\wedge^e = \dfrac{5nkT_e}{2m_e\omega_{ce}};$
stress tensor (either species)	$P_{xx} = -\dfrac{\eta_0}{2}(W_{xx} + W_{yy}) - \dfrac{\eta_1}{2}(W_{xx} - W_{yy}) - \eta_3 W_{xy};$

$$P_{yy} = -\frac{\eta_0}{2}(W_{xx} + W_{yy}) + \frac{\eta_1}{2}(W_{xx} - W_{yy}) + \eta_3 W_{xy};$$

$$P_{xy} = P_{yx} = -\eta_1 W_{xy} + \frac{\eta_3}{2}(W_{xx} - W_{yy});$$

$$P_{xz} = P_{zx} = -\eta_2 W_{xz} - \eta_4 W_{yz};$$

$$P_{yz} = P_{zy} = -\eta_2 W_{yz} + \eta_4 W_{xz};$$

$$P_{zz} = -\eta_0 W_{zz}$$

(here the z axis is defined parallel to $\mathbf{B}$);

ion viscosity	$\eta_0^i = 0.96 nkT_i\tau_i; \quad \eta_1^i = \dfrac{3nkT_i}{10\omega_{ci}^2\tau_i}; \quad \eta_2^i = \dfrac{6nkT_i}{5\omega_{ci}^2\tau_i};$
	$\eta_3^i = \dfrac{nkT_i}{2\omega_{ci}}; \quad \eta_4^i = \dfrac{nkT_i}{\omega_{ci}};$
electron viscosity	$\eta_0^e = 0.73 nkT_e\tau_e; \quad \eta_1^e = 0.51 \dfrac{nkT_e}{\omega_{ce}^2\tau_e}; \quad \eta_2^e = 2.0 \dfrac{nkT_e}{\omega_{ce}^2\tau_e};$
	$\eta_3^e = -\dfrac{nkT_e}{2\omega_{ce}}; \quad \eta_4^e = -\dfrac{nkT_e}{\omega_{ce}}.$

For both species the rate-of-strain tensor is defined as

$$W_{jk} = \frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} - \frac{2}{3}\delta_{jk} \nabla \cdot \mathbf{v}.$$

When $\mathbf{B} = 0$ the following simplifications occur:

$$\mathbf{R_u} = ne\mathbf{j}/\sigma_\parallel; \quad \mathbf{R}_T = -0.71 n\nabla(kT_e); \quad \mathbf{q}_i = -\kappa_\parallel^i \nabla(kT_i);$$

$$\mathbf{q_u}^e = 0.71 nkT_e\mathbf{u}; \quad \mathbf{q}_T^e = -\kappa_\parallel^e \nabla(kT_e); \quad P_{jk} = -\eta_0 W_{jk}.$$

For $\omega_{ce}\tau_e \gg 1 \gg \omega_{ci}\tau_i$, the electrons obey the high-field expressions and the ions obey the zero-field expressions.

Collisional transport theory is applicable when (1) macroscopic time rates of change satisfy $d/dt \ll 1/\tau$, where τ is the longest collisional time scale, and (in the absence of a magnetic field) (2) macroscopic length scales L satisfy $L \gg l$, where $l = \bar{v}\tau$ is the mean free path. In a strong field, $\omega_{ce}\tau \gg 1$, condition (2) is replaced by $L_\parallel \gg l$ and $L_\perp \gg \sqrt{lr_e}$ ($L_\perp \gg r_e$ in a uniform field), where $L_{\parallel}$ is a macroscopic scale parallel to the field $\mathbf{B}$ and $L_{\perp}$ is the smaller of $B/|\nabla_{\perp} B|$ and the transverse plasma dimension. In addition, the standard transport coefficients are valid only when (3) the Coulomb logarithm satisfies $\lambda \gg 1$; (4) the electron gyroradius satisfies $r_e \gg \lambda_D$, or $8\pi n_e m_e c^2 \gg B^2$; (5) relative drifts $\mathbf{u} = \mathbf{v}_\alpha - \mathbf{v}_\beta$ between two species are small compared with the thermal velocities, i.e., $u^2 \ll kT_\alpha/m_\alpha$, kT_β/m_β; and (6) anomalous transport processes owing to microinstabilities are negligible.

Weakly Ionized Plasmas

Collision frequency for scattering of charged particles of species α by neutrals is

$$\nu_\alpha = n_0 \sigma_s^{\alpha\backslash 0} (kT_\alpha/m_\alpha)^{1/2},$$

where n_0 is the neutral density and $\sigma_s^{\alpha\backslash 0}$ is the cross section, typically $\sim 5 \times 10^{-15}$ cm^2 and weakly dependent on temperature.

When the system is small compared with a Debye length, $L \ll \lambda_D$, the charged particle diffusion coefficients are

$$D_\alpha = kT_\alpha/m_\alpha \nu_\alpha,$$

In the opposite limit, both species diffuse at the ambipolar rate

$$D_A = \frac{\mu_i D_e - \mu_e D_i}{\mu_i - \mu_e} = \frac{(T_i + T_e) D_i D_e}{T_i D_e + T_e D_i},$$

where $\mu_\alpha = e_\alpha/m_\alpha \nu_\alpha$ is the mobility. The conductivity σ_α satisfies $\sigma_\alpha = n_\alpha e_\alpha \mu_\alpha$.

In the presence of a magnetic field $\mathbf{B}$ the scalars μ and σ become tensors,

$$\mathbf{J}^\alpha = \boldsymbol{\sigma}^\alpha \cdot \mathbf{E} = \sigma_\parallel^\alpha \mathbf{E}_\parallel + \sigma_\perp^\alpha \mathbf{E}_\perp + \sigma_\wedge^\alpha \mathbf{E} \times \mathbf{b},$$

where $\mathbf{b} = \mathbf{B}/B$ and

$$\sigma_\parallel^\alpha = n_\alpha {e_\alpha}^2/m_\alpha \nu_\alpha;$$

$$\sigma_\perp^\alpha = \sigma_\parallel^\alpha {\nu_\alpha}^2/({\nu_\alpha}^2 + \omega_{c\alpha}^2);$$

$$\sigma_\wedge^\alpha = \sigma_\parallel^\alpha \nu_\alpha \omega_{c\alpha}/({\nu_\alpha}^2 + \omega_{c\alpha}^2).$$

Here $\sigma_\perp$ and $\sigma_\wedge$ are the Pedersen and Hall conductivities, respectively.

Electromagnetic Waves in a Cold Plasma

Assume $\underset{\sim}{E} = Re\{ \hat{E} e^{-i(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{x})} \}$

$\underset{\sim}{u}_{e,i} = Re\{ \hat{u}_{e,i} e^{-i(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{x})} \}$

Take $\underset{\sim}{B} = B_0 \underset{\sim}{a}_z$ $\qquad B_0 = \text{const}$

Solve for $\underset{\sim}{J} = \sum_{e,i} q_{e,i} n_{e,i} \underset{\sim}{u}$

Linearize fluid equations

$$\frac{\partial}{\partial t}\underset{\sim}{u} + \underbrace{\underset{\sim}{u}\cdot\nabla\underset{\sim}{u}}_{\text{non linear}} = +\frac{q}{m}\left(\underset{\sim}{E} + \frac{\underset{\sim}{u}\times B_0 a_z}{c}\right)$$

$$-i\omega\hat{\underset{\sim}{u}} = \frac{q}{m}\hat{\underset{\sim}{E}} + \underset{\sim}{u}\times\underset{\sim}{a}_z\Omega_c \qquad \Omega_c = \frac{qB}{mc}$$

solve for $\hat{\underset{\sim}{u}} = \hat{\underset{\sim}{u}}_\perp + u_\parallel \hat{\underset{\sim}{a}}_z$

$$\cancel{-i\omega(\hat{\underset{\sim}{u}}\times\underset{\sim}{a}_z) = \frac{q}{m}\hat{\underset{\sim}{E}}\times\underset{\sim}{a}_z + \Omega_c(\underset{\sim}{u}\times\underset{\sim}{a}_z)\times\underset{\sim}{a}_z}$$

$$\boxed{-i\omega u_\parallel = \frac{q}{m}\underset{\sim}{a}_z\cdot\hat{\underset{\sim}{E}}}$$

motion parallel to magnetic field unaffected by $\underset{\sim}{B}$

$$-i\omega\,\hat{\mathbf{u}}_\perp = \frac{q}{m}\mathbf{E}_\perp + \hat{\mathbf{u}}_\perp \times \mathbf{a}_z\,\Omega_c$$

$$\hat{\mathbf{u}}_\perp = \frac{1}{(-i\omega)}\left[\frac{q}{m}\mathbf{E}_\perp + \Omega_c\,\hat{\mathbf{u}}_\perp \times \mathbf{a}_z\right]$$

$$\Omega_c(\mathbf{u}_\perp \times \mathbf{a}_z) = \frac{\Omega_c}{-i\omega}\left[\frac{q}{m}\mathbf{E}_\perp \times \mathbf{a}_z - \Omega_c\,\hat{\mathbf{u}}_\perp\right]$$

$$-i\omega\,\mathbf{u}_\perp = \frac{q}{m}\mathbf{E}_\perp - \frac{\Omega_c}{i\omega}\left[\frac{q}{m}\mathbf{E}_\perp \times \mathbf{a}_z - \Omega_c\mathbf{u}_\perp\right]$$

$$(\omega^2 - \Omega_c^2)\,\mathbf{u}_\perp = i\omega\frac{q}{m}\mathbf{E}_\perp - \Omega_c\frac{q}{m}\mathbf{E}_\perp \times \mathbf{a}_z$$

$$\boxed{\hat{\mathbf{u}}_\perp = \frac{1}{\omega^2 - \Omega_c^2}\left[i\omega\frac{q}{m}\hat{\mathbf{E}}_\perp - \Omega_c\frac{q}{m}\hat{\mathbf{E}}_\perp \times \mathbf{a}_z\right]}$$

Check against known limits

#1) $\Omega_c \to 0$ (unmagnetized)

$$\mathbf{u}_\perp = \frac{i}{\omega}\frac{q}{m}\mathbf{E}_\perp$$ OK (just like $u_\parallel$)

#2) $\Omega_c \to \infty$ (~~...~~ strong B)

$$\mathbf{u}_\perp = \frac{1}{\Omega_c}\frac{q}{m}\mathbf{E}_\perp \times \mathbf{a}_z = \frac{c}{B}\mathbf{E}_\perp \times \mathbf{a}_z$$

"E×B" drift

#3) next term in ω/Ω

$$\underset{\sim}{u}_\perp = -\frac{i\omega}{\Omega_c^2}\frac{q}{m}\underset{\sim}{E}_\perp + \frac{1}{\Omega_c}\frac{q}{m}\underset{\sim}{E}_\perp \times \underline{a}_z$$

"ExB"

Polarization drift

Solution looks good

Current density

$$\hat{\underset{\sim}{J}} = \sum_q q n \hat{\underset{\sim}{u}}_q$$

IN AMPERE'S LAB

$$\nabla \times \underline{B} = \frac{4\pi}{c}\underset{\sim}{J} + \frac{1}{c}\frac{\partial}{\partial t}\underset{\sim}{E}$$

$$i\underset{\sim}{k} \times \hat{\underset{\sim}{B}} = \frac{4\pi}{c}\hat{\underset{\sim}{J}} - \frac{i\omega}{c}\hat{\underset{\sim}{E}}$$

$$4\pi\hat{\underset{\sim}{J}} = \sum_q \frac{4\pi q^2 n_q}{m}\left[\frac{i\omega}{\omega^2-\Omega_c^2}\underset{\sim}{E}_\perp - \frac{\Omega_c}{\omega^2-\Omega_c^2}\underset{\sim}{E}_\perp \times \underset{\sim}{a}_z - \frac{\underline{a}_z}{i\omega}\underline{a}_z \cdot E\right]$$

$$i\underset{\sim}{k} \times \underset{\sim}{\hat{B}} = -\frac{i\omega}{c}\left[\underset{\sim}{\hat{E}} - \sum_q \frac{\omega_{pq}^2}{\omega^2-\Omega_c^2}\underset{\sim}{\hat{E}}_\perp - \frac{i\Omega_c}{\omega}\frac{\omega_{pq}^2}{\omega^2-\Omega_c^2}\underset{\sim}{E}_\perp \times \underset{\sim}{\hat{a}}_z\right.$$

$$\left. - \sum_q \frac{\omega_{pq}^2}{\omega^2}\, \underset{\sim}{a}_z \underset{\sim}{a}_z \cdot \underset{\sim}{\hat{E}}\right] = -\frac{i\omega}{c}\,\underset{=}{\epsilon}\cdot\underset{\sim}{E}$$

THIS CAN BE cast in matrix form
dielectric tensor

$$\underset{=}{\epsilon} = \begin{bmatrix} \epsilon_\perp & -i\epsilon_x & 0 \\ +i\epsilon_x & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{bmatrix}$$

$$\boxed{\epsilon_\perp = 1 - \sum_q \frac{\omega_{pq}^2}{\omega^2-\Omega_c^2} \qquad \epsilon_x = \sum_q \frac{\Omega_c}{\omega}\frac{\omega_{pq}^2}{\omega^2-\Omega_c^2} \qquad \epsilon_\parallel = 1 - \sum_q \frac{\omega_{pq}^2}{\omega^2}}$$

note $(\epsilon^{t})^{*} = \epsilon$ — transpose & conjugate

matrix is <u>Hermitian</u> ||

Wave Propagation

Ampere's Law

$$i\underset{\sim}{k} \times \hat{B} = -i\frac{\omega}{c}\underline{\underline{\epsilon}}\cdot\hat{\underset{\sim}{E}}$$

Faraday's Law

$$i\frac{\omega}{c}\hat{B} = i\underset{\sim}{k}\times\hat{E}$$

(Do not need to solve Poisson Equation)

$$\boxed{\underset{\sim}{k}\cdot\underline{\underline{\epsilon}}\cdot\hat{\underset{\sim}{E}} = 0}$$

$$\underset{\sim}{k}\times(\underset{\sim}{k}\times\underset{\sim}{E}) = -\frac{\omega^2}{c^2}\underline{\underline{\epsilon}}\cdot\underset{\sim}{E}$$

$$\frac{c^2}{\omega^2}\underset{\sim}{k}\times(\underset{\sim}{k}\times\underset{\sim}{E}) = -\underline{\underline{\epsilon}}\cdot\underset{\sim}{E}$$

wave no

$$\underset{\sim}{n} = \frac{c\underset{\sim}{k}}{\omega}$$

$\cancel{(n^2 - \underset{\sim}{n}\underset{\sim}{n})}$

$$\underset{\sim}{k}\times(\underset{\sim}{k}\times\underset{\sim}{E}) = \underset{\sim}{k}\underset{\sim}{k}\cdot\underset{\sim}{E} - k^2\underset{\sim}{E}$$

$$\left[\frac{c^2k^2}{\omega^2}\underset{\sim}{E} - \frac{c^2}{\omega^2}\underset{\sim}{k}\underset{\sim}{k}\cdot\underset{\sim}{E} - \underline{\underline{\epsilon}}\cdot\underset{\sim}{E}\right]$$

dispersion relation

$$\det\left[\underline{\underline{I}}\frac{k^2c^2}{\omega^2} - \frac{c^2\underset{\sim}{k}\underset{\sim}{k}}{\omega^2} - \underline{\underline{\epsilon}}\right] = 0$$

~~quadratic equation in~~

determines $\omega(\underset{\sim}{k})$

Special cases $\underset{\sim}{k} \perp$ to $\underset{\sim}{B}$

$k \parallel$ to $\underset{\sim}{B}$

$\boxed{k_{\parallel} \text{ to } B}$

$$\det \begin{bmatrix} \frac{k_{\parallel}^2 c^2}{\omega^2} - \epsilon_{\perp} & +i\epsilon_x & 0 \\ -i\epsilon_x & \frac{k_{\parallel}^2 c^2}{\omega^2} - \epsilon_{\perp} & 0 \\ 0 & 0 & \epsilon_{\parallel} \end{bmatrix} = 0$$

$$\epsilon_{\parallel} \left[\left[\left(\frac{k_{\parallel}^2 c^2}{\omega^2} \right) - \epsilon_{\perp} \right]^2 - \epsilon_x^2 \right] = 0$$

THREE solutions

~~Pole~~

$$\epsilon_{\parallel} = 0$$

$$\frac{k_{\parallel}^2 c^2}{\omega^2} = \epsilon_{\perp} \pm \epsilon_x \quad \| \quad \text{Light waves}$$

Polarization

$$\begin{bmatrix} \frac{k_{\parallel}^2 c^2}{\omega^2} - \epsilon_{\perp} & +i\epsilon_x & 0 \\ -i\epsilon_x & \frac{k_{\parallel}^2 c^2}{\omega^2} - \epsilon_{\perp} & 0 \\ 0 & 0 & \epsilon_{\parallel} \end{bmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

-) $\epsilon_{\parallel} = 0$ $\quad E_x, E_y = 0$ $\quad E_z \neq 0$

* THIS is an electrostatic wave $\underset{\sim}{k}$ is parallel to $\underset{\sim}{E}$ $\quad \underset{\sim}{E} = -i\underset{\sim}{k}\phi$

* $1 - \sum_q \frac{\omega_p^2}{\omega^2} = 0$ unmagnetized plasma wave

motion along B unaffected by B_0

Second solution

$$\frac{k_\parallel^2 c^2}{\omega^2} - \epsilon_\perp = \pm \epsilon_x \qquad E_z = 0$$

$\underset{\sim}{E}$ is $\perp$ to $\hat{k}$ (electromagnetic wave)

$$\left(\frac{k_\parallel^2 c^2}{\omega^2} - \epsilon_\perp\right) E_x + i\epsilon_x E_y = 0$$

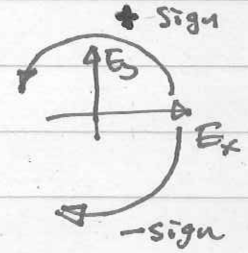
$$\pm \epsilon_x E_x + i\epsilon_x E_y = 0$$

$$\boxed{\frac{E_y}{E_x} = \pm i}$$

circulary polarized with differening rotations

$$\underset{\sim}{E} = \mathrm{Re}\left\{ \hat{E}_x (\hat{x} \pm i\hat{y})\, e^{i(k_\parallel z - \omega t)} \right\}$$

$-i \quad i e^{-ic}$



R $\quad E_x = \cos(\omega t) \pm E_y \sin\omega t$

Check light waves.

SPECIAL case $B \to 0$ $\epsilon_\perp = 1 - \frac{\omega_p^2}{\omega^2}$

$\epsilon_x \to 0$

$\epsilon_\parallel = 0$ $E_x, E_y = 0$ $E_z \neq 0$

$$\frac{k_\parallel^2 c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$ Two solutions

$E_z = 0$ $E_x, E_y \neq 0$

Rotation of particles

$$\frac{dV_x}{dt} = \Omega_q V_y \qquad \Omega_q = \frac{qB}{mc}$$

$$V_x = \cos \Omega_q t \qquad \text{let's assume } B>0$$

$$V_y = -\sin \Omega_q t$$

if $\Omega_q > 0$ clockwise ions

if $\Omega_q < 0$ counter clockwise electrons

Thus (if $B>0$) + sign rotates in electron sense
− sign " in ion sense

$$\frac{k_{\parallel}^2 c^2}{\omega^2} = 1 - \sum_q \frac{\omega_{pq}^2}{\omega^2 - \Omega_c^2} \pm \sum_q \frac{\Omega_c}{\omega} \frac{\omega_{pq}^2}{\omega^2 - \Omega_c^2}$$

$$= 1 - \sum_q \frac{\omega_{pq}^2}{\omega(\omega \pm \Omega_c)}$$

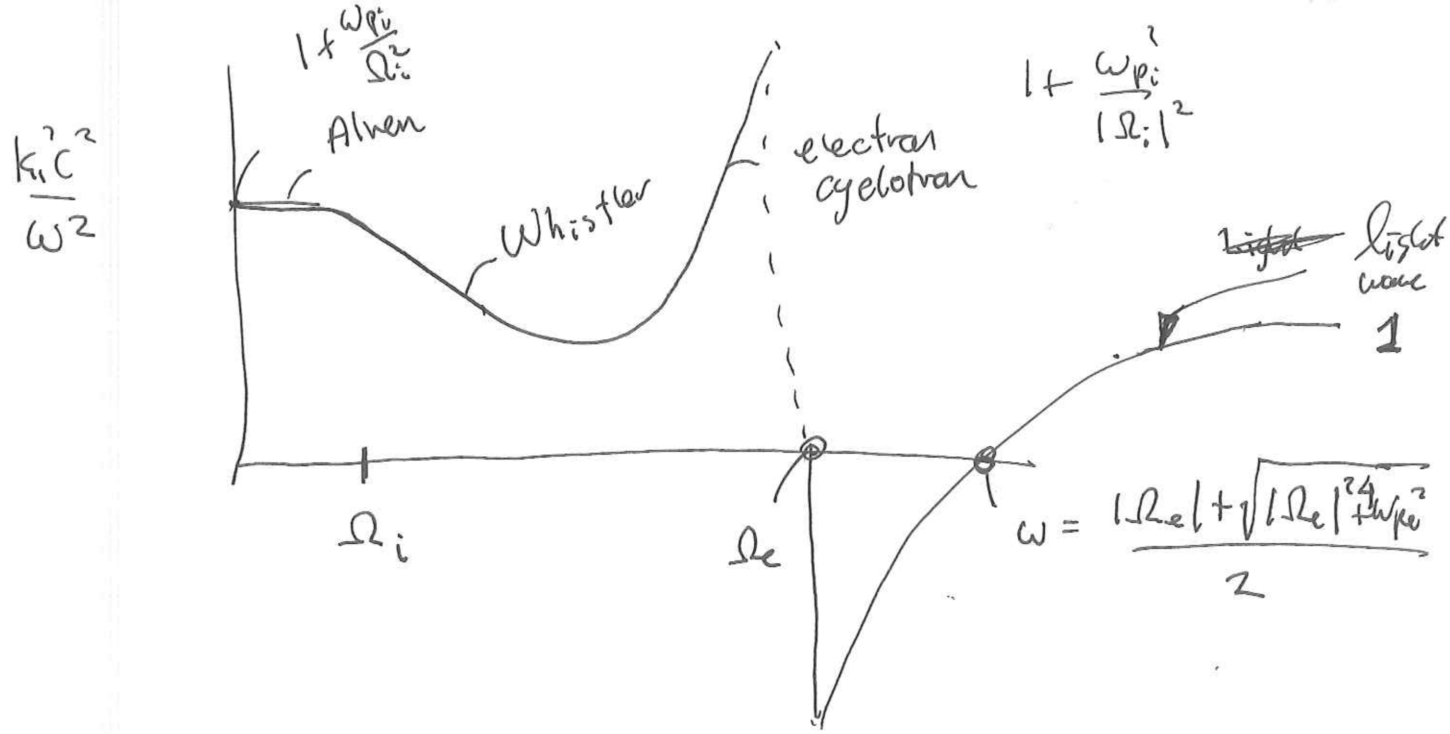
$\Omega_e < 0$
$\Omega_i > 0$

$$\frac{k_\parallel^2 c^2}{\omega^2} = 1 - \frac{\omega_{pi}^2}{\omega(\omega+|\Omega_i|)} - \frac{\omega_{pe}^2}{\omega(\omega-|\Omega_e|)}$$

$$= 1 - \frac{\omega_{pi}^2}{|\Omega_i|}\left[\frac{|\Omega_i|}{\omega(\omega+|\Omega_i|)} + \frac{|\Omega_e|}{\omega(\omega-|\Omega_e|)}\right]$$

$$= 1 - \frac{\omega_{pi}^2}{|\Omega_i|}\frac{1}{\omega}\left[\frac{(\omega-|\Omega_e|)|\Omega_i|) + (\omega+|\Omega_i|)|\Omega_e|}{(\omega+|\Omega_i|)(\omega-|\Omega_e|}\right]$$

$$= 1 + \frac{\omega_{pi}^2}{|\Omega_i|}\left[\frac{|\Omega_e|-|\Omega_i|}{(|\Omega_e|-\omega)(\omega+|\Omega_i|)}\right]$$

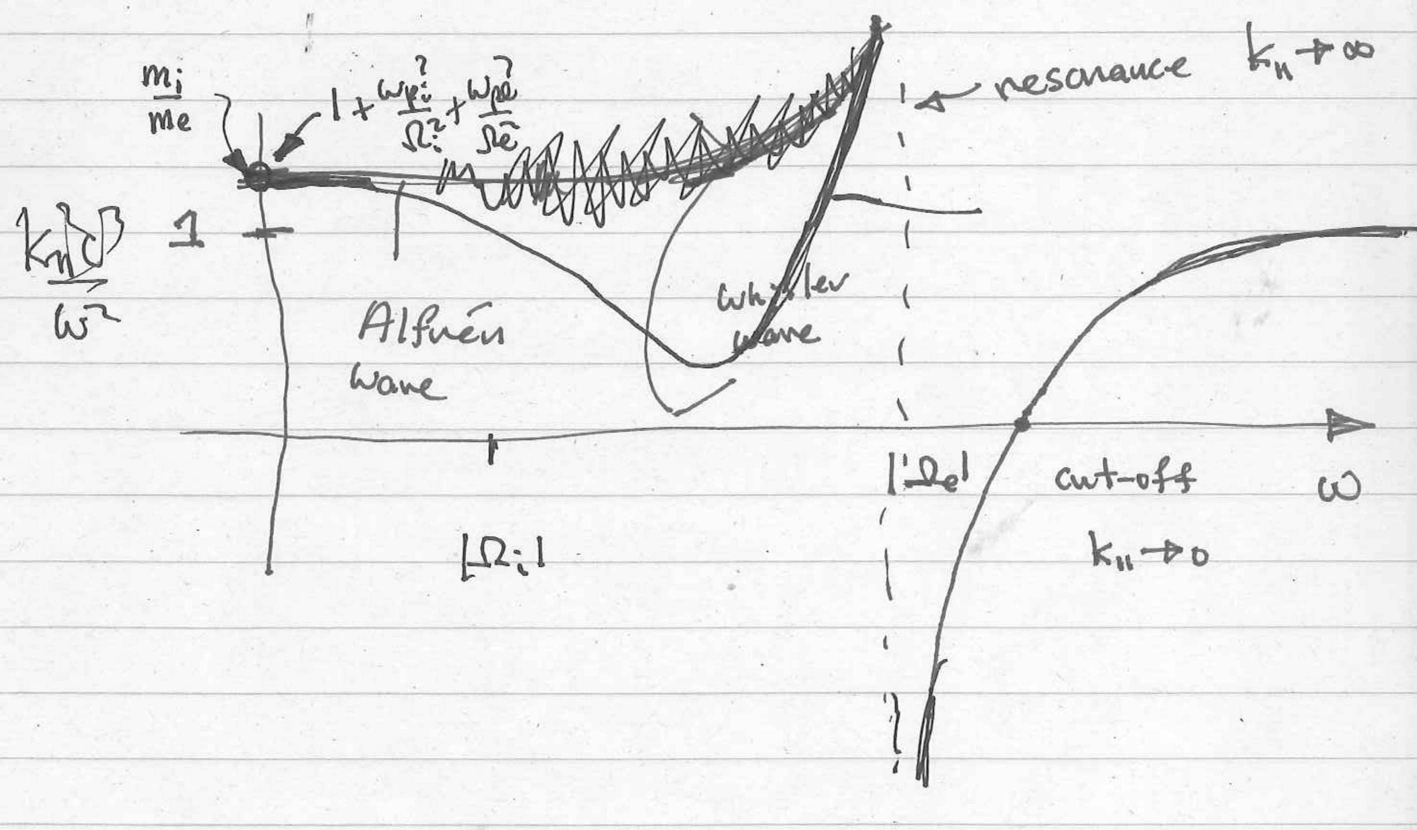


+ sign resonates with electrons
− sign resonates with ions

For + sign

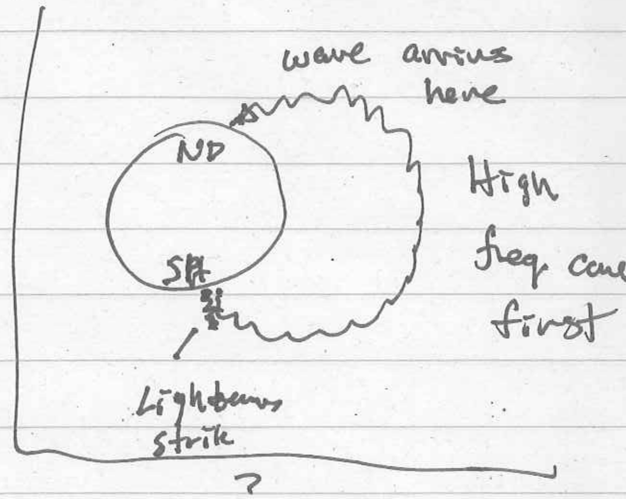
$$\omega_{pe}^2 = \frac{|\Omega_e|^2}{2}$$

$$\frac{k_{\parallel}^2 c^2}{\omega^2} = 1 - \frac{\omega_{pi}^2}{\omega(\omega+|\Omega_i|)} - \frac{\omega_{pe}^2}{\omega(\omega-|\Omega_e|)}$$



for small $\omega << |\Omega_i|$

$$\frac{k_{\parallel}^2 c^2}{\omega^2} = 1 - \frac{\omega_{pi}^2}{\omega|\Omega_i|}\left[1-\frac{\omega}{|\Omega_i|}\right] + \frac{\omega_{pe}^2}{\omega|\Omega_e|}\left[1+\frac{\omega}{|\Omega_e|}\right]$$



Whistler Wave

$\Omega_e \gg \omega \gg \Omega_i, \omega_{pi}$

$$\frac{k_\parallel^2 c^2}{\omega^2} = 1 + \frac{\omega_p^2}{\omega|\Omega_e|} \simeq \frac{\omega_{pe}^2}{\omega|\Omega_e|}$$

$$k_\parallel = \frac{1}{c}\sqrt{\frac{\omega\,\omega_{pe}^2}{|\Omega_e|}}$$

$$\frac{\partial k_\parallel}{\partial \omega} = \frac{1}{V_g} = \frac{1}{2c}\sqrt{\frac{\omega_{pe}^2}{|\Omega_e|}}\,\frac{1}{\sqrt{\omega}}$$

$$V_g = 2c\,\frac{\sqrt{\omega}}{\sqrt{\frac{\omega_{pe}^2}{|\Omega_e|}}} = 2c\sqrt{\frac{\omega|\Omega_e|}{\omega_{pe}^2}} \sim \omega^{1/2}$$

Low freq Branch

$$\frac{k_{\parallel}^2 c^2}{\omega^2} = 1 - \underbrace{\frac{\omega_{pi}^2}{\omega|\Omega_i|} + \frac{\omega_{pe}^2}{\omega|\Omega_e|}}_{\text{cancel if } n_e = n_i} + \frac{\omega_{pi}^2}{\Omega_i^2} + \frac{\omega_{pe}^2}{\Omega_e^2}$$

$$\frac{\omega_{pi}^2}{\Omega_i^2} \gg \frac{\omega_{pe}^2}{\Omega_e^2} \qquad \frac{\omega_{pi}^2}{\omega_{pe}^2} = \frac{m_e}{m_i} \qquad \frac{|\Omega_i|}{|\Omega_e|} = \frac{m_e}{m_i}$$

$$\cancel{k_{\parallel}^2 = \frac{\omega^2}{c^2}\left(1 + \omega_{pi}^2\right.}$$

$$\omega^2 = k_{\parallel}^2 \frac{c^2}{\left(1 + \frac{\omega_{pi}^2}{\Omega_i^2}\right)} \cong k_{\parallel}^2 V_a^2$$

Alfven velocity

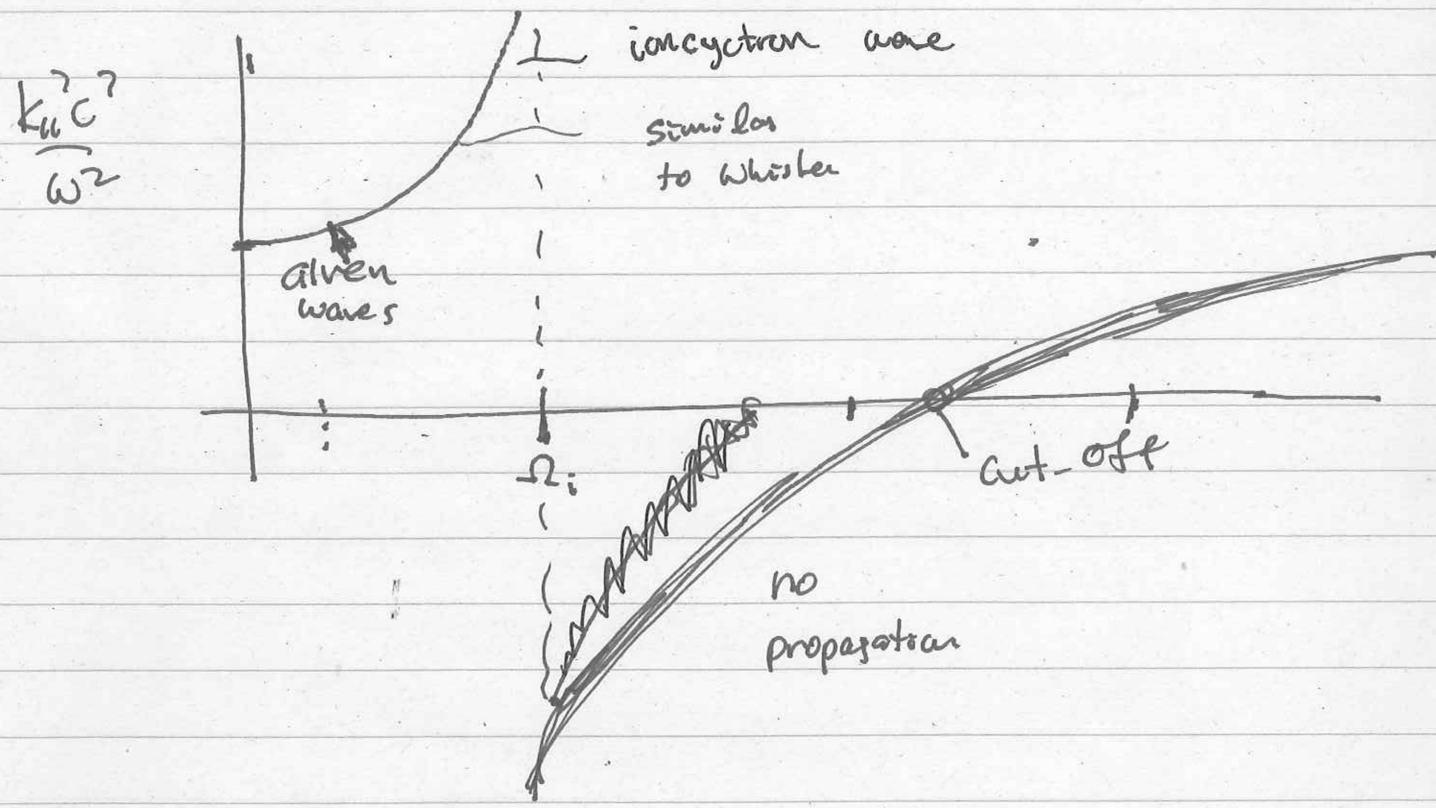
$$V_a = \sqrt{\frac{c^2 \Omega_i^2}{\omega_{pi}^2}} = \sqrt{\frac{B^2}{4\pi \underbrace{n_i M_i}_{\text{mass Density}}}}$$

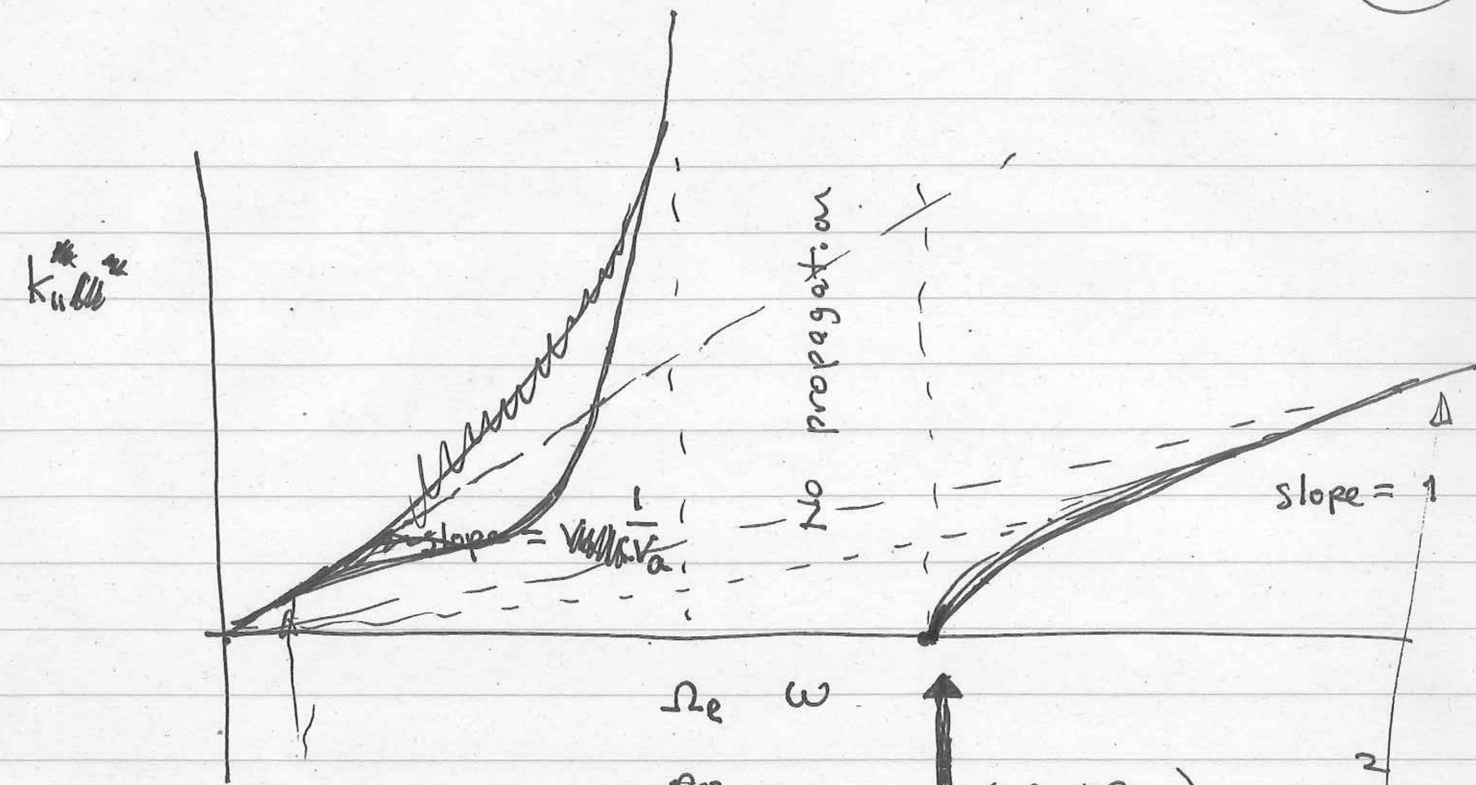
Like a sound wave

but $P \rightarrow \frac{B^2}{4\pi}$ (magnetic pressure)

$$\frac{k_{\parallel}^2 c^2}{\omega^2} = 1 - \frac{\omega_{pi}^2}{\omega(\omega-\Omega_i)} - \frac{\omega_{pe}^2}{\omega(\omega+|\Omega_e|)}$$

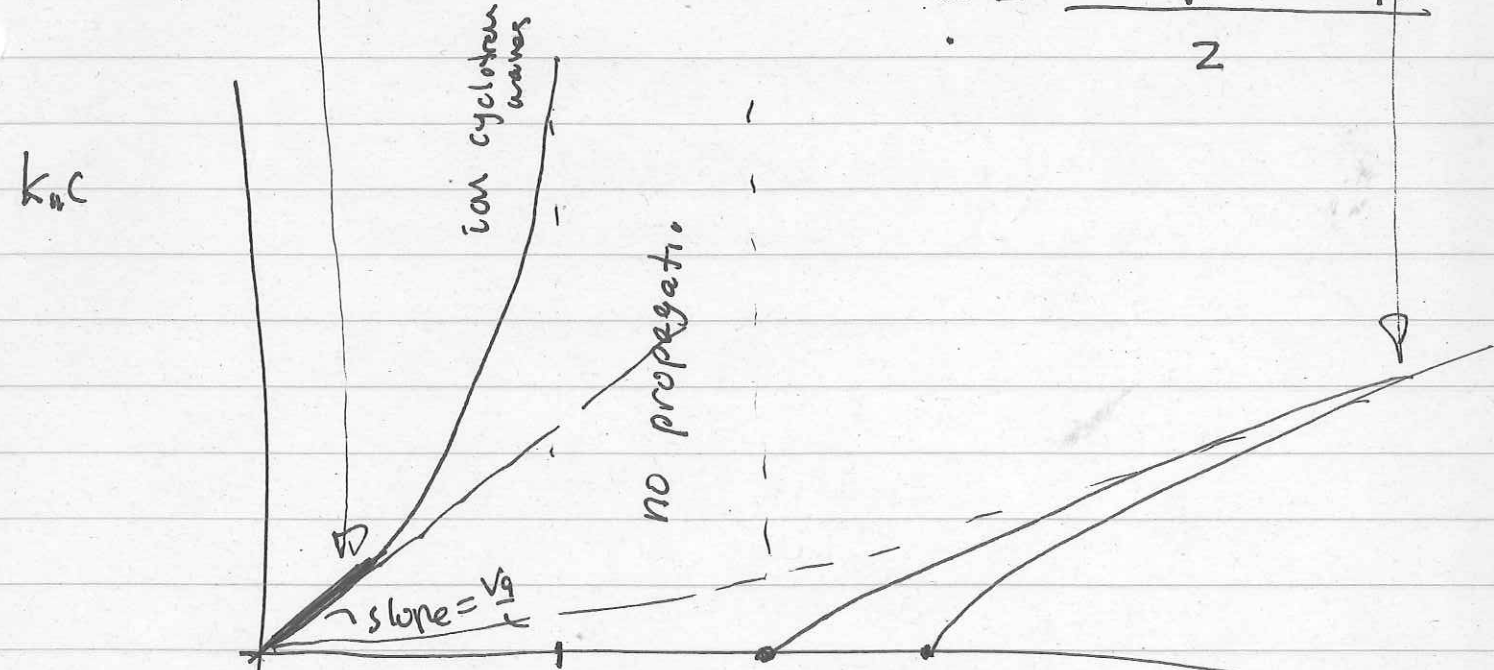
For $-$ sign





$$\cancel{\omega(\omega - |\Omega_e|) = \omega_{pe}^2}$$

$$\omega = \frac{|\Omega_e| \pm \sqrt{|\Omega_e|^2 + 4\omega_{pe}^2}}{2}$$



$$\omega(\omega + \Omega_e) = \omega_{pe}^2$$

L

$$\omega = \frac{\sqrt{|\Omega_e|^2 + 4\omega_{pe}^2} - |\Omega_e|}{2}$$

Propagation $\perp$ to B

Take

$$\underline{\underline{M}} = \underline{\underline{1}}\frac{k^2c^2}{\omega^2} - \frac{\underline{k}\,\underline{k}\,c^2}{\omega^2} - \underline{\underline{\epsilon}}$$

$$\underline{k} = k_x \underline{a}_x$$

$$\det \begin{bmatrix} -\epsilon_\perp & +i\epsilon_x & 0 \\ -i\epsilon_x & \frac{k_x^2c^2}{\omega^2} - \epsilon_\perp & 0 \\ 0 & 0 & \frac{k_x^2c^2}{\omega^2} - \epsilon_\parallel \end{bmatrix}$$

$$\det = \left(\frac{k_x^2c^2}{\omega^2} - \epsilon_\parallel\right)\left(-\epsilon_\perp\left(\frac{k_x^2c^2}{\omega^2} - \epsilon_\perp\right) - \epsilon_x^2\right) = 0$$

solution

$$\frac{k_x^2c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2} \qquad \omega^2 = k_x^2c^2 + \omega_p^2$$

ORDINARY MODE
PROPAGATION in
unmagnetized

$$\boxed{\begin{aligned} E_z &\neq 0 \\ E_x &= E_y = 0 \end{aligned}}$$

$$\epsilon_\perp\left(\frac{k_x^2c^2}{\omega^2}-\epsilon_\perp\right) = -\epsilon_x^2$$

$E_z = 0$

$E_x, E_y \neq 0$

$$\frac{k_x^2c^2}{\omega^2} = \frac{(\epsilon_\perp-\epsilon_x)(\epsilon_\perp+\epsilon_x)}{\epsilon_\perp}$$

Extra-ordinary mode X-mode

cut-off's $(k_x \to 0)$ $\epsilon_\perp = \pm\epsilon_x$

resonances $(k_x \to \infty)$ $\epsilon_\perp = 0$

$$\epsilon_\perp = 1 - \frac{\omega_{pi}^2}{\omega^2-\Omega_i^2} - \frac{\omega_{pe}^2}{\omega^2-\Omega_e^2} = 0$$

multiply out

two solutions for ω^2

$$(\omega^2-\Omega_i^2)(\omega^2-\Omega_e^2) - \omega_{pi}^2(\omega^2-\Omega_e^2) - \omega_{pe}^2(\omega^2-\Omega_i^2) = 0$$

$$\cancel{\omega^4 - \omega^2\left(\Omega_e^2+\Omega_i^2\right) + \left(\omega_{pe}^2+\omega_{pi}^2\right)} + \omega_{pi}^2\Omega_e^2 + \omega_{pe}^2\Omega_i^2 = 0$$

$$\cancel{\omega^4 - \omega^2\left[\Omega_e^2+\Omega_i^2\right] + \left(\omega_{pe}^2+\omega_{pi}^2\right)} + \omega_{pi}^2\Omega_e^2 + \omega_{pe}^2\Omega_i^2 = 0$$

$\omega \sim \Omega_e$ $\cancel{\frac{\omega_{pi}^2}{\Omega_e^2}} <$

~~assume~~ $\cancel{\omega_{pi}^2/\Omega_e^2 \ll 1}$

assume $\omega_{pe}^2 \sim \Omega_e^2 \gg \omega_{pi}^2 \gg \Omega_i^2$

high freq solution

$$1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} = 0 \qquad \omega^2 = \Omega_e^2 + \omega_{pe}^2$$

(upper hybrid resonance)

Low frequency solution (Lower hybrid resonance)

$$1 + \frac{\omega_{pe}^2}{\Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2} = 0$$

$$\omega^2 = \Omega_i^2 + \frac{\omega_{pi}^2}{1 + \frac{\omega_{pe}^2}{\Omega_e^2}} \simeq \frac{\omega_{pi}^2}{\left(1 + \frac{\omega_{pe}^2}{\Omega_e^2}\right)}$$

if $\omega_{pi}^2 \gg \Omega_e^2$

$\Omega_e > \omega \sim \omega_{pi} \gg \Omega_i$ $\qquad \omega^2 = \Omega_e \Omega_i$

involves both ions and electrons

Polarizations ~~At resonance~~

~~at cut-offs~~

$\cancel{E_y}$ $\quad \epsilon_\perp E_x + i\,\epsilon_x E_y = 0$

$$\frac{E_y}{E_x} = i\,\frac{\epsilon_\perp}{\epsilon_x}$$

at cut offs $\epsilon_\perp = \pm\epsilon_x$ circularly polarized

at resonance $\epsilon_x \to 0$ $\quad E_y \to 0$ $\quad E_x \neq 0$

wave is electrostatic $\quad \underline{k} \parallel \underline{E}$

also

$$\frac{\epsilon_\perp}{\epsilon_x} = -1 \quad \text{circularly polarized}$$

?

at cyclotron resonance

$\omega - \Omega_e$

High freq

$$1 - \frac{\omega_{pe}^2}{\omega_{uc}(\omega_{uc} - |\Omega_e|)} = 0$$

$$1 - \frac{\omega_{pe}^2}{\omega_{Lc}(\omega_{Lc} + |\Omega_e|)} = 0$$

for $\omega \sim \Omega_e, \omega_{pe} >> \Omega_i, \omega_{pi}$

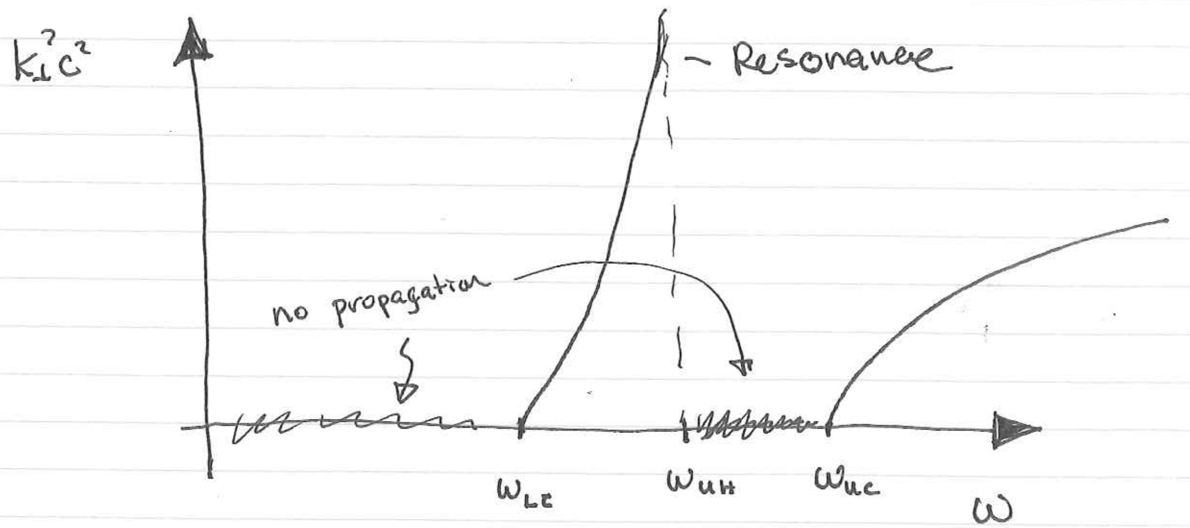
$$\frac{k_\perp^2 c^2}{[redacted]} = \frac{(\omega^2 + \omega\Omega_e - \omega_{pe}^2)(\omega^2 - \omega\Omega_e - \omega_{pe}^2)}{\omega^2 - (\omega_{pe}^2 + \Omega_e^2)}$$

upper hybrid resonance

cut-offs

$$\omega_{uc} = \frac{|\Omega_e| + \sqrt{|\Omega_e|^2 + 4\omega_{pe}^2}}{2} > \omega_{uH}$$

$$\omega_{Lc} = \frac{\sqrt{|\Omega_e|^2 + 4\omega_{pe}^2} - |\Omega_e|}{2} < \omega_{uH}$$



Note no cyclotron resonance!

wave adopts polarization $\rightarrow$

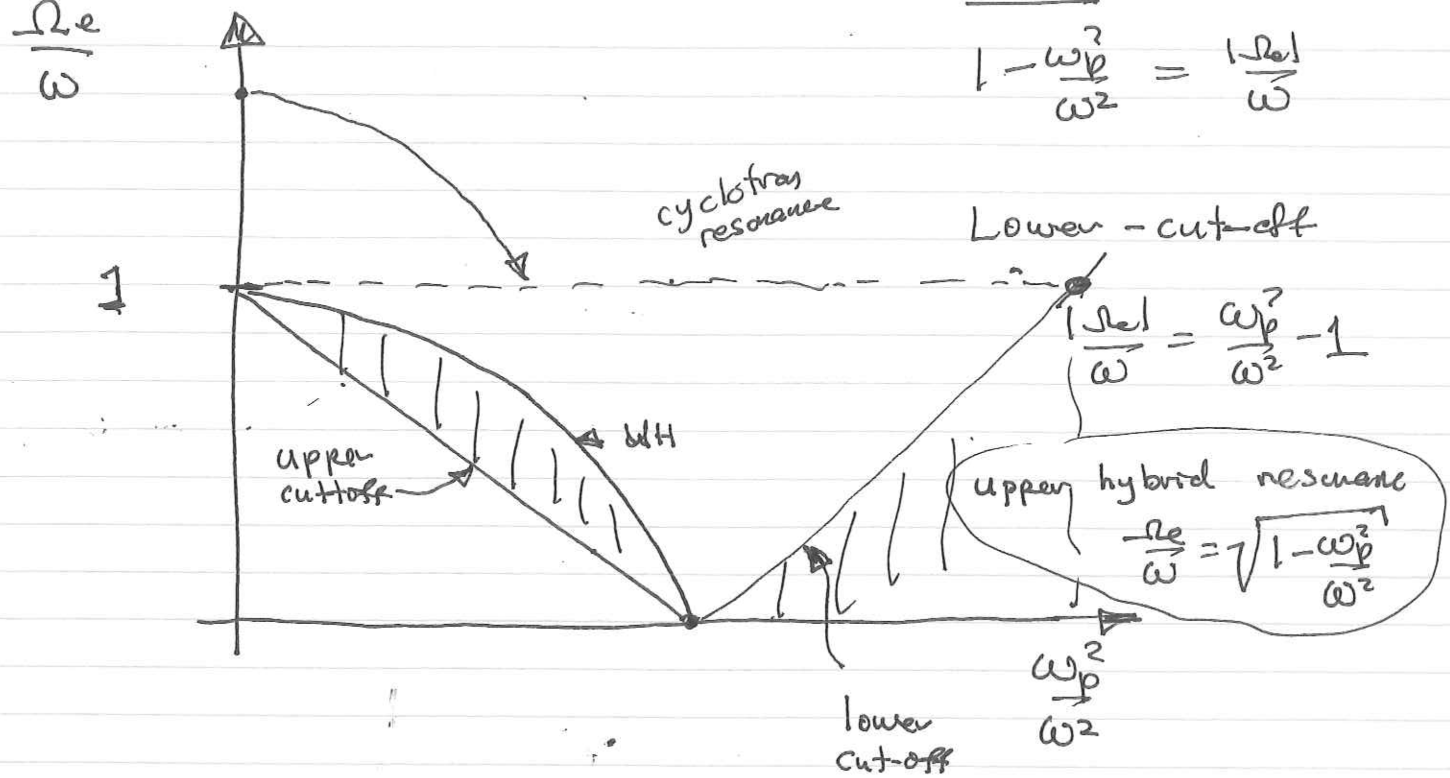
electrons +

at $\omega = \Omega_e$

upper cuttoff

$\cancel{\omega^2 - \omega_p^2}$

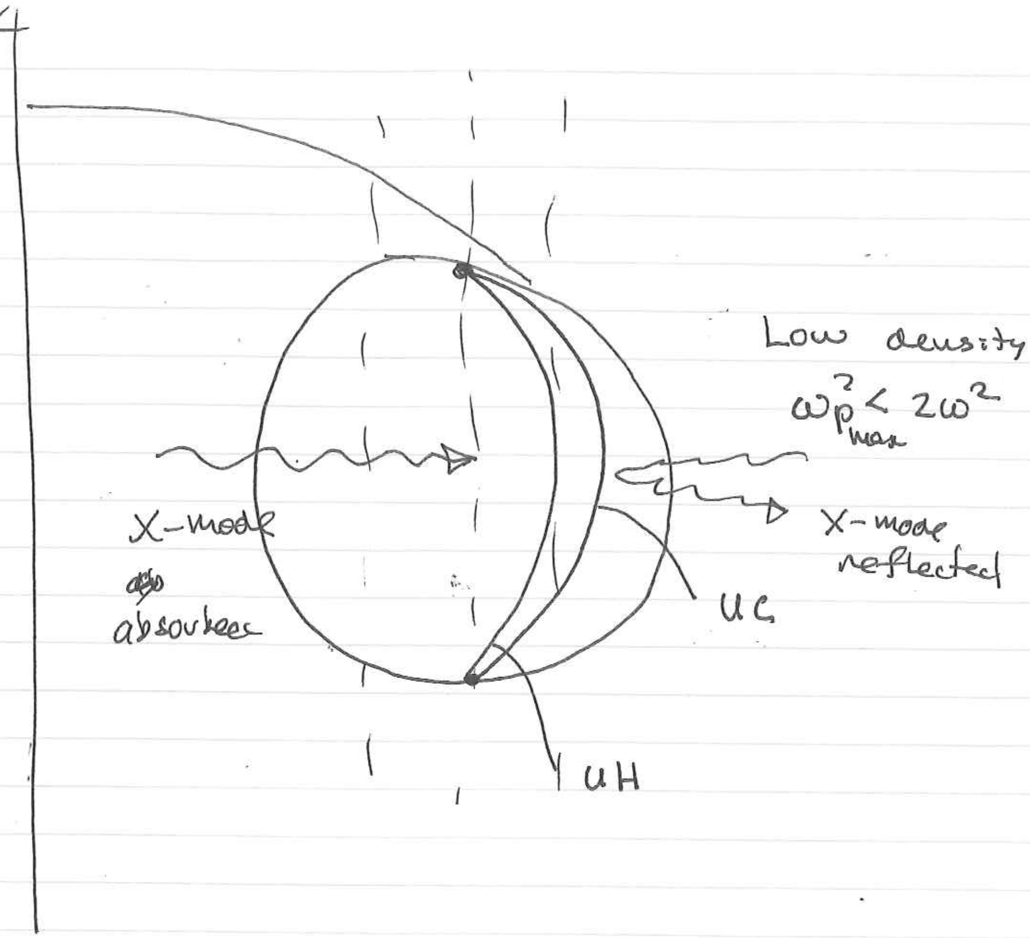
$$1 - \frac{\omega_p^2}{\omega^2} = \frac{|\Omega_e|}{\omega}$$



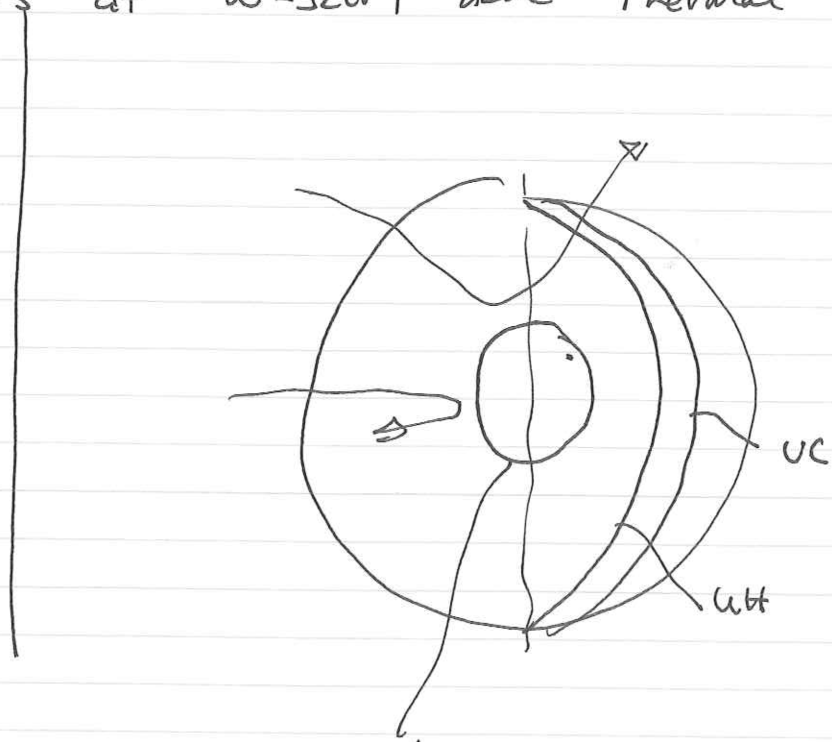
forbidden zones

density limit $\boxed{\frac{\omega_p^2}{\omega^2} < 2}$

Tokamak



Absorbtion occurs at $\omega = \Omega_{cr}$ due Thermal effects



Faraday Rotation

Consider case $\omega >> \Omega_e, \omega_{pe}$ (Ω_i, ω_{pi})

$$k_\parallel^2 c^2 = \omega^2 - \frac{\omega\omega_{pe}^2}{\omega - |\Omega_e|}$$

+) Rotates with electrons

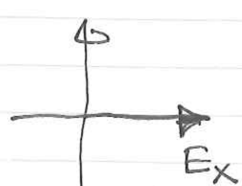
$$k_\parallel^2 c^2 = \omega^2 - \frac{\omega\omega_{pe}^2}{\omega + |\Omega_e|}$$

−) Rotates with ions

$\cancel{k_{\parallel(+)} \neq k_{\parallel(-)}}$

In limit ω_{pe}^2 or $\Omega_e \to 0$ $\quad k_{\parallel +} = k_{\parallel -}$

Right and left circularly polarized waves have the same wave vector. ~~Plane Polarized~~ Waves can also be represented as plane polarized.



$= \cancel{\frac{1}{2}(E_x}$

$$\hat{E} \;\cancel{E_x} = \frac{1}{2}(a_x + i a_y)E_+ + \frac{1}{2}(a_x - i a_y)E_-$$

if $E_+ = E_-$ wave is plane polarized in x-direction

if ~~E_+~~ $E_- = -E_+$ wave is plane polarized in y-direction

$$\underset{\sim}{E} = \mathrm{Re}\left\{ \left((\hat{a}_{\cancel{x}} + i a_y) \hat{E}_+ e^{i k_{\parallel +} z} + (\hat{a}_x - i a_y) \hat{E}_- e^{i k_{\parallel -} z} \right) e^{-i\omega t} \right.$$

if $k_{\parallel +} = k_{\parallel -}$ and wave is plane polarized at $z = 0$ it will be plane polarized for all z in the same direction

if $k_{\parallel +} \neq k_{\parallel -}$ polarization will change with z

suppose wave is plane polarized at $z = 0$ $\quad \hat{E}_+ = \hat{E}_-$

$$\overline{k_\parallel} = \tfrac{1}{2} k$$

$E = \cancel{\mathrm{Re}\{ a_x}$

also assume $\hat{E}_+ = \hat{E}_-$

$$k_{\parallel +} = \bar{k}_\parallel + \frac{1}{2}\Delta k$$

$$k_{\parallel -} = \bar{k}_\parallel - \frac{1}{2}\Delta k$$

Then

$$\underset{\sim}{E} = Re \left\{ \underset{\sim}{a}_x \left(E_+ e^{ik_{\parallel +} z} + E_+ e^{ik_{\parallel -} z} \right) + i\, a_y \left(E_+ e^{ik_{\parallel +} z} + E_+ e^{ik_{\parallel -} z} \right) \right\}$$

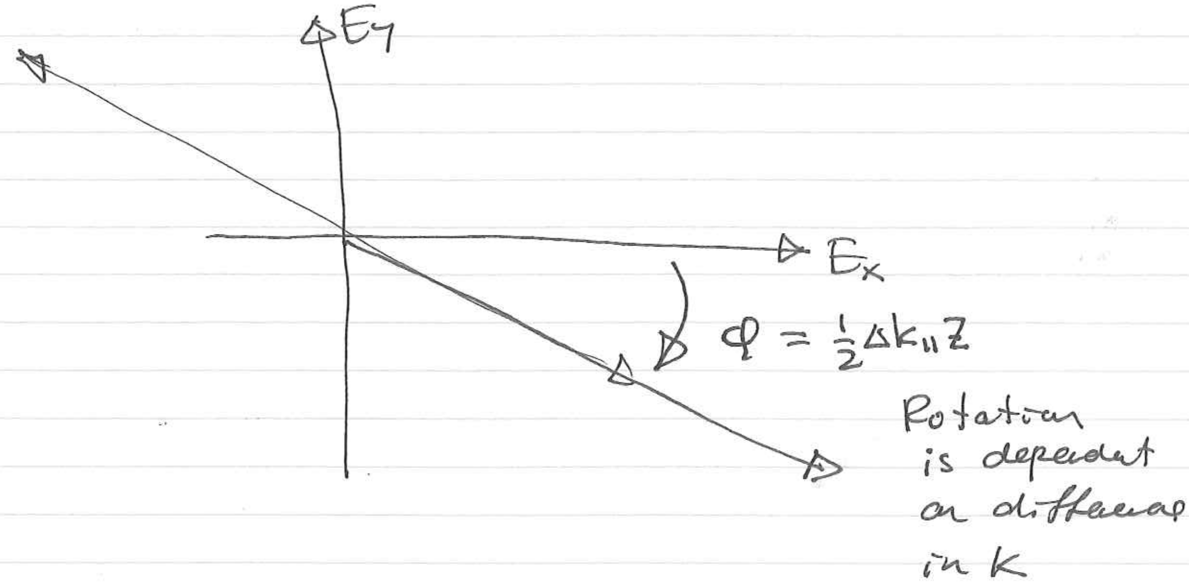
$$= Re \left\{ E_+ e^{i\bar{k}_\parallel z} \left[\underset{\sim}{a}_x \, 2\cos(\Delta k z/2) \cancel{+E_+ e} - \underset{\sim}{a}_y \, 2\sin(\Delta k z/2) \right] \right\}$$

4

$$E = \mathrm{Re}\left\{ 2\hat{E}_+ e^{i(\bar{k}_{\parallel} z - \omega t)} \left[\underset{\sim}{a}_x \cos\left(\tfrac{1}{2}\Delta k_{\parallel} z\right) - \underset{\sim}{a}_y \sin\left(\tfrac{1}{2}\Delta k_{\parallel} z\right) \right] \right\}$$

Note ~~a_z~~ E_x & E_y are in phase

so this is a plane polarized wave.



for $\omega \gg \omega_{pe}$

$k_{\parallel}$

Now find $\bar{k}_\parallel$ and Δk

$$k_{\parallel +}^2 c^2 = 1 - \frac{\omega_p^2}{\omega(\omega - |\Omega_e|)}$$

$$k_{\parallel -}^2 c^2 = 1 - \frac{\omega_p^2}{\omega(\omega + |\Omega_e|)}$$

$$\frac{(k_{\parallel +}^2 + k_{\parallel -}^2)c^2}{\omega^2} = 2 - \frac{\omega_p^2}{\omega}\left(\frac{1}{(\omega - |\Omega_e|)} + \frac{1}{\omega + |\Omega_e|}\right)$$

$$2\bar{k}_\parallel^2 c^2 + \frac{1}{2}\Delta k^2 c^2 = 2\left(1 - \frac{\omega_p^2}{\omega^2 - |\Omega_e|^2}\right)$$

$$\left(\bar{k}_\parallel + \tfrac{1}{2}\Delta k\right)^2 = \bar{k}_\parallel^2 + \bar{k}_\parallel \Delta k + \tfrac{1}{4}\Delta k^2$$

$$\frac{\bar{k}_\parallel c}{\omega} \simeq \sqrt{1 - \frac{\omega_p^2}{\omega^2 - |\Omega_e|^2}}$$

$$\frac{(k_{\parallel +}^2 - k_{\parallel -}^2)c^2}{\omega^2} = \frac{2\bar{k}_\parallel \Delta k\, c^2}{\omega^2} = -\frac{\omega_p^2}{\omega}\left(\frac{1}{\omega - |\Omega_e|} - \frac{1}{\omega + |\Omega_e|}\right)$$

$$= -\frac{2\omega_{pe}^2 |\Omega_e|}{\omega(\omega^2 - |\Omega_e|^2)}$$

THUS

$$\frac{\Delta k_c}{\omega} = -i \frac{\omega}{\bar{k}_{\parallel} c} \frac{\omega_{pe}^2 |\Omega_e|}{\omega(\omega^2 - |\Omega_e|^2)} \quad \propto \quad n$$

measurement of line integrated density. (if B is known)