Phys 762

Scattering Processes,

Begin with Fokker-Planck Equation

$$\frac{\partial f_\alpha}{\partial t} = -\frac{\partial}{\partial \underset{\sim}{v}} \cdot \sum_\beta \underset{\sim}{J}^{\alpha/\beta}$$

$\underset{\sim}{J}^{\alpha/\beta}$ = velocity space flux due to scattering with particles of type $\beta$

$$\underset{\sim}{J}^{\alpha/\beta} = C_{\alpha\beta} \left\{ f_\alpha(\underset{\sim}{v}) \nabla_v H^{\alpha/\beta} - \frac{1}{2} \nabla_{\underset{\sim}{v}} \cdot \left( f_\alpha \nabla_v \nabla_v G^{\alpha/\beta} \right) \right\}$$

where $H$ & $G$ are the Rosenbluth potentials satisfying

$$\nabla_v^2 G = \frac{2}{1 + \frac{m_\alpha}{m_\beta}} H \qquad \nabla_v^2 H = -4\pi \frac{m_\alpha + m_\beta}{m_\beta} f_\beta(\underset{\sim}{v})$$

$$C_{\alpha\beta} = \frac{4\pi q_\alpha^2 q_\beta^2}{m_\alpha^2} \ln\left(\frac{\lambda_d}{r_0}\right)$$

The hand out sheet given in class last week evaluates the Rosenbluth Potentials for the case in which the scatterers (particles of type $\beta$) are Maxwellian.

$$f_\beta(\underset{\sim}{v}) = \frac{n_\beta}{\pi^{3/2} V_\beta^3} \exp\left(-\frac{v^2}{V_\beta^2}\right)$$

The resulting expression for the particle flux $\underset{\sim}{J}^{\alpha/\beta}$ is then

Landau form

~~$J^{a/b} = 2\pi$ [illegible]~~

$$J^{a/b} = \frac{2\pi\,\lambda^{a/b}\,q_a^2\,q_b^2}{m_a}\int d^3v' \;\frac{u^2\underline{\underline{I}} - \underset{\sim}{u}\,\underset{\sim}{u}}{u^3}$$

$$\left[\frac{1}{m_b}\, f_a(\underset{\sim}{v})\,\frac{\partial}{\partial \underset{\sim}{v}'}\, f_b(\underset{\sim}{v}') - \frac{1}{m_a}\, f^b(\underset{\sim}{v}')\,\frac{\partial}{\partial \underset{\sim}{v}}\, f^a(\underset{\sim}{v})\right]$$

collision frequency

slowing down

diffusion in $|v|$ or energy

$$\underset{\sim}{J}^{\alpha/\beta} = - \overbrace{\frac{4\pi\, n_\beta\, e_\alpha^2 e_\beta^2}{m_\alpha^2\, v^3}}^{\nu_0^{\alpha/\beta}(v)} \left\{ \psi(x) \left[ \frac{m_\alpha}{m_\beta} \underset{\sim}{v} f^\alpha + \frac{1}{2} \frac{v_\beta^2}{v^2} \underset{\sim}{v}\underset{\sim}{v} \cdot \nabla_v f \right] \right.$$

$$\left. + \frac{1}{2}\phi(x) \left( v^2 \nabla_{\underset{\sim}{v}} f^\alpha - \underset{\sim}{v}\underset{\sim}{v} \cdot \nabla_v f^\alpha \right) \right\}$$

pitch angle scattering

where $\psi$ & $\phi$ are defined by

$$\psi(x) = \frac{4}{\pi^{1/2}} \int_0^x d\xi\, \xi^2 \exp(-\xi^2)$$

$$x^2 = \tfrac{1}{2} m_\beta v^2 / T_\beta$$

$$\phi(x) = \frac{2}{\pi^{1/2}} \int_0^x d\xi \left(1 - \frac{\xi^2}{x^2}\right) \exp(-\xi^2)$$

<u>note</u>

$$\psi(\infty) = \phi(\infty) = 1$$

The first term describes slowing down

The second term describes energy diffusion

The third term describes pitch angle scattering

# In spherical Coordinates

$$\left( v, \overset{\cos\theta}{\mu}, \phi \right)$$

write

$$\frac{\partial f^{\alpha}}{\partial t} = - \nabla_v \cdot \underset{\sim}{J}^{\alpha/\beta} \quad \text{in } \cancel{\text{polar}} \text{ spherical coordinates.}$$

$$\frac{\partial f^{\alpha}}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v} v^2 \nu_0^{\alpha/\beta} \psi(x) \left[ \overbrace{\frac{m_\alpha}{m_\beta} v f^{\alpha}}^{\text{slowing down}} + \overbrace{\frac{1}{2} v_\beta^2 \frac{\partial f^{\alpha}}{\partial v}}^{\cancel{\text{energy}}\ \cancel{\text{pitch}}\ \text{diffusion in energy}\ \cancel{\text{(velocity)}}} \right]$$

$$\frac{1}{2} \nu_0^{\alpha/\beta} \phi \underbrace{\left[ \frac{\partial}{\partial \mu} (1-\mu^2) \frac{\partial f^{\alpha}}{\partial \mu} + \frac{1}{1-\mu^2} \frac{\partial^2 f^{\alpha}}{\partial \phi^2} \right]}_{\text{diffusion in angle}}$$

Writing the F-P Equation in spherical Coordinates in velocity space we get the above equation (see hand out)

Next we investigate the three processes individually.

## Slowing Down

suppose only slowing down is present

$$\frac{\partial f^\alpha}{\partial t} \doteq \frac{1}{v^2}\frac{\partial}{\partial v} v^2 \nu_0^{\alpha/\beta} \psi(x) \frac{m_\alpha}{m_\beta} v f^\alpha$$

suppose at $t=0$ $\quad f^\alpha(v) = \dfrac{n^\alpha \delta(\underline{v}-v_0)}{4\pi v^2}$

all particles have the same $|v|$ also $\varepsilon = \frac{1}{2}mv^2$


solution at later time

$$f^\alpha = \frac{n^\alpha}{4\pi v^2}\,\delta(\underline{v}-v_0(t))$$

where $v_0(t)$

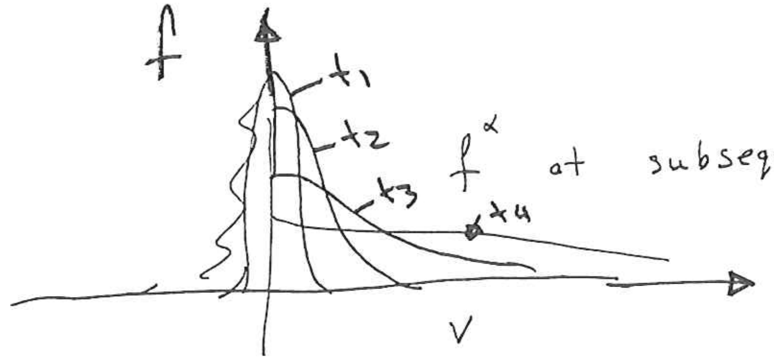
satisfies

$$\frac{\partial v_0}{\partial t} = -\overbrace{\nu_0^{\alpha/\beta}(v_0)\,\psi(x_0)}^{\text{slowing down rate}}\frac{m_\alpha}{m_\beta} v_0$$

<u>no dispersion of velocities!</u>

ow Suppose only energy diffusion were present

$$\frac{\partial f^\alpha}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v} v^2 \nu_0^{\alpha/\beta} \phi \frac{1}{2} V_\beta^2 \frac{\partial f^\alpha}{\partial v}$$




$f^\alpha$ at subsequent times

If only scattering were present temperature of type $\alpha$ particles would increase indefinitely. When both are present an equilibrium is reached

slowing down & energy diffusion

Solution for equilibrium

const

(sink)

~~Source of particles at v=0 sink at V=∞~~

$$\frac{\partial f^\kappa}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2 J_v^{\alpha/\beta}\right)$$

$$\boxed{\text{equilibrium} \quad J_v^{\alpha/\beta} = 0}$$ (no sources or sinks)

Setting $J_v^{\alpha/\beta} = 0$ gives

$$\frac{m_\alpha}{m_\beta} v f^\alpha + \frac{1}{2} V_\beta^2 \frac{\partial f^\alpha}{\partial v} = 0$$

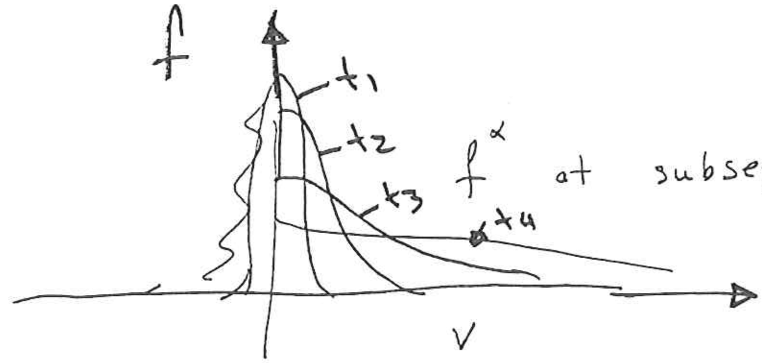
whose solution is

$$f^\alpha = \frac{n_\alpha}{\pi^{3/2} V_\alpha^3} \exp\left(-\frac{v^2}{V_\alpha^2}\right)$$

where $V_\alpha$ is determined by $\frac{m_\alpha}{m_\beta} = \frac{V_\beta^2}{V_\alpha^2}$ $\Rightarrow$ $\boxed{T_\alpha = T_\beta}$ equal temperatures

Now suppose only energy diffusion were present

$$\frac{\partial f^\alpha}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v} v^2 \nu_0^{\alpha/\beta} \phi \, \frac{1}{2} V_\beta^2 \frac{\partial f^\alpha}{\partial v}$$




$f^\alpha$ at subsequent times

If only scattering were present temperature of type $\alpha$ particles would increase indefinitely.
When both ~~are~~ are present an equilibrium is reached

slowing down & energy diffusion

## Solution for equilibrium

(sink)
~~Source of particles at v=0~~ ~~sink at $v=\infty$~~

$$\frac{\partial f^\alpha}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2 J_v^{\alpha/\beta}\right)$$

const

$$\boxed{\text{equilibrium} \quad J_v^{\alpha/\beta} = 0}$$ (no sources or sinks)

Setting $J_v^{\alpha/\beta}=0$ gives

$$\frac{m_\alpha}{m_\beta} v f^\alpha + \frac{1}{2} V_\beta^2 \frac{\partial f^\alpha}{\partial v} = 0$$

whose solution is

$$f^\alpha = \frac{n_\alpha}{\pi^{3/2} V_\alpha^3} \exp\left(-\frac{v^2}{V_\alpha^2}\right)$$

where $V_\alpha$ is determined by $\frac{m_\alpha}{m_\beta} = \frac{V_\beta^2}{V_\alpha^2} \Rightarrow \boxed{T_\alpha = T_\beta}$ equal temperatures

Thus, the balance of energy diffusion and slowing down causes $f_\alpha$ to come to a thermal equilibrium with $f_\beta$ with $T_\alpha = T_\beta$.

The approximate rate at which relaxation occurs is given by

$$\nu_0^{\alpha/\beta}(V_\alpha)\,\frac{m_\alpha}{m_\beta}\,\psi\left(\frac{V_\alpha}{V_\beta}\right) = \nu_{eq}$$

For electron-electron collisions this rate can be estimated

$$\nu_{eq}(e{-}e) \simeq \frac{4\pi n e^4 \lambda}{m_e^2\, V_{the}^3}\,\overbrace{\psi(1)}^{\simeq 1} \propto \frac{n}{T_e^{3/2}}$$

For ion-ion collisions the equilibration rate is

$$\nu_{eq}(i \to i) \simeq \frac{4\pi n_i Z^4 e^4 \lambda}{m_i^2 v_{thi}^3} \psi(1)$$

$$\sim \left(\frac{m_e}{m_i}\right)^{1/2} \nu_{eq}(e \to e)$$

Thus, ions come to equilibrium with themselves at a slower than electrons come to equilibrium with themselves.

Now consider electrons equilibrating with ions

atomic #

$$\nu_{eq}(e \to i) = \frac{4\pi n_i Z^2 e^4}{m_e^2 \, v_{the}^3} \, \frac{m_e}{m_i} \, \psi\left(\frac{v_{the}}{v_{thi}}\right)$$

$$\psi(\infty) = 1$$

thus $(Z n_i \sim n_e)$

$$\nu_{eq}(e \to i) \sim \nu_{eq}(e \to e) \, \frac{Z m_e}{m_i}$$

electrons come to thermal equilibrium with ions at a slower rate than ions equilibrate with themselves (same applies for ions equilibrating with electrons

What about pitch angle scattering?

pitch angle scattering rate

$$\frac{1}{2}\nu_0^{\alpha/\beta}\,\phi\left(\frac{v}{v_{th\beta}}\right) = \frac{1}{2}\,\frac{4\pi n_\beta q_\alpha^2 q_\beta^2}{m_\alpha^2 v^3}\,\Phi\left(\frac{v}{v_\beta}\right)$$

For the different possibilities we have

$$\nu_{scatt}(e \to e) \sim \frac{4\pi n_e e^4}{m_e^2 v_{the}^3} \sim \nu_{eq}(e \to e)$$

$$\nu_{scatt}(e \to i) \sim \text{~~the same~~}\ Z\,\nu_{scatt}(e \to e)$$

$$\nu_{scatt}(i \to i) \sim \nu_{eq}(i \to i) \sim Z^3\left(\frac{m_e}{m_i}\right)^{1/2}\nu_{eq}(e \to e)$$

$$\nu_{scatt}(i \to e) \sim \nu_{eq}(\overset{e \to i}{i \to e}) \sim \frac{m_e}{m_i}\,\nu_{eq}(e \to e)$$

## Interpretation

Think of the plasma as a gas consisting of bowling balls (ions) and ping pong balls (electrons).

Ping Pong balls collide with themselves and bowling balls. Bowling balls appear to be fixed scatterers

Bowling balls are unaffected by collisions with ping pong balls. ~~the~~ They are only affected by collisions with other bowling balls.

Evolution of $f^\alpha$ due to pitch angle scatter

$$\frac{\partial f^\alpha}{\partial t} = \frac{1}{2}\nu_0^{\alpha/\beta}(v)\left[\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu}f^\alpha + \frac{1}{1-\mu^2}\frac{\partial^2 f}{\partial\phi^2}\right]$$

$$f^\alpha(v,\mu,\phi) = \sum_m \sum_{l=-m}^{m} f_{lm}\, e^{im\phi}\, P_l^m(\mu)$$

$\angle$ Legendre fn

$$\left[\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu} + \frac{1}{1-\mu^2}\frac{\partial^2}{\partial\phi^2}\right]\left(e^{im\phi}P_l^m\right) = -l(l+1)\left(e^{im\phi}P_l^m\right)$$

$$\frac{\partial f_{lm}}{\partial t} = -l(l+1)\frac{1}{2}\nu_0^{\alpha/\beta} f_{lm} \qquad f_{lm} \sim e^{-(l(l+1)\nu_0^{\alpha/\beta}/2)t}$$

note decay rate $\sim l^2$

Diffusion wipes out

fine variations quickly

Evolution of $f^{\alpha}$ due to pitch angle scatter

$$\frac{\partial f^{\alpha}}{\partial t} = \frac{1}{2}\nu_0^{\alpha/\beta}(v)\left[\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu} f^{\alpha} + \frac{1}{1-\mu^2}\frac{\partial^2 f}{\partial\phi^2}\right]$$

$$f^{\alpha}(v,\mu,\phi) = \sum_m \sum_{\ell=m}^{m} f_{\ell m}\, e^{im\phi}\, P_\ell^m(\mu)$$

∟ Legendre fn

$$\left[\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu} + \frac{1}{1-\mu^2}\frac{\partial^2}{\partial\phi^2}\right]\left(e^{im\phi}P_\ell^m\right) = -\ell(\ell+1)\left(e^{im\phi}P_\ell^m\right)$$

$$\frac{\partial f_{\ell m}}{\partial t} = -\ell(\ell+1)\frac{1}{2}\nu_0^{\alpha/\beta} f_{\ell m} \qquad f_{\ell m} \sim e^{-(\ell(\ell+1)\nu_0^{\alpha/\beta}/2)t}$$

note decay rate $\sim \ell^2$

Diffusion wipes out

fine variations quickly

Simple Resistivity $\Rightarrow$ Run aways

Model 1D FP with electric Field

$$\frac{q}{m} E \frac{\partial f}{\partial v} = \frac{\partial}{\partial v} \nu(v)\left[v f + \frac{T}{m}\frac{\partial f}{\partial v}\right]$$

if $qE > 0$ assume $f \to 0$ as $v \to -\infty$

integrate over $v$

$$\frac{q}{m} E f = \nu(v)\left[v f + \frac{T}{m}\frac{\partial f}{\partial v}\right]$$

Solution $\quad f = f(0)\exp\left[-\frac{mv^2}{2T} + \int_0^v dv' \frac{qE}{T}\frac{1}{\nu(v')}\right]$

Suppose $E$ is weak

odd function of $v$

$$f \simeq f(0)\exp\left[-\frac{mv^2}{2T}\right]\left\{1 + \frac{qE}{T}\int_0^v \frac{dv'}{\nu}\right\}$$

$$J = \int dv\, q v f = \frac{q^2 E f_0}{T}\int dv \exp\left[-\frac{mv^2}{2T}\right]\left(\int_0^v \frac{dv'}{\nu}\right) v$$

Appears ok But