b) Boltzmann Equation

Preliminaries:

$f_\alpha(\underline{x},\underline{v},t)\,d^3x\,d^3v$ = THE NUMBER OF PARTICLES OF THE α^{th} TYPE (i.e. ELECTRONS, IONS, ETC.) IN THE VOLUME ELEMENT $d^3x\,d^3v$ OF PHASE SPACE AT THE TIME t.

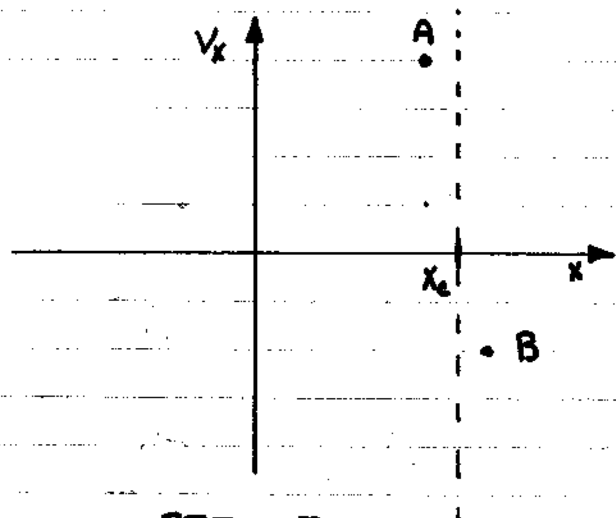
PHASE SPACE HAS SIX INDEPENDENT DIMENSIONS: THREE CONFIGURATION COORDINATES, (x,y,z); AND THREE VELOCITY COORDINATES, (v_x,v_y,v_z). THE BOLTZMANN EQUATION DESCRIBES THE TIME EVOLUTION OF THE DISTRIBUTION FUNCTION f IN PHASE SPACE. IT IS BASICALLY JUST THE CONTINUITY EQUATION FOR f, BUT IN THE SIX DIMENSIONAL PHASE SPACE. IN WRITING THE BOLTZMANN EQUATION ONE ASSUMES THAT MOST OF THE TIME A GIVEN PARTICLE OBEYS NEWTON'S LAW ($\underline{F}=m\underline{a}$) WHERE THE FORCE, $\underline{F}$, IS THE SUM OF THE FORCES APPLIED EXTERNALLY (~~as~~ GRAVITY, OR ELECTROMAGNETIC FIELDS FROM COILS ~~AND~~ OR CAPACITOR PLATES), AND THE MEAN FIELDS DUE TO THE COLLECTIVE ACTION OF THE CHARGED PARTICLES PRESENT; AND THAT OCCASIONALLY A PARTICLES SUFFERS A SEVERE COLLISION WITH ANOTHER PARTICLE.

THE CONTINUITY EQUATION FOR f IN PHASE SPACE IS

$$\frac{\partial f_\alpha}{\partial t} + \sum_{i=1}^{6} \frac{\partial}{\partial x_i} v_i f_\alpha = \left(\frac{d f_\alpha}{dt}\right)_{\text{collisions}}(\underline{x},\underline{v},t).$$

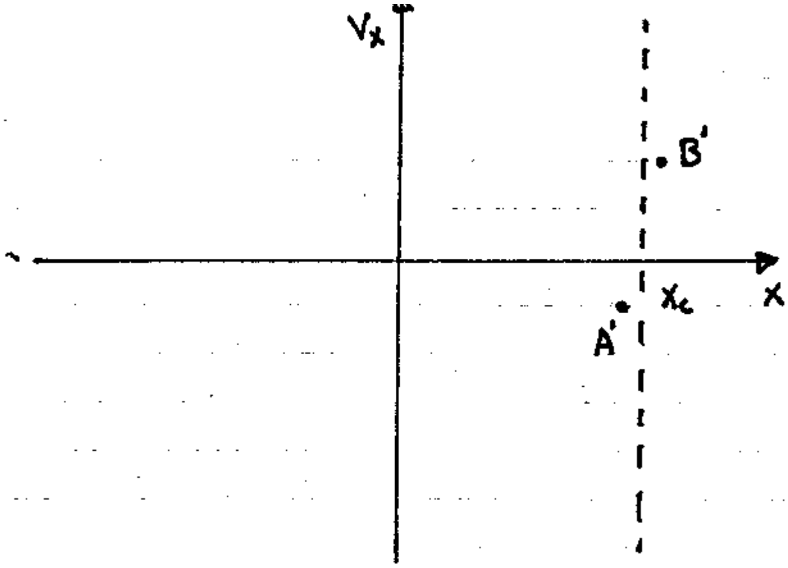
IF THE COLLISION TERM IN THE ABOVE EQUATION WERE ZERO, THE EQUATION WOULD SIMPLY STATE THAT f WAS CONTINUOUS, THAT IS THE ~~RATE~~ TIMED RATE OF ~~#~~CHANGE OF f INTEGRATED OVER A VOLUME ~~OF~~ PHASE SPACE MUST BE

As stated before, collisions which result from the interaction of two particles via a short range force can be thought to create and destroy particles in phase space. For instance, consider the following simple example: Two particles are constrained to move along a straight line (the x-axis). Particle A travels along the x coordinate with positive velocity; particle B travels with negative velocity, and they collide at X_c. Just prior to the collision the two particles positions in the two dimensional phase space are shown below.



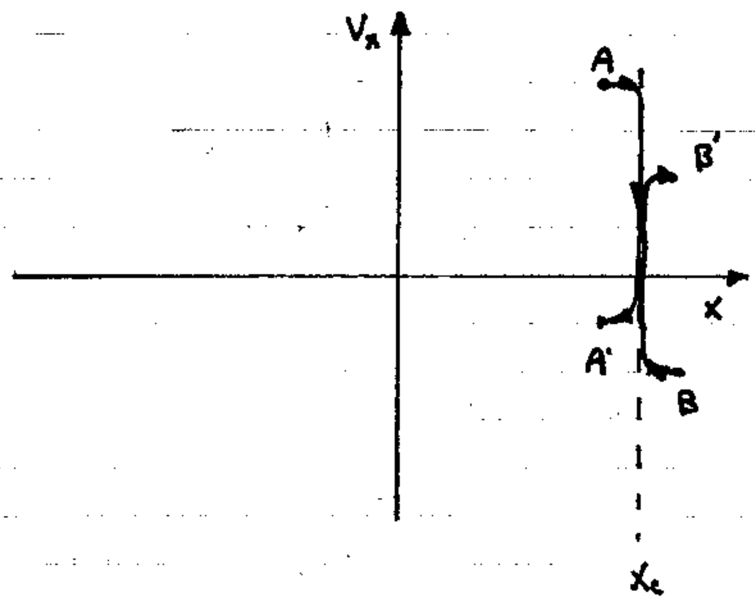
BEFORE COLLISION

Just after the collision the particles' location in phase space may be that given in the next figure. Here the positions are labeled A' and B', but they refer to the same particles as before.



AFTER COLLISION

THE ACTUAL TRAJECTORIES FOLLOWED BY THE PARTICLES ARE SHOWN IN THE THIRD FIGURE.



IF THE COLLISION OCCURED OVER A SHORT ENOUGH PERIOD OF TIME IT CAN BE VIEWED AS THE DISSAPPEARANCE OF PARTICLES AT A AND B, AND THE REAPPEARANCE OF THE PARTICLES AT A' AND B'. BECAUSE THE INTERACTION LENGTH IS SHORT A, A', B, AND B' SHOULD ALL HAVE THE SAME X COORDINATE, X_s. A MORE RIGOROUS DERIVATION OF THE BOLTZMANN EQUATION

Heuristic Derivation of the Boltzmann Equation

The following assumptions are made as a preliminary to the development of Boltzmann's equation:

(a) It is reasonable to assume that the state of the gas is described by a one-body distribution function,

(b) the density of particles is low enough for only two body interactions to be considered, i.e., $r_o \ll \ell_o$, where r_i is the range of interparticle forces and ℓ_o is the mean interparticle distance,

(c) the duration of an encounter between two particles is much smaller than the period of the free motion of the particles, i.e., $t_i \ll t_f$, where $t_i = r_o/v_{av}$ and $t_f = \lambda_o/v_{av}$, v_{av} is the mean speed of the particles and λ_o their mean free path,

(d) particles are assumed to be point centers of spherically symmetric fields, so that the one-body distribution function depends only on the position $\bar{x}$, velocity $\bar{v}$ of the particles and time t . In case of exceptional models for the particles other variables, e.g., the angular velocity, may be introduced.

A fifth assumption is made later on.

Let $\tilde{f}(\bar{x}, \bar{v}, t)\, \Delta^3x\ \Delta^3v$ be the expected number of particles to be found in a volume element $\Delta^3x\ \Delta^3v$ of phase space about $\bar{x}$ and $\bar{v}$ at the instant of time t . The volume element $\Delta\mu \equiv \Delta^3x\ \Delta^3v$ must be large enough to contain a sufficient number of particles in order that probability concepts can be applied at all. Thus

$$\tilde{f}(\bar{x}, \bar{v}, t) = \frac{1}{\Delta^3x\, \Delta^3v} \int_{\Delta\mu} f\, d^3x\, d^3v \qquad (1)$$

where $f = \sum_r \delta(\bar{x} - \bar{x}_r)\, \delta(\bar{v} - \bar{v}_r)$, r is a particle index.

Further, the changes in $\tilde{f}$ will be observed over a time Δt which is much larger than t_i . In what follows we will keep these restrictions in mind although we shall write $f(\bar{x}, \bar{v}, t)\, d^3x\, d^3v$ instead of $\tilde{f}(\bar{x}, \bar{v}, t)\, \Delta^3x\, \Delta^3v$.

We are concerned with developing an equation which determines the temporal evolution of f given its value at some initial time t_o for all $\bar{x}$ and $\bar{v}$. By definition the total number of particles

$$N = \int f(\bar{x}, \bar{v}, t)\, d^3x\, d^3v \qquad (2)$$

where the integration is carried over the volume V in configuration space occupied by the particles and over the accessible region of velocity space. Further, we define a density n

$$n = \frac{N}{V} = \int f(\bar{x}, \bar{v}, t)\, d^3v \qquad (3)$$

with V vanishingly small but large enough to contain several particles.

If we now assume that those particles contained in $d^3x\, d^3v$ do not interact with each other, then at a time $t + dt$ $(dt \gg t_i)$ we expect these particles to be in the volume element $d^3x'\, d^3v'$ about $\bar{x}'$ and $\bar{v}'$ where

$$\bar{x}' = \bar{x} + \bar{v}\, dt + O(dt)^2 \qquad (4a)$$

$$\bar{v}' = \bar{v} + \bar{a}\, dt + O(dt)^2 \qquad (4b)$$

where $\bar{a}$ is the acceleration suffered by the particles as a result of fields that may be applied by external means or those generated by the collective action of all particles excluding those whose trajectories are under examination. Thus

$$\bar{a} = \bar{a}^e + \bar{a}^i \qquad (5)$$

where the superscripts e and i classify the cause of the acceleration. The new volume element $d^3x'\, d^3v'$ is related to the old volume element $d^3x\, d^3v$ by the relation

$$d^3x'\, d^3v' = \left| J\left(\frac{\bar{x}', \bar{v}'}{\bar{x}, \bar{v}}\right) \right| d^3x\, d^3v \qquad (6)$$

where J is the Jacobian of the transformation written out in full as

$$J = \begin{vmatrix} \frac{\partial x_1'}{\partial x_1} & \frac{\partial x_2'}{\partial x_1} & \frac{\partial x_3'}{\partial x_1} & 0 & 0 & 0 \\ \frac{\partial x_1'}{\partial x_2} & \frac{\partial x_2'}{\partial x_2} & \frac{\partial x_3'}{\partial x_2} & 0 & 0 & 0 \\ \frac{\partial x_1'}{\partial x_3} & \frac{\partial x_2'}{\partial x_3} & \frac{\partial x_3'}{\partial x_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial v_1'}{\partial v_1} & \frac{\partial v_2'}{\partial v_1} & \frac{\partial v_3'}{\partial v_1} \\ 0 & 0 & 0 & \frac{\partial v_1'}{\partial v_2} & \frac{\partial v_2'}{\partial v_2} & \frac{\partial v_3'}{\partial v_2} \\ 0 & 0 & 0 & \frac{\partial v_1'}{\partial v_3} & \frac{\partial v_2'}{\partial v_3} & \frac{\partial v_3'}{\partial v_3} \end{vmatrix}$$

Making use of (4a) and (4b)

$$J = 1 + \frac{\partial a_\alpha}{\partial v_\alpha} dt + O(dt)^2 \quad (7)$$

$$d^3x\, d^3v' = \left[1 + \frac{\partial a_\alpha}{\partial v_\alpha} dt + O(dt)^2\right] d^3x\, d^3v \quad (8)$$

and further,

$$f(\bar{x}', \bar{v}', t + dt)\, d^3x'\, d^3v' = f(\bar{x}, \bar{v}, t)\, d^3x\, d^3v + \left(\frac{\delta f}{\delta t}\right)_c d^3x\, d^3v\, dt \quad (9)$$

which means that the same number of particles are in the new volume element as in the old element except for those gained or lost by interaction among the particles themselves denoted by

$$\left(\frac{\delta f}{\delta t}\right)_c d^3x\, d^3v\, dt\ .$$

Expanding the L.H.S. of (9) in a Taylor series about $(\bar{x}, \bar{v}, t)$ we have

$$\text{LHS (9)} = \left\{f(\bar{x}, \bar{v}, t) + \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_\alpha}\frac{dx_\alpha}{dt} + \frac{\partial f}{\partial v_\alpha}\frac{dv_\alpha}{dt}\right)dt\right\} d^3x\, d^3v \left(1 + \frac{\partial a_\alpha}{\partial v_\alpha} dt\right) + O(dt^2)$$

Thus $$\left\{\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_\alpha}\frac{dx_\alpha}{dt} + \frac{\partial f}{\partial v_\alpha}\frac{dv_\alpha}{dt} + f\frac{\partial a_\alpha}{\partial v_\alpha}\right\} = \left(\frac{\delta f}{\delta t}\right)_c$$

As we shall be confining ourselves to forces that are either independent of particle velocity or if they do depend, they are given by the Lorentz relation $q\,\bar{v} \times \bar{B}$, $\frac{\partial a_\alpha}{\partial v_\alpha}$ always vanishes and we have finally

$$\frac{\partial f}{\partial t} + v_\alpha \frac{\partial f}{\partial x_\alpha} + a_\alpha \frac{\partial f}{\partial v_\alpha} = \left(\frac{\delta f}{\delta t}\right)_c \quad (10)$$

<u>Collision term $\left(\frac{\delta f}{\delta t}\right)_c$</u>.

The volume swept per second by a particle having velocity $\bar{v}$ (class A) and a particle of velocity v_1 (class B) such that if they are found in this volume, a collision will certainly occur is given by

$$\int v_R\, \sigma(v_R, \Theta)\, d^2\Omega$$

$v_R = \left|\bar{v} - \bar{v}_1\right|$ is the relative velocity.

$\sigma(v_R, \Theta)$ is the differential cross-section for the two particles.

Θ is the scattering angle in the center of mass system of the two particles.

The total amount of such volume swept in a time dt by particles in volume elements $d^3x\, d^3v$ and $d^3x_1\, d^3v_1$ is

$$f_2(\bar{x}, \bar{v}, \bar{x}_1, \bar{v}_1, t)\, d^3x\, d^3x_1\, d^3v\, d^3v_1\, dt \int v_R\, \sigma\,(v_R, \Theta)\, d^2\Omega$$

where $f_2\, d^3x\, d^3x_1\, d^3v\, d^3v_1$ is the number of particles with velocity vectors $\bar{v}$ and $\bar{v}_1$ in $d^3x\, d^3v$ and $d^3x_1\, d^3v_1$ of phase space.

(e) If the particles are to interact $|\bar{x} - \bar{x}_1|$ must be of order r_o and since d^3x is much larger than r_o^3, we put $x = x_1$ and further, we make the assumption that the particles are uncorrelated, i.e.,

$$f_2(\bar{x}, \bar{v}, \bar{x}_1\, \bar{v}_1, t) = f(\bar{x}, \bar{v}, t)\, f(\bar{x}, \bar{v}_1, t)$$

(Assumption of molecular chaos)

The total number of collisions that result in particles being knocked out of $d^3x\, d^3v$ in dt

$$L \equiv d^3x\, d^3v\, dt \int f(\bar{x}, \bar{v}, t)\, f(\bar{x}, \bar{v}_1, t)\, \rho\, d^2\Omega\, d^3v_1$$

with $\rho \equiv v_R\, \sigma\,(v_R, \Theta)$

Likewise the number of collisions between particles having velocities $\bar{v}'$ and $\bar{v}_1'$ which finally end up in $d^3x\, d^3v$

$$G \equiv d^3x\, d^3v'\, dt \int f(\bar{x}, \bar{v}', t)\, f(\bar{x}, \bar{v}_1', t)\, \rho\, d^2\Omega\, d^3v_1'$$

Net loss of particles from $d^3x\, d^3v$ is then

$$\left(\frac{\delta f}{\delta t}\right)_c d^3x\, d^3v\, dt = dt\, d^3x \left\{ d^3v \int_{v_1', \Omega} f(\bar{x}, \bar{v}', t)\, f(\bar{x}, \bar{v}_1', t)\, \rho\, d^2\Omega\, d^3v_1' \right.\,.$$

$$- d^3v \int_{v_1, \Omega} f(\bar{x}, \bar{v}, t)\, f(\bar{x}, \bar{v}_1, t)\, \rho\, d^2\Omega\, d^3v\, d^3v_1 \qquad (11)$$

The velocities $\bar{v}'$, $\bar{v}_1'$ and $\bar{v}$, $\bar{v}_1$ are related because these are the velocities of the interacting particles before and after the collision respectively. Because we have assumed an elastic collision the following conservation relations hold,

$$m\bar{v}' + m\bar{v}_1' = m\bar{v} + m\bar{v}_1$$

$$\frac{1}{2}\, mv'^2 + \frac{1}{2}\, mv_1'^2 = \frac{1}{2}\, mv^2 + \frac{1}{2}\, mv_1^2$$

DUE TO THE TOTAL FLUX OF f THROUGH THE SURFACE SURROUNDING THE VOLUME. THE PRESENCE OF THE COLLISION TERM INDICATES THAT AT THE LEVEL OF APPROXIMATION AT WHICH WE ARE CONSIDERING THE PROBLEM COLLISIONS ACT AS A SOURCE OR SINK OF PARTICLES. THIS WILL BE ILLUSTRATED LATER.

THE SIX INDEPENDENT COORDINATES ARE

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (x, y, z, V_x, V_y, V_z).$$

THE $v_i(\underline{x},\underline{V},t)$ ARE THE VELOCITY COMPONENTS IN PHASE SPACE OF THE PARTICLES LOCATED AT $(\underline{x}, \underline{V})$ (i.e. $(x_1, x_2, x_3, x_4, x_5, x_6)$). WE KNOW FROM CLASSICAL MECHANICS

$$v_1 = V_x \,, \quad v_2 = V_y \,, \quad v_3 = V_z$$

$$v_4 = \frac{dV_x}{dt} = \frac{F_x}{m} \,, \quad v_5 = \frac{F_y}{m} \,, \quad v_6 = \frac{F_z}{m}.$$ THUS,

$$\frac{\partial f_\alpha}{\partial t} + \frac{\partial}{\partial \underline{x}} \cdot \underline{V} f_\alpha + \frac{\partial}{\partial \underline{V}} \cdot \frac{\underline{F}}{m} f_\alpha = \left(\frac{d f_\alpha}{dt}\right)_{\text{COLLISIONS}}$$

$$\frac{\partial}{\partial \underline{x}} \cdot \underline{V} f_\alpha = \underline{V} \cdot \frac{\partial f_\alpha}{\partial \underline{x}} + f_\alpha \frac{\partial}{\partial \underline{x}} \cdot \underline{V} \quad (\to 0)$$

$\underline{x}$ & $\underline{V}$ ARE INDEPENDENT VARIABLES

$$= \underline{V} \cdot \frac{\partial f_\alpha}{\partial \underline{x}}$$

$$\frac{\partial}{\partial \underline{V}} \cdot \frac{\underline{F}}{m} f_\alpha = \frac{\underline{F}}{m} \cdot \frac{\partial f_\alpha}{\partial \underline{V}} + f_\alpha \frac{\partial}{\partial \underline{V}} \cdot \frac{\underline{F}}{m} \quad (\to 0)$$

$\frac{\partial}{\partial \underline{V}} \cdot \underline{F} = 0$ FOR LORENTZ FORCES

THEREFORE

$$\frac{\partial f_\alpha}{\partial t} + \underline{V} \cdot \frac{\partial f_\alpha}{\partial \underline{x}} + \frac{\underline{F}}{m} \cdot \frac{\partial f_\alpha}{\partial \underline{V}} = \left(\frac{d f_\alpha}{dt}\right)_{\text{COLLISIONS}}.$$

The collision is described aptly by Fig. 1. The relative velocity $\bar{v}_R$ rotates through an angle θ and the center of mass velocity OG is unchanged; for this transformation from $\bar{v}'$, $\bar{v}_1'$ to $\bar{v}$, $\bar{v}_1$ the Jacobian is unity and therefore

$$d^3v_1' \, d^3v' = d^3v \, d^3v_1 \qquad (12)$$

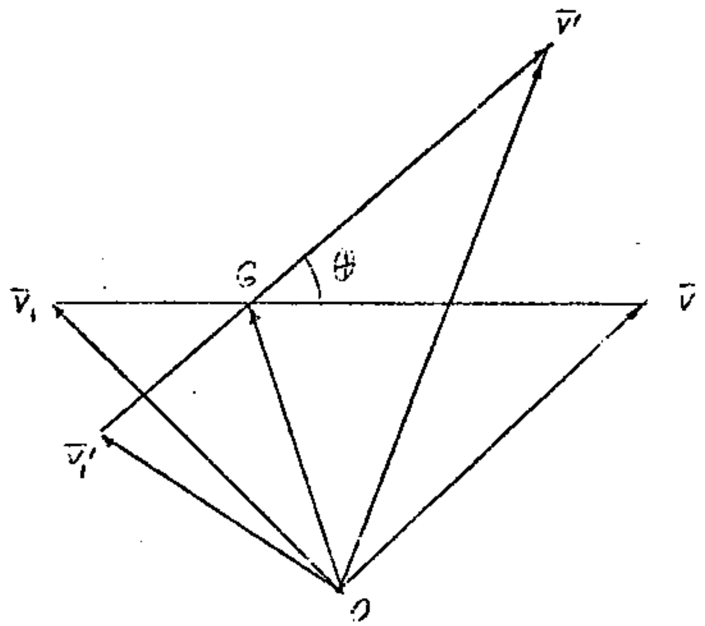


Figure 1

Substituting this in (11) and taking (10) into account,

$$\frac{\partial f}{\partial t} + v_\alpha \frac{\partial f}{\partial x_\alpha} + a_\alpha \frac{\partial f}{\partial v_\alpha} = \int \Big\{ f(\bar{x}, \bar{v}', t) \, f(\bar{x}, \bar{v}_1', t) - f(\bar{x}, \bar{v}, t) \, f(\bar{x}, \bar{v}_1, t) \Big\} \, \rho \, d^2\Omega \, d^3v_1 \qquad (13)$$

which is the standard way of expressing Boltzmann's equation.

If the gas contains several kinds of particles then each kind of particle has its own distribution function and we can readily generalize Boltzmann's equation to

$$\frac{\partial f^A}{\partial t} + v_\alpha \frac{\partial f^A}{\partial x_\alpha} + a_\alpha^A \frac{\partial f^A}{\partial v_\alpha} = \sum_B J\,(f^A f^B) \qquad (14)$$

A and B refer to the particle species, $J\,(f^A f^B)$ denotes the collision operator and the summation over B includes A .