Radiation by Moving Charges

4/22/21

Three Generic Forms

Radiation due to Acceleration

- Bremsrahlung   “braking radiation”

- Synchrotron

Radiation due to “faster than light” motion

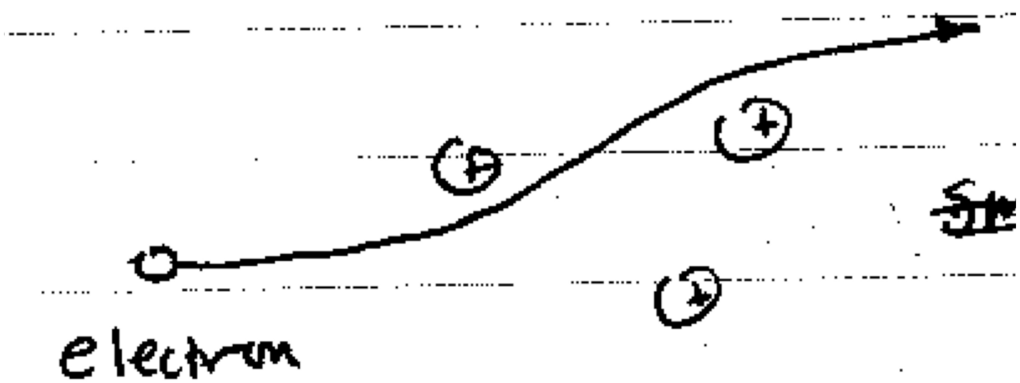
- Cherenkov

Radiation due to induced currents

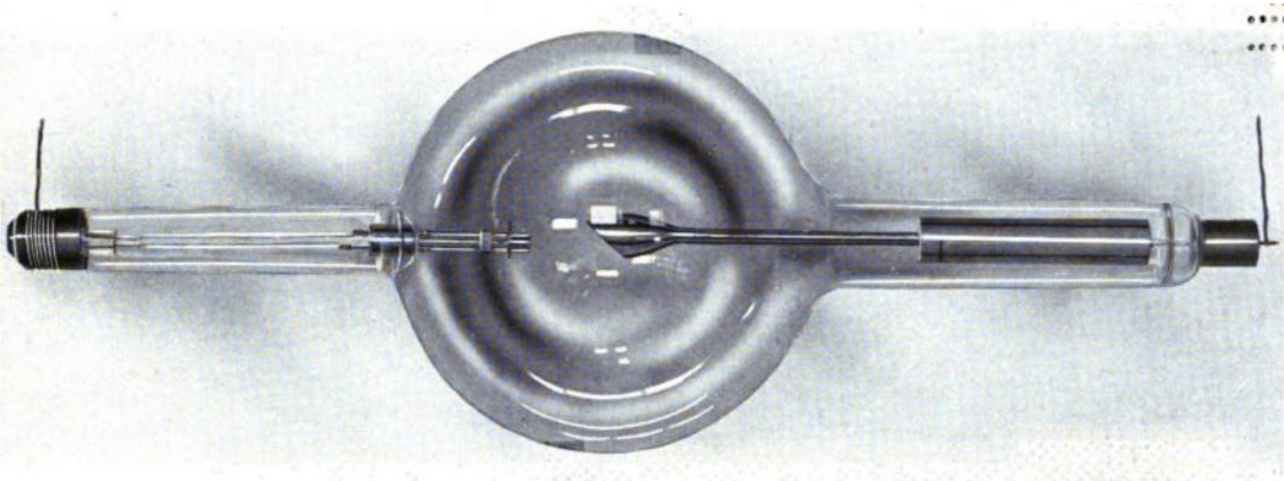
- Transition Radiation

Bremstrahlung

Braking radiation

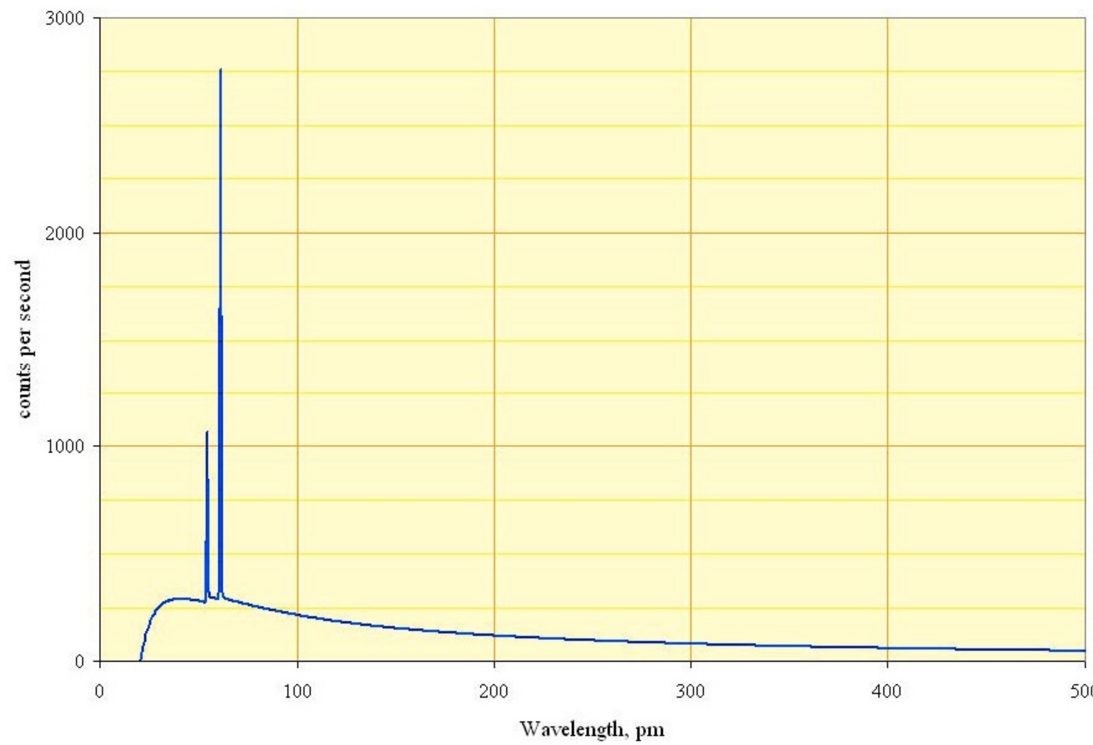


As charged particles (usually electrons) slow down in matter they are accelerated both longitudinally and transversely.



X-Ray Source

Coolidge X-ray tube, from around 1917. The heated cathode is on the left, and the anode is right. The X-rays are emitted downwards. Downloaded from Daniel Frost Comstock & Leonard T. Troland (1917) The Nature of Matter and Electricity



Spectrum of the X-rays emitted by an X-ray tube with a rhodium target, operated at 60 kV. The smooth, continuous curve is due to bremsstrahlung, and the spikes are characteristic K lines for rhodium atoms. - Wikipedia

Synchrotron Radiation

radiation comes out in a small angle $\sim \gamma^{-1}$

for a relativistic particle

Energy radiated per unit frequency per unit solid angle

$$\frac{dU}{d\omega\, d\Omega} = \frac{Z_0}{32\pi^3}\left|\mathbf{k}\times\bar{\mathbf{C}}(\mathbf{k},\omega)\right|^2$$

$$\bar{\mathbf{C}}(\mathbf{k},\omega) = \int_{-\infty}^{\infty} dt\, q\,\mathbf{v}(t)\, e^{i\omega t}\, e^{-i\,\mathbf{k}\cdot\mathbf{r}(t)} = \int_{-\infty}^{\infty} dt\, q\,\mathbf{v}(t)\, e^{i\omega(t\,-\,\hat{\mathbf{n}}\cdot\mathbf{r}(t)/c)}$$

Larmor's Formula

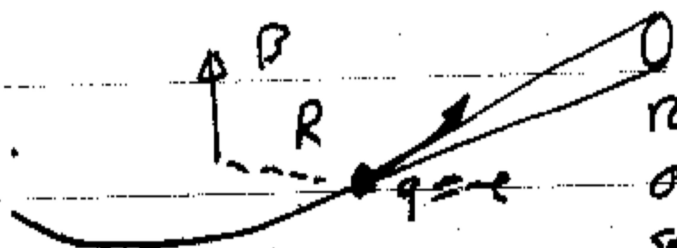
The total instantaneous power radiated

$$\boxed{P_T(t) = \frac{Z_0}{6\pi}\frac{q^2}{c^2}\left|\mathbf{a}(t)\right|^2}$$

Nonrelativistic $\quad |\boldsymbol{a}| = \dfrac{v^2}{R}$

Relativistic $\quad |\boldsymbol{a}| = \gamma^2 \dfrac{v^2}{R}$

$$\gamma = \frac{1}{\sqrt{1 - v^2 / c^2}}$$



radiation comes out in a small angle $\sim \gamma^{-1}$ for a relativistic particle

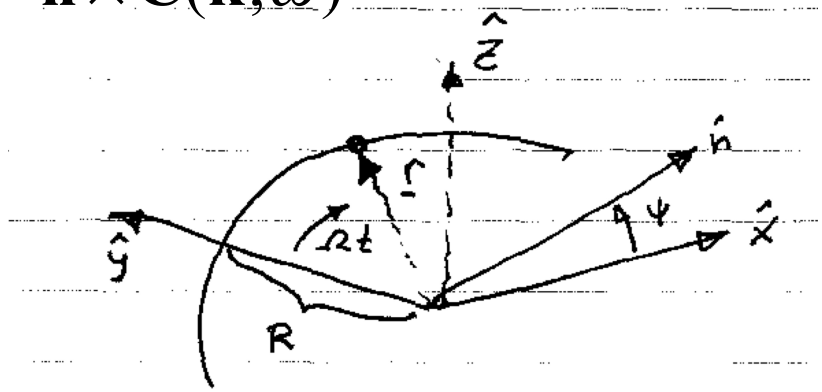
For Electrons

$$\gamma = 1 + \frac{E_{beam}}{mc^2} = 1 + \frac{E_{beam}[eV]}{512 \times 10^3}$$

Circular Motion

$$\bar{\mathbf{C}}(\mathbf{k},\omega) = \int_{-\infty}^{\infty} dt\, q\, \mathbf{v}(t) e^{i\omega t} e^{-i\mathbf{k}\cdot\mathbf{r}(t)} = \int_{-\infty}^{\infty} dt\, q\, \mathbf{v}(t) e^{i\omega(t - \hat{\mathbf{n}}\cdot\mathbf{r}(t)/c)}$$

$$\mathbf{n}\times\bar{\mathbf{C}}(\mathbf{k},\omega)$$



$$\underline{n} = \hat{\underline{x}} \cos\psi + \hat{z} \sin\psi$$

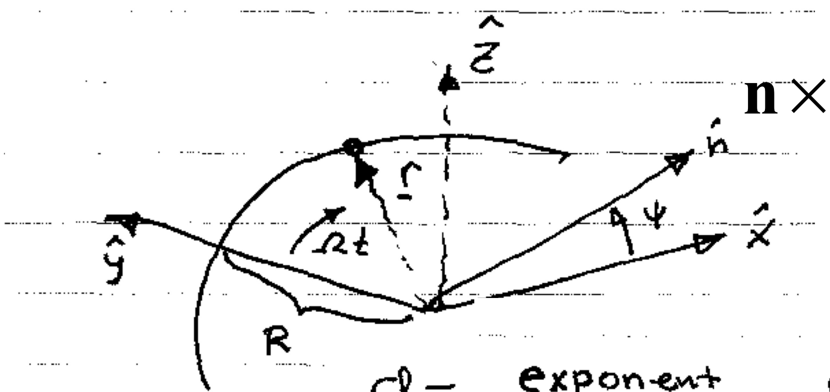
$$\underline{r}(t) = R[\hat{x} \sin\Omega t + \hat{y} \cos\Omega t]$$

$$\underline{v}(t) = R\Omega[\hat{x} \cos\Omega t - \hat{y} \sin\Omega t]$$

$$\underline{n} \times \underline{v} \cong -\Omega R[\hat{z} \sin\Omega t \cos\psi + \hat{y} \cos\Omega t \sin\psi] \qquad -v\left[\hat{z} (\Omega\tau) \tfrac{t}{\tau} + \hat{y}\,\psi\right]$$

$$\underline{n} \cdot \underline{r}(t) = R\cos\psi \{ \sin\Omega t$$

Rapidly Varying Exponent



$$\mathbf{n}\times\bar{\mathbf{C}}(\mathbf{k},\omega) = q\int_{-\infty}^{\infty} dt\,\mathbf{n}\times\mathbf{v}(t)e^{i\omega(t-\hat{\mathbf{n}}\cdot\mathbf{r}(t)/c)}$$

$\phi =$ exponent $\quad \omega\left(t - \frac{\underline{r}(t)\cdot\underline{n}}{c}\right)$

Expand exponent for small t and angle

$$\phi = \omega\big(t - \boldsymbol{n}\cdot\boldsymbol{r}(t)/c\big) \simeq \omega\left[\left(1-\beta+\frac{1}{2}\beta\psi^2\right)t + \frac{1}{6}\frac{R}{c}\big(\Omega t\big)^3\right]$$

$$\beta = v/c$$

Redefine variables in exponent

$$\phi \simeq \omega \left\{ \left[1 - \beta + \frac{1}{2}\beta\psi^2 \right] t + \frac{1}{6}\frac{R}{c}(\Omega t)^3 \right\}$$

Introduce: $\varepsilon = 2(1-\beta) + \beta\psi^2$

$$\tau = \Omega\, t\varepsilon^{-1/2} \qquad\qquad \xi = \frac{\omega}{3\Omega}\varepsilon^{3/2}$$

Exponent becomes:

$$\phi = \frac{3}{2}\xi\left(\tau + \frac{1}{3}\tau^3\right)$$

Energy Radiated

$$n \times \bar{\mathbf{C}} = -qR \int_{-\infty}^{\infty} d\tau \left[\hat{\mathbf{z}}\tau + \hat{y}\frac{\psi}{\varepsilon} \right] \exp\left[i\frac{3}{2}\xi\left(\tau + \frac{1}{3}\tau^3 \right) \right]$$

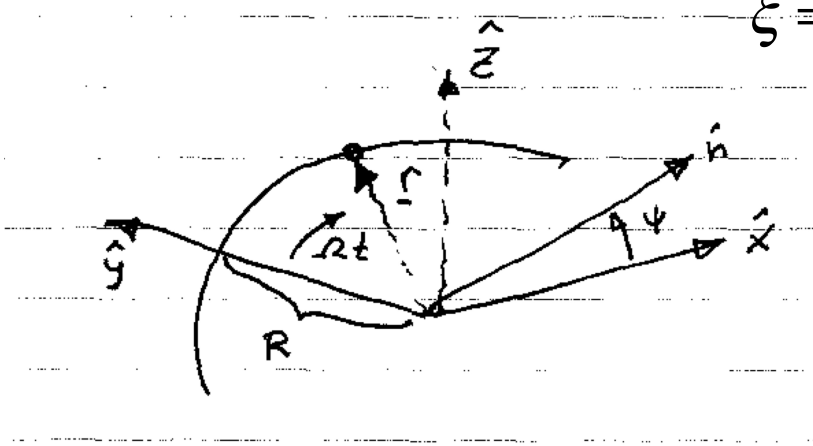
Modified Bessel Functions

This gives

$$\frac{dU}{d\omega d\Omega} = \frac{3Z_0 q^2}{32\pi^3}\frac{\xi^2}{\varepsilon}\left[K_{2/3}^2(\xi) + \left(\frac{\psi}{\varepsilon}\right)^2 K_{1/3}^2(\xi) \right]$$

$$\varepsilon = 2(1-\beta) + \beta\psi^2$$

$$\xi = \frac{\omega}{3\Omega}\left[2(1-\beta) + \beta\psi^2 \right]^{3/2}$$



Width in angle

$$|\psi| < \left[\frac{2(1-\beta)}{\beta} \right]^{1/2}$$

Ultra Relativistic Limit

$$\frac{dU}{d\omega d\Omega} = \frac{3Z_0 q^2}{32\pi^3}\frac{\xi^2}{\varepsilon}\left[K_{2/3}^2(\xi) + \left(\frac{\psi}{\varepsilon}\right)^2 K_{1/3}^2(\xi)\right]$$

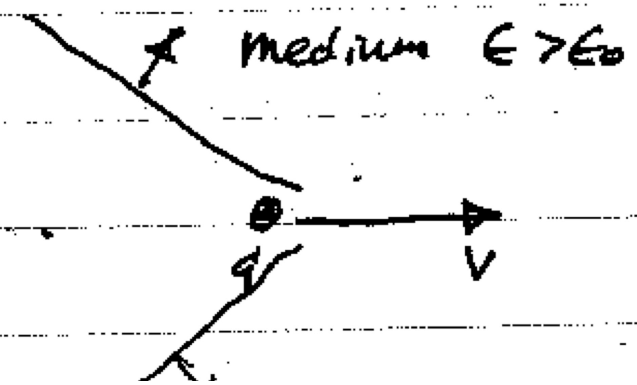
For high energy $\quad 2(1-\beta) = 2\frac{(1-\beta)(1+\beta)}{1+\beta} \to \frac{1}{\gamma^2}$

Argument

$$\xi = \frac{\omega}{3\Omega}\left[2(1-\beta) + \beta\psi^2\right]^{3/2} \to \frac{\omega}{3\Omega\gamma^3}\left[1+\gamma^2\psi^2\right]^{3/2}$$

$$\omega \simeq \Omega\gamma^3 \qquad\qquad |\psi| < \frac{1}{\gamma}$$

Cherenkov Radiation



Radiation if $v > \frac{1}{\sqrt{\epsilon\mu}}$

Current Density $\quad \underline{J} = q\,\delta(x)\delta(y)\,v\hat{z}\,\delta(z-vt)$

$$\tilde{\underline{E}}(\underline{k}_\perp, z, \omega) = \int d^2x_\perp\, dt\, \exp[i\omega t - i\underline{k}_\perp\cdot\underline{x}_\perp]\, \underline{E}(\underline{x}_\perp, z, t)$$

$$\underline{E}(\underline{x}_\perp, z, t) = \int \frac{d^2k_\perp\, d\omega}{(2\pi)^3} \exp[i\underline{k}_\perp\cdot\underline{x}_\perp - i\omega t]\, \tilde{\underline{E}}$$

Likewise for $\underline{H}$, $\underline{J}$

Maxwell's Equations

$$\left(i\underline{k}_\perp + \hat{z}\frac{\partial}{\partial z}\right)\times \tilde{\underline{E}} = i\omega\mu\,\tilde{\underline{H}} \qquad \left(i\underline{k}_\perp + \hat{z}\frac{\partial}{\partial z}\right)\times \tilde{\underline{H}} = \tilde{\underline{J}} - i\omega\epsilon\,\tilde{\underline{E}}$$

Evaluate current transform

Current Densit $\underline{J} = q\,\delta(x)\delta(y)\,v\hat{z}\,\delta(z-vt)$

$$\bar{\underline{J}} = \int d^2x_\perp dt\, \exp(-i\underline{k}_\perp\cdot\underline{x}_\perp + i\omega t)\,\underline{J}$$

$$\bar{\underline{J}} = q\,\hat{z}\,\exp(i k_v z) \qquad k_v = \frac{\omega}{v}$$

Solution of ME

Particular Solution

$$\frac{\partial}{\partial z} \rightarrow i k_v \qquad \bar{\underline{E}} = \bar{\bar{\underline{E}}}\, e^{i k_v z}$$

$$\bar{\bar{\underline{E}}}_\perp = \frac{k_v\,\underline{k}_\perp}{k_v^2 - \omega^2\epsilon\mu}\,\bar{\bar{E}}_z$$

$$\bar{\bar{E}}_z = q\frac{(k_v^2 - \omega^2\epsilon\mu)}{i\omega\epsilon\,[k_v^2 + k_\perp^2 - \omega^2\epsilon\mu]}$$

Inverse Transform

$$E_z = \int \frac{d^2k_\perp d\omega}{(2\pi)^3} \exp\left[i\underline{k}_\perp \cdot \underline{x}_\perp - i\omega t + ik_v z \right] \frac{q(k_v^2 - \omega^2\epsilon\mu)}{i\omega\epsilon(k_\perp^2 + k_v^2 - \omega^2\epsilon\mu)}$$

Replace ω by $k_v = \frac{\omega}{v}$

$ik_v z - i\omega t = ik_v(z - vt)$ $\qquad$ $\omega^2\epsilon\mu = k_v^2\beta^2$

$\beta^2 = \epsilon\mu v^2 = v^2/V_p^2$ $\qquad$ V_p = phase velocity $= \frac{1}{\sqrt{\epsilon\mu}}$

$V_p = c$ for vacuum

v=0

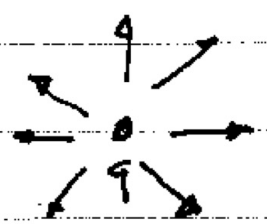
$$E_z = \int \frac{d^2k_\perp dk_v}{(2\pi)^3} \frac{k_v^2(1-\beta^2)q}{i k_v \epsilon \left(k_\perp^2 + k_v^2(1-\beta^2)\right)} \exp\left[i \underline{k}_\perp \cdot \underline{x}_\perp + i k_v z'\right]$$

$$z' = z - vt$$

$E_z(\underline{x}_\perp, z-vt)$

$E_z(\underline{x}_\perp, z' = z-vt)$

Solution when $\beta = v/v_p = 0$



$$E_z = -\frac{\partial}{\partial z} \frac{q}{4\pi\epsilon_0 |x|}$$

$$E_z = -\int \frac{d^2k_\perp dk_v}{(2\pi)^3} i k_v \frac{q\, e^{i(\underline{k}_\perp \cdot \underline{x}_\perp + k_v z)}}{\epsilon(k_\perp^2 + k_v^2)}$$

$V \neq 0$

$$E_z = \int \frac{d^2k_\perp dk_v}{(2\pi)^3} \frac{k_v^2(1-\beta^2)q}{ik_v\sigma\left(k_\perp^2+k_v^2(1-\beta^2)\right)} \exp\left[i\underline{k}_\perp\cdot\underline{x}_\perp + ik_v z'\right]$$

$z' = z - vt$

$E_z(\underline{x}_\perp, z-vt)$

$E_z(\underline{x}_\perp, z' = z-vt)$

Solution when $\beta = v/v_p \neq 0$

Let ~~~~ $k_v(1-\beta^2)^{1/2} = k_v'$

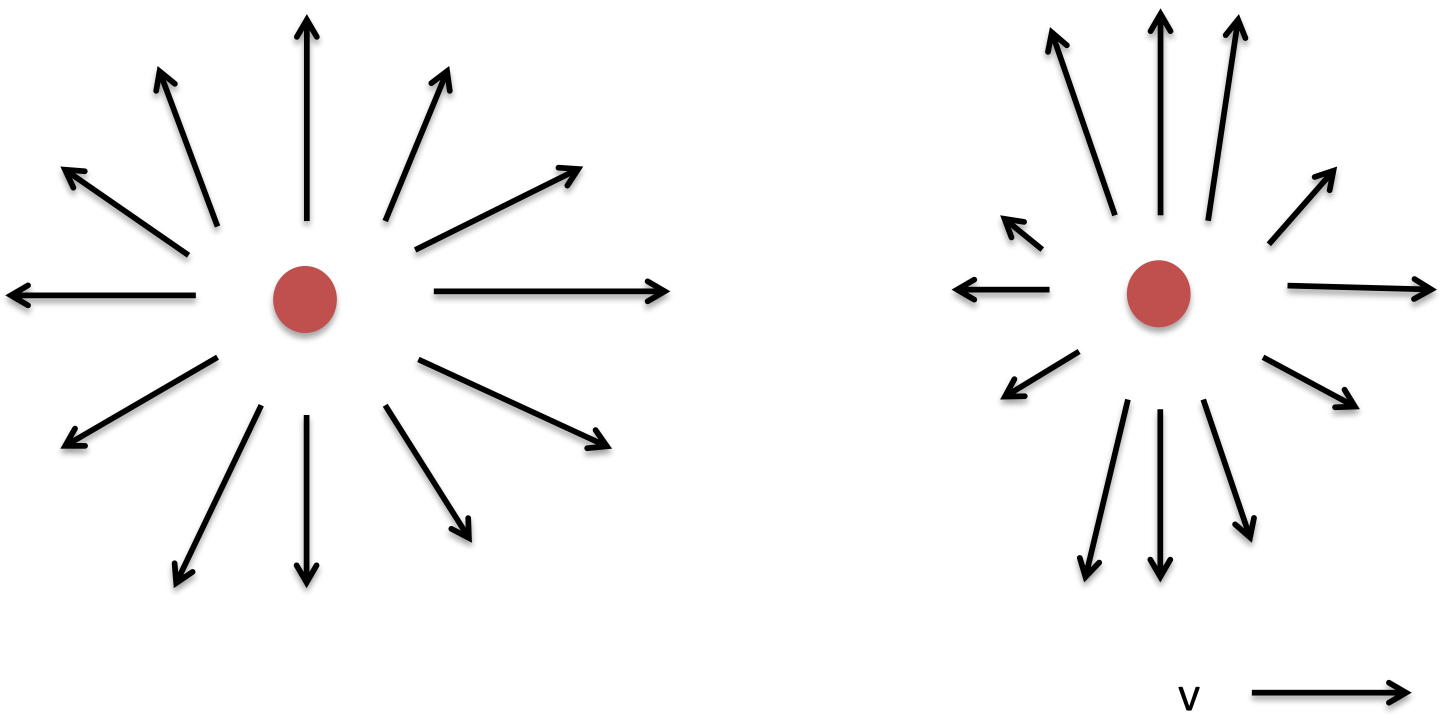
$z'/(1-\beta^2)^{1/2} = z''$

$$E_z = \int \frac{d^2k_\perp dk_v'}{(2\pi)^3} \frac{q\, k_v'^2}{ik_v'\sigma\left(k_\perp^2+k_v'^2\right)} \exp\left[i\underline{k}_\perp\cdot\underline{x}_\perp + ik_v' z''\right]$$

Lorentz contraction

$$E_z = E_{z0}(\underline{x}_\perp, \gamma(z-vt)) \qquad \gamma = (1-\beta^2)^{-1/2}$$

Field Lines Near a charge



Stationary Charge

Moving Charge

$V>V_p$

$$E_z = \int \frac{d^2k_\perp dk_v}{(2\pi)^3} \frac{k_v^2(1-\beta^2)q}{ik_v\sigma\left(k_\perp^2+k_v^2(1-\beta^2)\right)} \exp\left[i\underline{k}_\perp\cdot\underline{x}_\perp + ik_v z'\right]$$

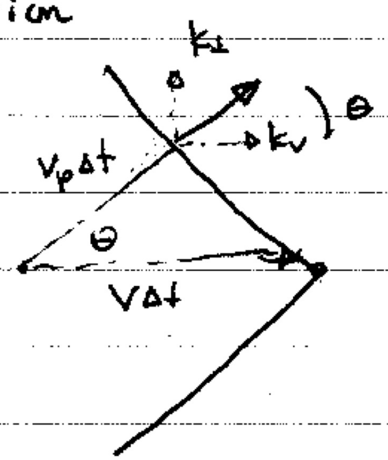
$$z' = z - vt$$

$E_z(\underline{x}_\perp, z-vt)$

What happens when $\beta > 1$?

Denominator $\to 0$ when $|\underline{k}_\perp|^2 = k_v^2(\beta^2-1)$

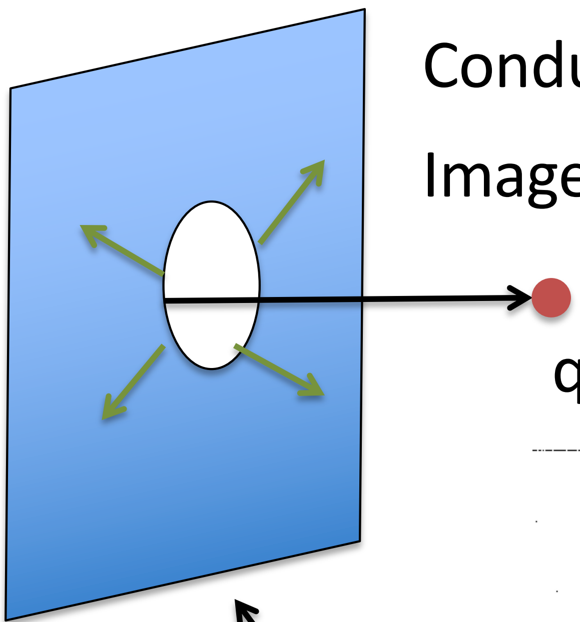
Radiation



$$\cos^2\theta = \frac{k_v^2}{k_v^2+k_\perp^2} = \frac{1}{\beta^2}$$

$$\cos\theta = \frac{V_p}{V} = \frac{1}{\beta}$$

Transition Radiation



Conductor

Image current

Previous solution

Solution of ME

Particular Solution

$$\frac{\partial}{\partial z} \rightarrow i k_v \qquad \bar{E} = \bar{\bar{E}}\, e^{i k_v z}$$

$$\bar{\bar{E}}_\perp = \frac{k_v k_\perp}{k_v^2 - \omega^2 \epsilon \mu} \bar{\bar{E}}_z$$

$$\bar{\bar{E}}_z = q \frac{(k_v^2 - \omega^2 \epsilon \mu)}{i \omega \epsilon [k_v^2 + k_\perp^2 - \omega^2 \epsilon \mu]}$$

Should be 0 at z=0

Add homogeneous solution

For Particular Solution

$$\bar{E}_\perp(z=0) = \frac{q\, k_v k_\perp}{i\omega\epsilon\left[k_v^2(1-\beta^2)+k_\perp^2\right]}$$

For Homogeneous Solution — outgoing radiation

$$E_{\perp h} = \frac{-q\, k_v k_\perp}{i\omega\epsilon\left[k_v^2(1-\beta^2)+k_\perp^2\right]} \exp\left[i\, k_\perp \cdot x_\perp + k_z z\right]$$

$$k_z = \sqrt{\omega^2\epsilon\mu - k_\perp^2}$$

Special Relativity

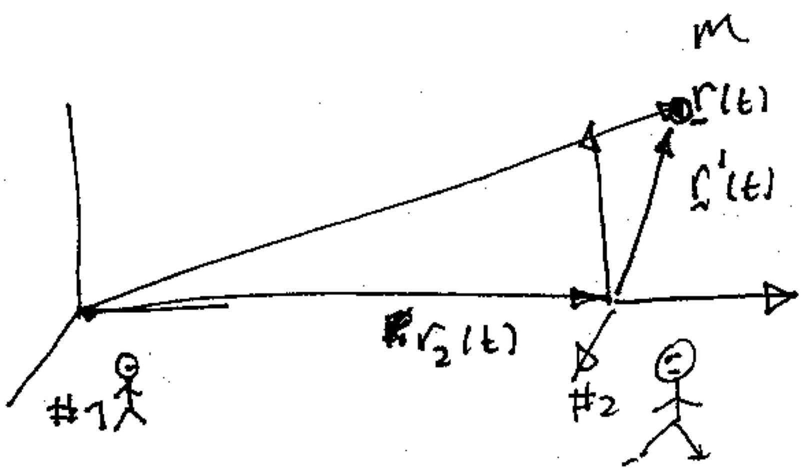
- The formulation of classical mechanics we have considered so far is based on Newton's law. It is thus limited to situations where

$$v = |\dot{\underset{\sim}{r}}| \ll c$$

where c is speed of light.

⊛ Associated with the use of Newton's law is the concept of Galilean invariance. Two observers ~~t~~ in reference frames that are moving with respect to each other with constant velocity both agree that particles follow trajectories in accord with Newton's Laws and both agree that time passes at the same rate.

A consequence of Galilean invariance is law of addition of velocities

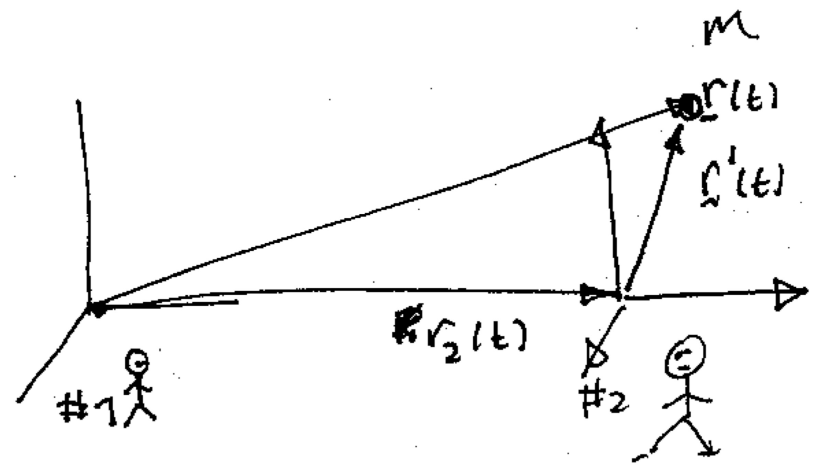


observer #1 sees $\underset{\sim}{r}(t)$ as position of m

observer #2 sees $\underset{\sim}{r}'(t) = \underset{\sim}{r}(t) - \underset{\sim}{r}_2(t)$

$\underset{\sim}{r}_2(t) = \underset{\sim}{V} t$ ~~velocity of~~ $\underset{\sim}{V}$ = Velocity of #2 as measured by #1 observer #1 sees velocity

$$\underset{\sim}{u} = \frac{d\underset{\sim}{r}}{dt}$$



observer #2 sees velocity $\underset{\sim}{u}'$

$$\underset{\sim}{u}' = \frac{d\underset{\sim}{r}}{dt} - \underset{\sim}{V} = \underset{\sim}{u} - \underset{\sim}{V}$$ velocities add

note same time

A consequence of Galilean invariance is that ~~[illegible]~~ observers will disagree on the speed of light. Who is correct?

Einstein introduce theory of special relativity which deals with these problems. Solution requires correct transformations of space and time measurements of the two observers.

Correct transformations satisfy two postulates

* The laws of physics are the same for all inertial observers

* The speed of light is the same for all inertial observers

Skipping to the End

Newton's Law in an ~~specified~~ inertial ref frame

$$\frac{d}{dt}\underset{\sim}{p} = \underset{\sim}{F}$$

$$\underset{\sim}{p} = m\underset{\sim}{v}$$

+ Rules for $\underset{\sim}{F}$

EM - force $\qquad \underset{\sim}{F} = q\,(\underset{\sim}{E} + \underset{\sim}{v} \times \underline{B})$

+ Maxwell's Equations

Skipping to the End

Newton's Law in an ~~[illegible]~~ ~~specified~~ inertial ref frame

$$\frac{d}{dt}\underset{\sim}{p} = \underset{\sim}{F}$$

$$\gamma = \frac{1}{(1-V^2/c^2)^{1/2}}$$

$$\underset{\sim}{p} = \gamma m \underset{\sim}{V}$$

$$\underset{\sim}{p} - m\underset{\sim}{V}$$

\+ Rules for $\underset{\sim}{F}$

EM – force $\qquad \underset{\sim}{F} = q\,(\underset{\sim}{E} + \underset{\sim}{V} \times \underset{\sim}{B})$

\+ Maxwell's Equations