Electrodynamics

Topics to be covered

Antennas:
Arrays, Impedance, Gain, Reciprocity

Radiation from Moving Charges

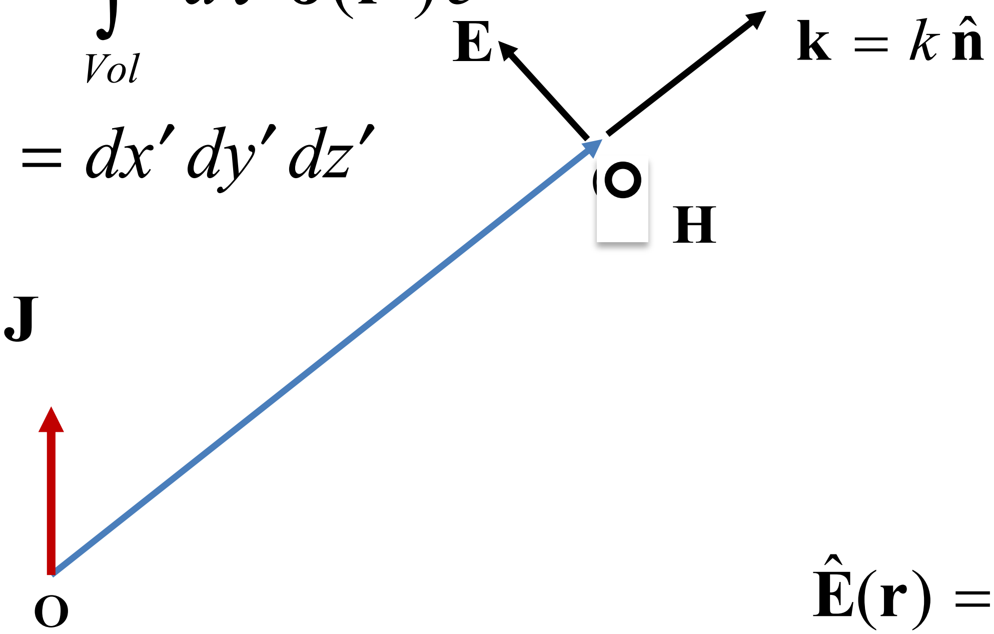
Notes Courtesy of Professor Phil Sprangle

Radiation

Fourier transform of the current density

$$\hat{\mathbf{C}}(\mathbf{k}) = \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

$$d\tau' = dx' \, dy' \, dz'$$



$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \, \hat{\mathbf{C}}(\mathbf{k})$$

$$\hat{\mathbf{B}}(\mathbf{r}) = i \frac{\mu_0}{4\pi r} e^{ikr} \, \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k})$$

$$\hat{\mathbf{E}}(\mathbf{r}) = -\, i \frac{e^{ikr}}{4\pi r} \frac{1}{\varepsilon_0 \, \omega} \mathbf{k} \times \left(\mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right)$$

$\hat{\mathbf{B}}(\mathbf{r})$ is transverse to $\hat{\mathbf{J}}$, $\mathbf{k} = k\,\hat{\mathbf{n}}$ and $\hat{\mathbf{E}}$

Radiated Power Flux

$$\langle \mathbf{S} \rangle = \frac{Z_0}{32\pi^2} \frac{\left| \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right|^2}{r^2} \hat{\mathbf{n}}$$

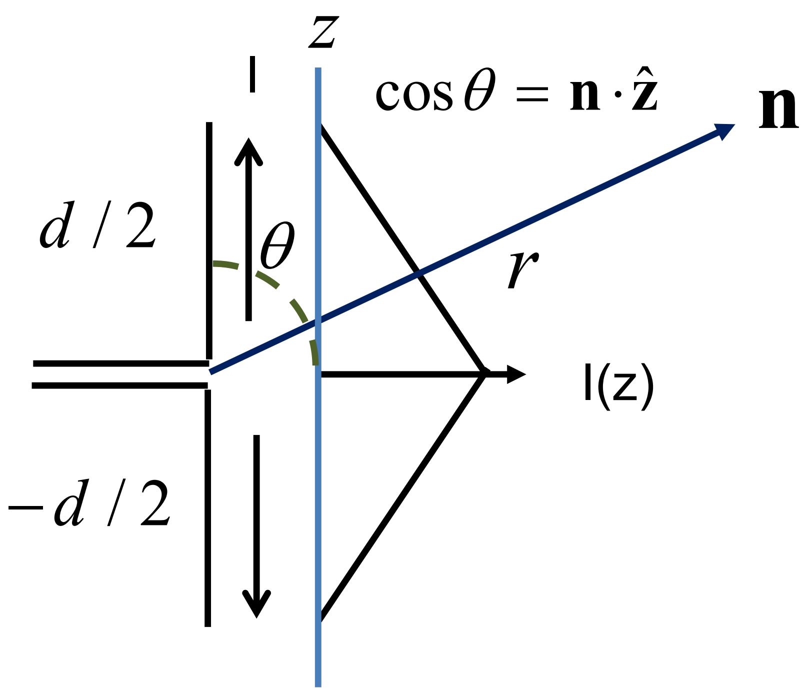
The power flux falls off like $1/r^2$ and is in the direction of $\mathbf{k} = k\,\hat{\mathbf{n}}$

Power radiated into the solid angle $d\Omega$

$$\frac{dP_T}{d\Omega} = \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle \mathrm{r}^2 = \frac{Z_0}{32\pi^2} \left| \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right|^2$$

$Z_0 = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = \dfrac{1}{c\,\varepsilon_0} = 377\,\Omega$ impedance of vacuum $\qquad c = \dfrac{1}{\sqrt{\varepsilon_0\,\mu_0}}$

Center Fed Linear Antenna



In short antennas current varies ~ linearly with z

Current density $\hat{\mathbf{J}}(\mathbf{r}) = I_0\,\delta(x)\delta(y)(1-2\frac{|z|}{d})\,\hat{\mathbf{z}} \qquad |z| \le \frac{d}{2}$

for $r >> r'$

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau'\,\hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'} = \frac{\mu_0}{4\pi r} e^{ikr} I_0\,\hat{\mathbf{z}} \int_{-d/2}^{d/2} dz'(1-2\frac{|z'|}{d}) e^{-ikz'\cos\theta}$$

Center Fed Linear Antenna

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq I_0\,\hat{\mathbf{z}} \int_{-d/2}^{d/2} dz' (1 - 2\frac{|z'|}{d}) e^{-ikz'\cos\theta}$$

use Euler's equ.

even odd

$$= I_0\,\hat{\mathbf{z}} \int_{-d/2}^{d/2} dz' \left(1 - 2\frac{|z'|}{d}\right)\left(\cos(kz'\cos\theta) - i\sin(kz'\cos\theta)\right)$$

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq 2I_0\,\hat{\mathbf{z}} \int_{0}^{d/2} dz' \left(1 - 2\frac{z'}{d}\right)\cos(kz'\cos\theta)$$

Center Fed Linear Antenna

To carry out the integration, let $\rho' = k\,z'\cos\theta$ and $\rho_0 = \dfrac{k\,d\cos\theta}{2}$

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq 2\,I_0\,\hat{\mathbf{z}}\frac{d}{\rho_0}\int_0^{\rho_0} d\rho'\left(1-\frac{\rho'}{\rho_0}\right)\cos\rho'$$

using $\int x\cos x\,dx = \cos x + x\sin x$

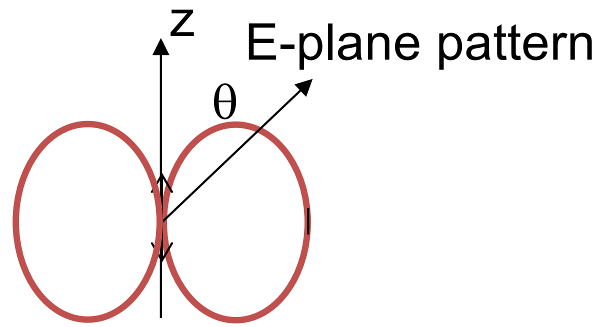
$$\hat{\mathbf{A}}(\mathbf{r}) \simeq I_0\,\hat{\mathbf{z}}\frac{d}{\rho_0^2}(1-\cos\rho_0) \quad \text{where } \rho_0 = \frac{k\,d\cos\theta}{2} = \frac{\pi\,d\cos\theta}{\lambda}$$

Antenna in the Dipoles Limit

In the dipole limit $\lambda >> d \quad (\rho_0 << 1)$

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} I_0 \frac{d}{2}\hat{\mathbf{z}} \qquad \hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \hat{\mathbf{C}}(\mathbf{k})$$

hence, $\mathbf{C}(\mathbf{k}) = I_0 \frac{d}{2}\hat{\mathbf{z}}$



Power radiated into the solid angle $d\Omega$

$$\frac{dP_T}{d\Omega} = \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle \mathrm{r}^2 = \frac{Z_0}{32\pi^2} \left| \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right|^2 = \frac{Z_0}{32\pi^2} k^2 \frac{I_0^2 d^2}{4} \sin^2\theta$$

Total Power Radiated and Radiation Resistance

The total power radiated is $P_T = \oint_S \frac{dP_T}{d\Omega} d\Omega$

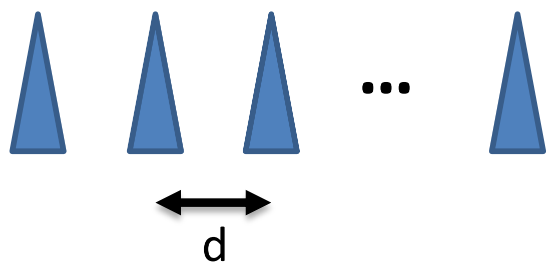
where $d\Omega = \sin\theta \, d\theta \, d\varphi$ is the solid angle

$$\frac{dP_T}{d\Omega} = \frac{Z_0}{32\pi^2} k^2 \frac{I_0^2 d^2}{4} \sin^2\theta \qquad \int_0^{2\pi} d\varphi \int_0^{\pi} \sin^3\theta \, d\theta = 2\pi \frac{4}{3}$$

$$P_T = \frac{Z_0}{48\pi} k^2 d^2 I_0^2 = \frac{1}{2} R_{rad} I_0^2$$

where the radiation resistance is $R_{rad} = \frac{Z_0}{24\pi} k^2 d^2$ [Ω]

Phased Array



$$\hat{\mathbf{C}}(\mathbf{k}) = \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

$$\frac{dP_T}{d\Omega} = \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle \mathrm{r}^2 = \frac{Z_0}{32\pi^2} \left| \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right|^2$$

N – identical antennas displaced by distance *d* and driven with different phases

$$\hat{\mathbf{C}}(\mathbf{k}) = \int_{Vol} d\tau' \sum_{n=1}^{N} \hat{\mathbf{J}}(\mathbf{r}' - \hat{\mathbf{x}} n d) e^{in\Delta\phi} \, e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

$$\hat{\mathbf{C}}(\mathbf{k}) = \hat{\mathbf{C}}_1(\mathbf{k}) \sum_{n=1}^{N} \exp\left[-i\left(kd\cos\theta - \Delta\phi \right) n \right]$$

Single unit

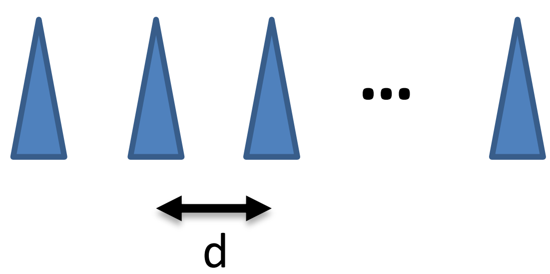
Effect of multiple units unit

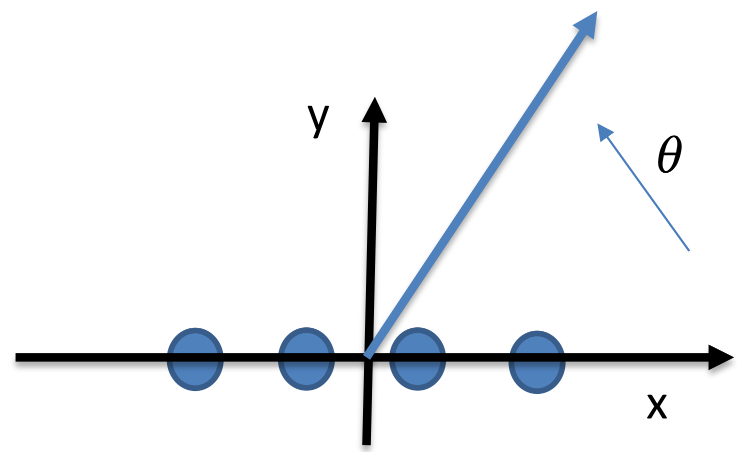
Radiated Power

$$\frac{dP_T}{d\Omega} = \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle \mathrm{r}^2 = \frac{Z_0}{32\pi^2} \left| \mathbf{k} \times \hat{\mathbf{C}}_1(\mathbf{k}) \right|^2 f(k,\theta)$$

$$f(k,\theta) = \left| \sum_{n=1}^{N} \exp\left[-i\left(kd\cos\theta - \Delta\phi \right) n \right] \right|^2 = \frac{1-\cos\left(N\psi\right)}{1-\cos\psi}$$

$$\psi = kd\cos\theta - \Delta\phi$$



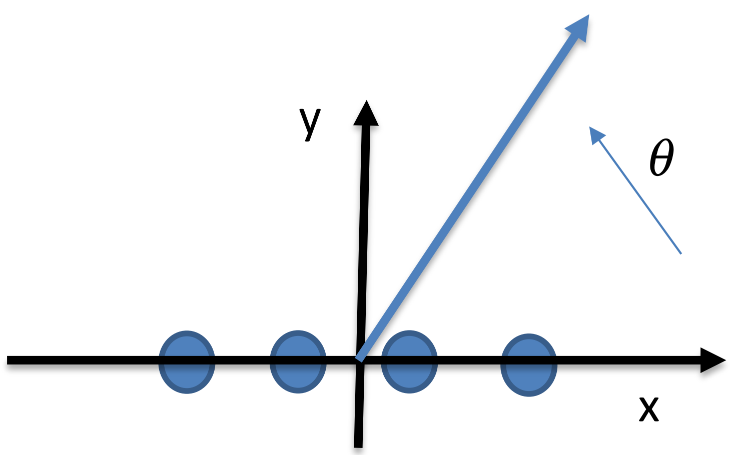


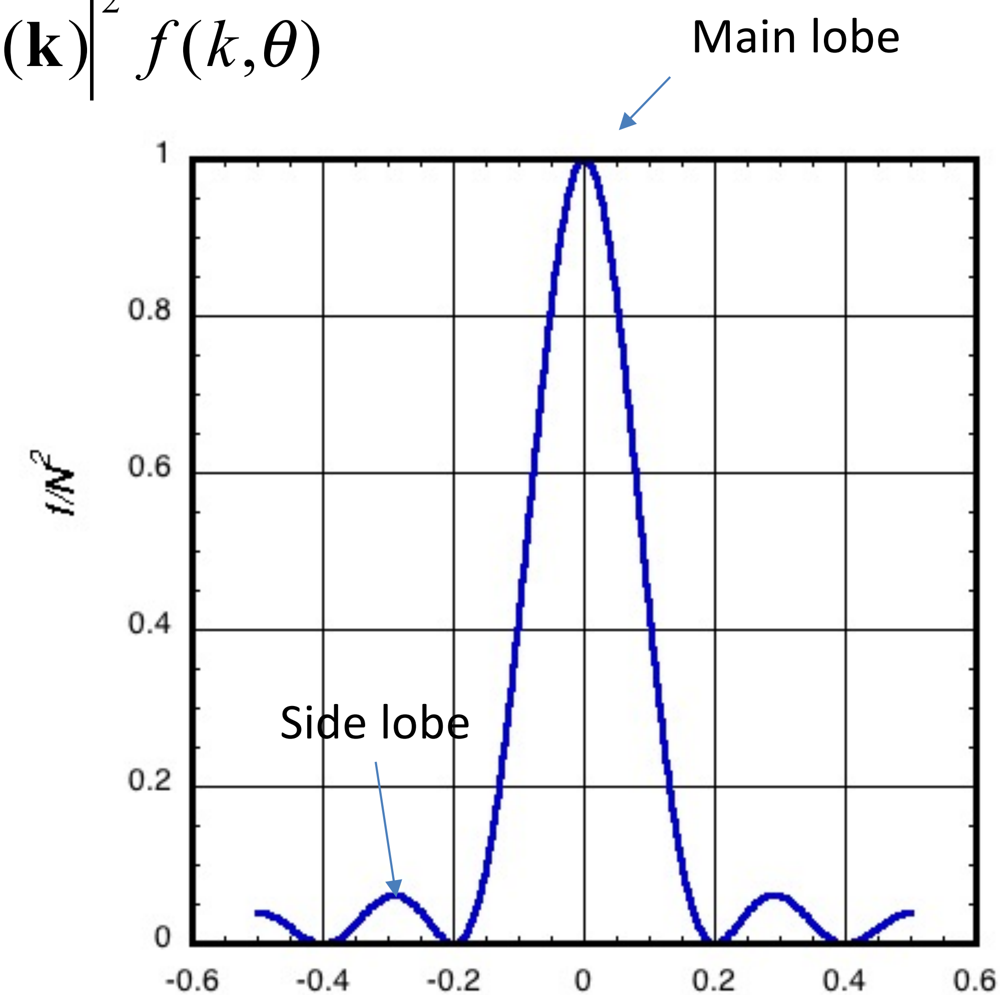
$$\psi = 0, \quad f = N^2$$

Top view

Radiation Pattern

$$\frac{dP_T}{d\Omega} = \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle r^2 = \frac{Z_0}{32\pi^2} \left| \mathbf{k} \times \hat{\mathbf{C}}_1(\mathbf{k}) \right|^2 f(k,\theta)$$

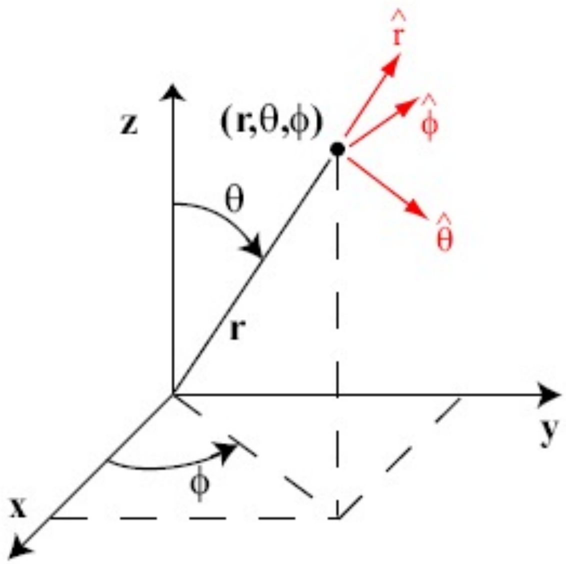


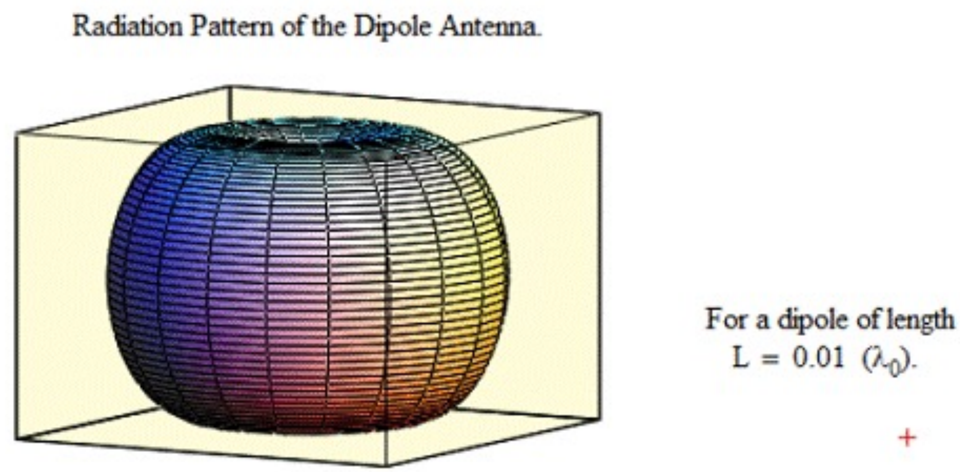


$$\psi = kd\cos\theta - \Delta\phi$$

Antenna Directivity and Gain

The total power radiated is $P_T = \oint_S \frac{dP_T}{d\Omega} d\Omega$ where $d\Omega = \sin\theta \, d\theta \, d\varphi$





Antenna Efficiency: (Radiated power/Input power) $\varepsilon = P_T / P_{in}$

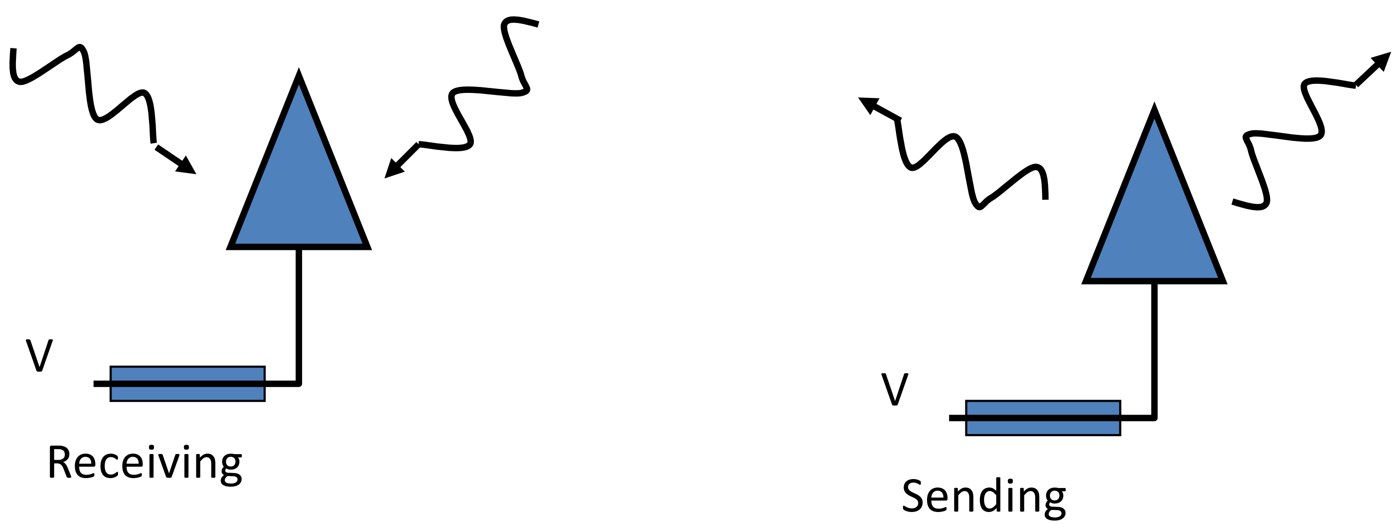
Antenna Directivity $D(\Omega) = \frac{dP_T}{d\Omega} / \frac{P_T}{4\pi}$, $\int \frac{d\Omega}{4\pi} D(\Omega) = 1$

Antenna Gain $G(\Omega) = \varepsilon D(\Omega)$

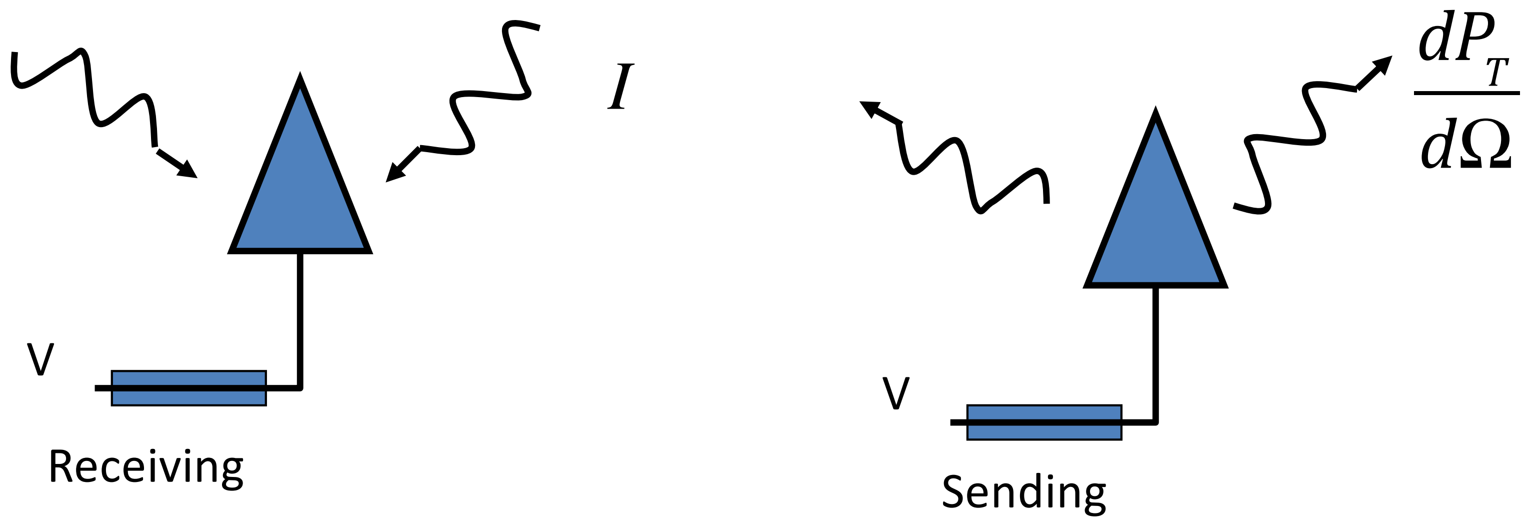
EM Reciprocity

Example:

- Antenna sending and receiving radiation patterns are equal due to time reversal symmetry of ME.
- Direct calculation of receiving pattern requires many simulations
- Instead, calculate sending pattern and invoke reciprocity



Effective Area – Antenna Gain



Power received → $P_R = A_e(\Omega) I$ ← Incident intensity

Effective area → $A_e(\Omega) = \frac{\lambda^2 G(\Omega)}{4\pi}$ ↔ gain

$$G(\Omega) = \frac{dP_T}{d\Omega} / \langle P_T \rangle$$

← Power per unit solid angle

$$\langle P_T \rangle = \int \frac{dP_T}{d\Omega} d\Omega$$

Defining Antenna Impedance

Suppose there are multiple antennas, each with its own current profile, u_p

$$\mathbf{J}(\mathbf{x}) = \sum_p \mathbf{u_p}(\mathbf{x}) I_p \ .$$

We can define a voltage for each antenna, V_p

$$V_p = -\int d^3x \ \mathbf{u_p}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) \ .$$

With this definition power balance is preserved

$$P = \frac{1}{2} \mathrm{Re}\left[\int d^3x \, \mathbf{E} \cdot \mathbf{J}^* \right] = -\frac{1}{2} \mathrm{Re}\left[\sum_p V_p I_p^* \right].$$

Solving Maxwell's Equations gives

$$V_p = \sum_{p'} Z_{pp'}^{rad}(k_0) I_{p'} \ ,$$

$$Z_{pp'}^{rad} = Z_{p'p}^{rad}$$

Formal Expression of Reciprocity

Consider two solutions of Maxwell's Equations in the same medium

$$\nabla\times\hat{\mathbf{E}}_1 = i\omega\mu(x)\hat{\mathbf{H}}_1$$
$$\nabla\times\hat{\mathbf{H}}_1 = -i\omega\varepsilon(x)\hat{\mathbf{E}}_1 + \hat{\mathbf{J}}_1$$

$$\nabla\times\hat{\mathbf{E}}_2 = i\omega\mu(x)\hat{\mathbf{H}}_2$$
$$\nabla\times\hat{\mathbf{H}}_2 = -i\omega\varepsilon(x)\hat{\mathbf{E}}_2 + \hat{\mathbf{J}}_2$$

Can Show

$$\int_V d^3x\left[\hat{\mathbf{J}}_2\cdot\hat{\mathbf{E}}_1 - \hat{\mathbf{J}}_1\cdot\hat{\mathbf{E}}_2\right] = \int_S da\,\mathbf{n}\cdot\left[\hat{\mathbf{H}}_2\times\hat{\mathbf{E}}_1 - \hat{\mathbf{H}}_1\times\hat{\mathbf{E}}_2\right]$$

E due to J1 at J2 same as
E due to J2 at J1

=0 if conducting BC
Or outgoing waves

Expression for Impedance Elements

Fourier transform in space

Ampere: $-i\omega\,\varepsilon\bar{\mathbf{E}}(\mathbf{k}) = i\mathbf{k}\times\bar{\mathbf{H}}(\mathbf{k}) - \sum_p I_p \bar{\mathbf{u}}_p(\mathbf{k})\,,$

Faraday: $i\omega\,\mu\bar{\mathbf{H}}(\mathbf{k}) = i\mathbf{k}\times\bar{\mathbf{E}}(\mathbf{k})$

Solve for Electric field

$$(k_0^2 - k^2)\bar{\mathbf{E}} + \mathbf{k}\mathbf{k}\cdot\bar{\mathbf{E}} = -ik\eta\sum_{p'} I_{p'}\bar{\mathbf{u}}_{p'}$$

Project Electric field onto current profile function

$$Z_{pp'}^{rad}(k_0 = \omega / c) = \sqrt{\frac{\mu}{\varepsilon}}\int \frac{d^3k}{(2\pi)^3}\frac{ik_0}{k_0^2 - k^2}\ \bar{\mathbf{u}}_p^* \cdot \underline{\underline{\Delta}}_1 \cdot \bar{\mathbf{u}}_{p'}\,.$$

$$\underline{\underline{\Delta}}_1 = \frac{1k^2 - \mathrm{k\,k}}{k^2} + \frac{\mathrm{k\,k}}{k^2 k_0^2}(k_0^2 - k^2)$$

Radiation from transient currents

Suppose we have a transient, time-dependent current,

$$\boldsymbol{J}(\boldsymbol{x},t)$$

How do we treat that?

Fourier Transform in time

$$\bar{\mathbf{J}}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} \mathbf{J}(\mathbf{r},t)e^{i\omega t}\, d\omega \qquad \mathbf{J}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\mathbf{J}}(\mathbf{r},\omega)e^{-i\omega t}\, d\omega$$

Everything follows from steady state equations

Radiation from Moving (Accelerating) Charges

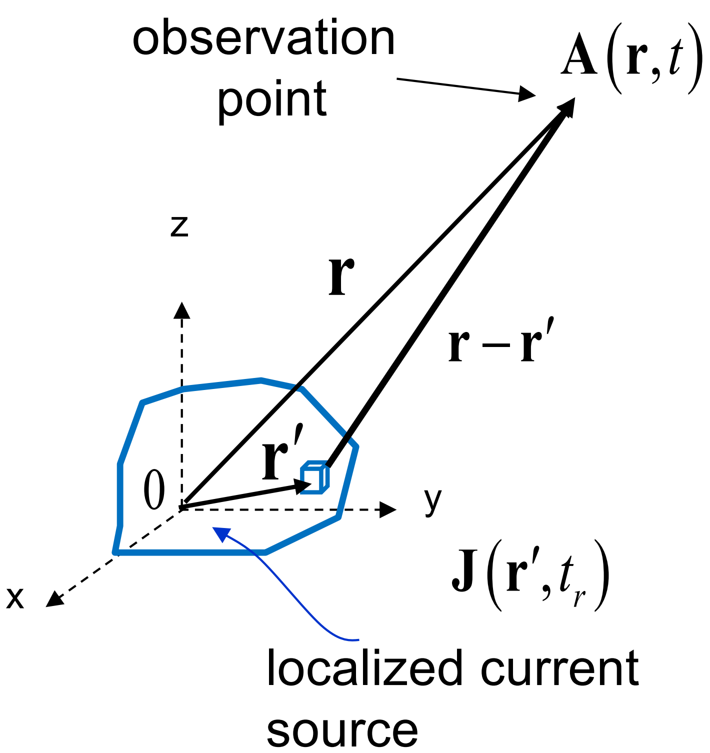
The fields and sources can be written in terms of their Fourier transforms in time

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\mathbf{E}}(\mathbf{r},\omega)\, e^{-i\omega t}\, d\omega \qquad \mathbf{H}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\mathbf{H}}(\mathbf{r},\omega)\, e^{-i\omega t}\, d\omega$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\mathbf{A}}(\mathbf{r},\omega)\, e^{-i\omega t}\, d\omega \qquad \mathbf{J}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\mathbf{J}}(\mathbf{r},\omega)\, e^{-i\omega t}\, d\omega$$

Radiation from Moving (Accelerating) Charges

In the **far field zone** $kr = \frac{2\pi}{\lambda} r >> 1$



$$\bar{\mathbf{A}}(\mathbf{r},\omega) \simeq \frac{\mu_0}{4\pi r} e^{ikr}\, \bar{\mathbf{C}}(\mathbf{k},\omega)$$

where $\bar{\mathbf{C}}(\mathbf{k},\omega) = \int_{Vol} d\tau' \, \bar{\mathbf{J}}(\mathbf{r}',\omega)\, e^{-i\mathbf{k}\cdot\mathbf{r}'}$

$$\bar{\mathbf{H}}(\mathbf{r},\omega) = \frac{i}{4\pi r} e^{ikr}\, \mathbf{k}\times\bar{\mathbf{C}}(\mathbf{k},\omega)$$

$$\bar{\mathbf{E}}(\mathbf{r},\omega) = -\,i\,\frac{e^{ikr}}{4\pi r}\frac{1}{\varepsilon_0\,\omega}\mathbf{k}\times\left(\mathbf{k}\times\bar{\mathbf{C}}(\mathbf{k},\omega)\right)$$

Poynting Flux (Far field zone)

$$S(\mathbf{r},t) = \hat{\mathbf{n}}\cdot(\mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t))$$

$$= \hat{\mathbf{n}}\cdot\left(\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\bar{\mathbf{E}}(\mathbf{r},\omega)\,e^{-i\omega t}\times\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\bar{\mathbf{H}}(\mathbf{r},\omega')\,e^{-i\omega' t}\right)$$

$$= \hat{\mathbf{n}}\cdot\left(\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}e^{-i(\omega+\omega')t}\left\{-i\frac{e^{ikr}}{4\pi r}\frac{1}{\varepsilon_0\,\omega}\mathbf{k}\times\left(\mathbf{k}\times\bar{\mathbf{C}}(\mathbf{k},\omega)\right)\right\}\times\left\{\frac{i}{4\pi r}e^{ik'r}\,\mathbf{k}'\times\bar{\mathbf{C}}(\mathbf{k}',\omega')\right\}\right)$$

$$\mathbf{k} = \frac{\omega}{c}\hat{\mathbf{n}}\;,\quad \mathbf{k}' = \frac{\omega'}{c}\hat{\mathbf{n}}$$

The total energy radiated is $U = \int_{-\infty}^{\infty} dt \int d\Omega\; r^2\, S(\mathbf{r},t)$

since $\int_{-\infty}^{\infty} dt\; e^{-i(\omega+\omega')t} = 2\pi\,\delta(\omega+\omega')$

$\int d\omega\,\delta(\omega+\omega')\,F(\omega,\omega') = F(\omega,-\omega)$ hence $\omega' \to -\omega$ and $k' \to -k$

Radiated Energy

$$U = \int d\Omega \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(\frac{1}{4\pi}\right)^2 \frac{1}{\varepsilon_0\, \omega} \hat{\mathbf{n}} \cdot \left\{ \mathbf{k} \times \left(\mathbf{k} \times \bar{\mathbf{C}}(\mathbf{k}, \omega) \right) \right\} \times \left\{ -\mathbf{k} \times \bar{\mathbf{C}}(-\mathbf{k}, -\omega) \right\}$$

$$\mathbf{k} \times \left(\mathbf{k} \times \bar{\mathbf{C}}(\mathbf{k}, \omega) \right) \times \mathbf{k} \times \bar{\mathbf{C}}^*(\mathbf{k}, \omega) = - \left| \mathbf{k} \times \bar{\mathbf{C}}(\mathbf{k}, \omega) \right|^2 \mathbf{k} \quad \text{and} \quad \hat{\mathbf{n}} \cdot \mathbf{k} = k$$

$$U = \int d\Omega \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(\frac{1}{4\pi}\right)^2 \frac{k}{\varepsilon_0\, \omega} \left| \mathbf{k} \times \bar{\mathbf{C}}(\mathbf{k}, \omega) \right|^2 = \int d\Omega \int_{-\infty}^{\infty} d\omega \underbrace{\left(\frac{Z_0}{32\pi^3} \left| \mathbf{k} \times \bar{\mathbf{C}}(\mathbf{k}, \omega) \right|^2 \right)}_{\frac{dU}{d\omega d\Omega}}$$

Energy radiated per unit frequency per unit solid angle

$$\frac{dU}{d\omega d\Omega} = \frac{Z_0}{32\pi^3} \left| \mathbf{k} \times \bar{\mathbf{C}}(\mathbf{k}, \omega) \right|^2$$

Accelerating Charges Radiate

The current density of a moving charge is $\mathbf{J}(\mathbf{r}',t) = q\,\mathbf{v}(t)\,\delta^3(\mathbf{r}'-\mathbf{r}(t))$

trajectory

In the far field zone

$$\overline{\mathbf{C}}(\mathbf{k},\omega) = \int_{Vol} d\tau'\,\overline{\mathbf{J}}(\mathbf{r}',\omega)\,e^{-i\,\mathbf{k}\cdot\mathbf{r}'} = \int_{-\infty}^{\infty} dt\,e^{i\omega t}\int_{Vol} d\tau'\,\mathbf{J}(\mathbf{r}',t)\,e^{-i\,\mathbf{k}\cdot\mathbf{r}'}$$

$$\overline{\mathbf{C}}(\mathbf{k},\omega) = \int_{-\infty}^{\infty} dt\,e^{i\omega t}\int_{Vol} d\tau' q\,\mathbf{v}(t)\,\delta^3(\mathbf{r}'-\mathbf{r}(t))\,e^{-i\,\mathbf{k}\cdot\mathbf{r}'}$$

$$\overline{\mathbf{C}}(\mathbf{k},\omega) = \int_{-\infty}^{\infty} dt\,q\,\mathbf{v}(t)\,e^{i\omega t}\,e^{-i\,\mathbf{k}\cdot\mathbf{r}(t)} = \int_{-\infty}^{\infty} dt\,q\,\mathbf{v}(t)\,e^{i\omega(t\,-\,\hat{\mathbf{n}}\cdot\mathbf{r}(t)/c)}$$

Radiation from Constant Velocity Charges

Charges moving at constant velocity in vacuum do not radiate
However, a constant velocity charge can radiate in a medium if its velocity is greater than the phase velocity of light v > ω/k (Cherenkov radiation)

For constant velocity in vacuum

$$\bar{\mathbf{C}}(\mathbf{k},\omega) = \int_{-\infty}^{\infty} dt\, q\, \mathbf{v}(t)\, e^{i\omega(t - \hat{\mathbf{n}}\cdot\mathbf{r}(t)/c)} = q\,\mathbf{v}_0 \int_{-\infty}^{\infty} dt\; e^{i\omega(1 - \hat{\mathbf{n}}\cdot\mathbf{v}_0/c)t}$$

$$\bar{\mathbf{C}}(\mathbf{k},\omega) = 2\pi\, q\, \mathbf{v}_0\, \delta(\omega(1 - \hat{\mathbf{n}}\cdot\mathbf{v}_0 / c)) = 0$$

argument of delta function can not be zero except for $\omega = 0$

Derivation of Larmor's Formula

If the velocity is nonrelativistic $\left|\mathbf{v}/c\right| << 1$

$$\bar{\mathbf{C}}(\mathbf{k},\omega) = \int_{-\infty}^{\infty} dt\, q\, \mathbf{v}(t)\, e^{i\omega(t - \hat{\mathbf{n}}\cdot\mathbf{r}(t)/c)} = \int_{-\infty}^{\infty} dt\, q\, \mathbf{v}(t)\, e^{i\omega t}$$

The energy radiated per unit frequency per unit solid angle

$$\frac{dU}{d\omega\, d\Omega} = \frac{Z_0}{32\pi^3}\left|\mathbf{k}\times\bar{\mathbf{C}}(\mathbf{k},\omega)\right|^2 \qquad \mathbf{k} = \frac{\omega}{c}\hat{\mathbf{n}}$$

$$\mathbf{k}\times\bar{\mathbf{C}}(\mathbf{k},\omega) = \frac{q}{c}\hat{\mathbf{n}}\times\int_{-\infty}^{\infty} dt\; \mathbf{v}(t)\,\omega\, e^{i\omega t} = -i\frac{q}{c}\hat{\mathbf{n}}\times\int_{-\infty}^{\infty} dt\; \mathbf{v}(t)\frac{\partial e^{i\omega t}}{\partial t}$$

Derivation of Larmor's Formula

$$\mathbf{k}\times\overline{\mathbf{C}}(\mathbf{k},\omega) = -i\frac{q}{c}\hat{\mathbf{n}}\times\int_{-\infty}^{\infty} dt\ \mathbf{v}(t)\frac{\partial e^{i\omega t}}{\partial t}$$

Integrating by parts $\mathbf{k}\times\overline{\mathbf{C}}(\mathbf{k},\omega) = -i\frac{q}{c}\hat{\mathbf{n}}\times\int_{-\infty}^{\infty} dt\, e^{i\omega t}\frac{\partial\,\mathbf{v}(t)}{\partial t}$

Total energy radiated by the nonrelativistic charge

$$U = \int d\Omega\int_{-\infty}^{\infty} d\omega\,\frac{dU}{d\omega\, d\Omega} = \int d\Omega\int_{-\infty}^{\infty} d\omega\left(\frac{Z_0}{32\pi^3}\left|\mathbf{k}\times\overline{\mathbf{C}}(\mathbf{k},\omega)\right|^2\right)$$

Derivation of Larmor's Formula

$$U = \int d\Omega \int_{-\infty}^{\infty} d\omega \left(\frac{1}{2\pi} \frac{Z_0}{16\pi^2} \frac{q^2}{c^2} \left| \hat{\mathbf{n}} \times \int_{-\infty}^{\infty} dt\, e^{i\omega t} \frac{\partial \mathbf{v}(t)}{\partial t} \right|^2 \right)$$

Parseval's Theorem states that $\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left|\overline{g}(\omega)\right|^2 = \int_{-\infty}^{\infty} dt\, g^2(t)$

$$U = \int d\Omega \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(\frac{Z_0}{16\pi^2} \frac{q^2}{c^2} \left| \int_{-\infty}^{\infty} dt\, e^{i\omega t} \hat{\mathbf{n}} \times \frac{\partial \mathbf{v}(t)}{\partial t} \right|^2 \right)$$

but $\int_{-\infty}^{\infty} dt\, e^{i\omega t} \hat{\mathbf{n}} \times \frac{\partial \mathbf{v}(t)}{\partial t} = \hat{\mathbf{n}} \times \overline{\mathbf{a}}(\omega)$ where $\overline{\mathbf{a}}(\omega)$ is F-T of the acceleration $\mathbf{a}$(t)

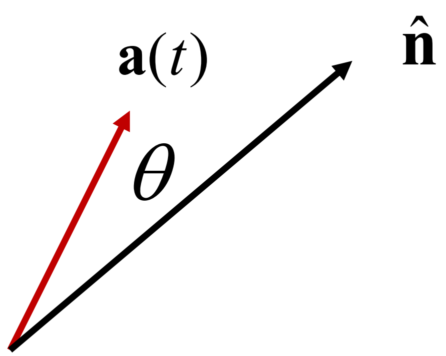
Derivation of Larmor's Formula

Using Parseval's theorem

$$U = \int_{-\infty}^{\infty} dt \left[\int d\Omega \left(\frac{Z_0}{16\pi^2} \frac{q^2}{c^2} \left| \hat{\mathbf{n}} \times \mathbf{a}(t) \right|^2 \right) \right] = \int_{-\infty}^{\infty} dt\, P_T(t)$$

$$P_T(t) = \int d\Omega \left(\frac{Z_0}{16\pi^2} \frac{q^2}{c^2} \left| \hat{\mathbf{n}} \times \mathbf{a}(t) \right|^2 \right) \rightarrow \text{ total power radiated}$$

by the accelerating charge



$$d\Omega = \sin\theta\, d\theta\, d\varphi$$

$$P_T(t) = \int \sin\theta\, d\theta\, d\varphi\, \frac{Z_0}{16\pi^2} \frac{q^2}{c^2} \sin^2\theta \left| \mathbf{a}(t) \right|^2$$

Larmor's Formula

The total instantaneous power radiated

$$P_T(t) = \frac{Z_0}{6\pi}\frac{q^2}{c^2}\left|\mathbf{a}(t)\right|^2$$

$$P_T(t) \sim q^2$$

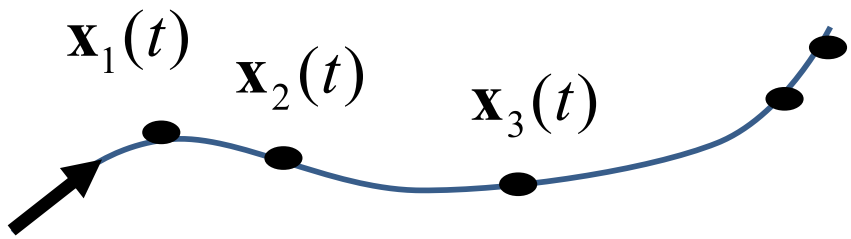
The radiation is polarized in the plane defined by $\hat{\mathbf{n}}$ and $\mathbf{a}(t)$

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{1}{c\,\varepsilon_0} = 377\ \Omega \text{ impedance of vacuum}$$

Radiation from Multiple Charges

Consider a beam of individual charges all having the <u>same trajectory</u>

t_j is the entrance time of the j th charge



$$\mathbf{x}_j(t) = \mathbf{x}(t - t_j)$$

$$\mathbf{v}_j(t) = \mathbf{v}(t - t_j)$$

We want to obtain the energy radiated per unit frequency per unit solid angle

$$\frac{dU}{d\omega\, d\Omega} = \frac{Z_0}{32\pi^3}\left|\mathbf{k}\times\bar{\mathbf{C}}(\mathbf{k},\omega)\right|^2$$

$\bar{\mathbf{C}}(\mathbf{k},\omega)$ is the $F-T$ in space and time of the current density

$$\bar{\mathbf{C}}(\mathbf{k},\omega) = \sum_{j=1}^{N}\int_{-\infty}^{\infty} dt\, q\, \mathbf{v}_j(t)\, e^{i\omega(t - \hat{\mathbf{n}}\cdot\mathbf{r}_j(t)/c)} \qquad N \text{ charges}$$

Radiation from Multiple Charges

Since all the charges have the same trajectories

$$\bar{\mathbf{C}}(\mathbf{k},\omega) = \sum_{j=1}^{N} \int_{-\infty}^{\infty} dt\, q\, \mathbf{v}(t-t_j)\, e^{i\omega(t - \hat{\mathbf{n}}\cdot\mathbf{r}(t-t_j)/c)}$$

letting $\tau = t - t_j$

$$\bar{\mathbf{C}}(\mathbf{k},\omega) = \sum_{j=1}^{N} e^{i\omega t_j} q \underbrace{\int_{-\infty}^{\infty} d\tau\, \mathbf{v}(\tau)\, e^{i\omega(\tau - \hat{\mathbf{n}}\cdot\mathbf{r}(\tau)/c)}} = \sum_{j=1}^{N} q\, e^{i\omega t_j} \mathbf{F}(\omega)$$

this term is indepentent of t_j

Radiation from Multiple Charges

Consider the case were the charges flow in <u>continuously</u> not discretely

$$\bar{\mathbf{C}}(\mathbf{k},\omega) = \sum_{j=1}^{N} \Delta t_j \, q \frac{e^{i\omega t_j}}{\Delta t_j} \mathbf{F}(\omega) \rightarrow \int dt \; I(t) e^{i\omega t} \, \mathbf{F}(\omega) = \bar{I}(\omega) \mathbf{F}(\omega)$$

where $I(t)$ is the beam current (time rate of change of charge)

If $I(t)$ varies slowly in time, $\bar{I}(\omega)$ will have only low frequency components

$$\bar{\mathbf{C}}(\mathbf{k},\omega) = \bar{I}(\omega)\mathbf{F}(\omega) \simeq 0 \;\rightarrow\; \text{little or no radiation}$$

Things are very different if the charges are randomly distributed in entrance times

Radiation from Multiple Charges

Consider the case were the charges have random entrance times

$$\bar{\mathbf{C}}(\mathbf{k},\omega) = \sum_{j=1}^{N} q\, e^{i\omega t_j}\, \mathbf{F}(\omega)$$

Energy radiated per unit frequency per unit solid angle

$$\frac{dU}{d\omega\, d\Omega} = \frac{Z_0}{32\pi^3}\left|\mathbf{k}\times\bar{\mathbf{C}}(\mathbf{k},\omega)\right|^2$$

$$\frac{dU}{d\omega\, d\Omega} = \frac{Z_0}{32\pi^3}\frac{\omega^2}{c^2} q^2 \left|\hat{\mathbf{n}}\times\mathbf{F}(\omega)\right|^2 \sum_{j=1}^{N}\sum_{k=1}^{N} e^{i\omega(t_j - t_k)}$$

$$\frac{dU}{d\omega\, d\Omega} = \frac{Z_0}{32\pi^3}\frac{\omega^2}{c^2} q^2 \left|\hat{\mathbf{n}}\times\mathbf{F}(\omega)\right|^2 \left[\, \underset{j=k}{N} + N(N-1)\underset{j\neq k}{\left\langle e^{i\omega(t_j - t_k)}\right\rangle}\right]$$

Radiation from Multiple Charges

For random entrance times $\left\langle e^{i\omega(t_j - t_k)} \right\rangle = 0 \qquad j \neq k$

Energy radiated per unit frequency per unit solid angle

$$\frac{dU}{d\omega\, d\Omega} = \frac{Z_0}{32\pi^3} \frac{\omega^2}{c^2} q^2 \left| \hat{\mathbf{n}} \times \mathbf{F}(\omega) \right|^2 N \sim N$$