Electrodynamics

Topics to be covered

- Radiation
- Scattering
- Antennas

Outline

Radiation basics

- Fields due to a localized source
- Fields far from the source
- Radiated power distribution
- Electric and magnetic dipole radiation

Scattered radiation

- coherent vs incoherent

Antennas

- Center-fed linear antenna
- Directivity
- Radiation resistance

Maxwell's Equations for Vector and Scalar Potentials

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi$$

In the Lorentz gauge $\left(\nabla \cdot \mathbf{A} = -\mu_0 \varepsilon_0 \frac{\partial \Phi}{\partial t} \right)$ the vector and scalar potentials obey wave equations

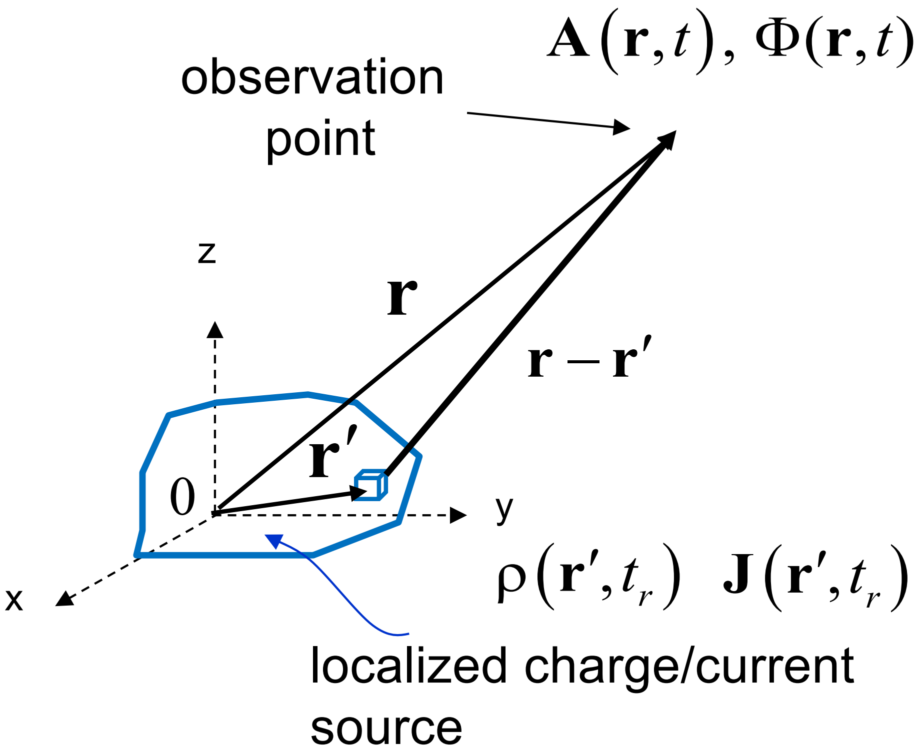
$$\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \qquad \sqrt{\varepsilon_0 \mu_0} = 1/c$$

$$\nabla^2 \Phi - \mu_0 \varepsilon_0 \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$$

where $\mathbf{J}$ and ρ are the current and charge densities

The solutions to the wave equations (in the absence of boundaries) are

Solution to Wave Equations



$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_{Vol} d\tau' \left.\frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}\right|_{t_r = t - |\mathbf{r}-\mathbf{r}'|/c}$$

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int_{Vol} d\tau' \left.\frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}\right|_{t_r = t - |\mathbf{r}-\mathbf{r}'|/c}$$

where $t_r = t - |\mathbf{r}-\mathbf{r}'|/c$ is the retarded time (earlier time)

$$d\tau' = dx'\,dy'\,dz'$$

Sinusoidal Dependence on Time

If we assume harmonic (sinusoid dependence on time)

for all the fields and sources $e^{-i\omega t_r} = e^{-i\omega t} e^{ik|\mathbf{r}-\mathbf{r}'|}$

$$\mathbf{A}(\mathbf{r},t) = \mathrm{Re}\left[\hat{\mathbf{A}}(\mathbf{r})\, e^{-i\omega t}\right] \qquad \Phi(\mathbf{r},t) = \mathrm{Re}\left[\hat{\Phi}(\mathbf{r})\, e^{-i\omega t}\right]$$

$$\mathbf{J}(\mathbf{r},t) = \mathrm{Re}\left[\hat{\mathbf{J}}(\mathbf{r})\, e^{-i\omega t}\right] \qquad \rho(\mathbf{r},t) = \mathrm{Re}\left[\hat{\rho}(\mathbf{r})\, e^{-i\omega t}\right]$$

In phasor notation

$$\hat{\mathbf{A}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{Vol} d\tau' \frac{\hat{\mathbf{J}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, e^{ik|\mathbf{r}-\mathbf{r}'|} \qquad \hat{\Phi}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{Vol} d\tau' \frac{\hat{\rho}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, e^{ik|\mathbf{r}-\mathbf{r}'|}$$

where $k = \omega / c = \dfrac{2\pi}{\lambda}$ is the wavenumber

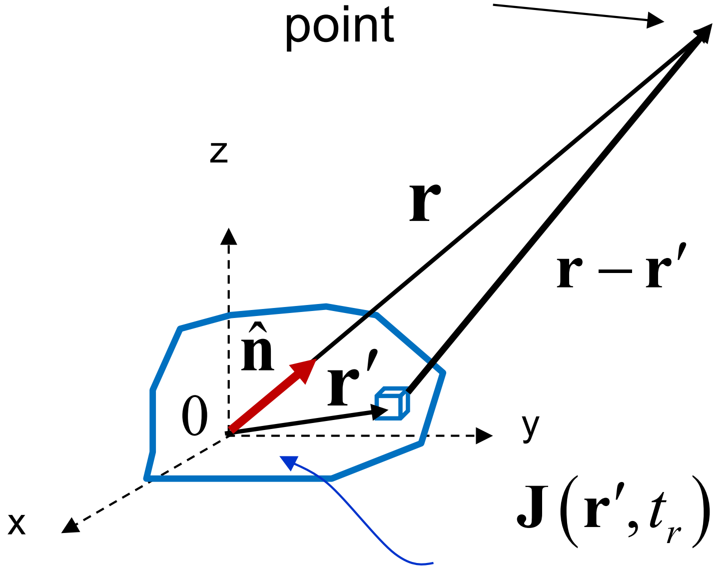
Far Field Approximation

$$\left|\mathbf{r}-\mathbf{r}'\right| = \sqrt{r^2 + r'^2 - 2\mathbf{r}\cdot\mathbf{r}'} \simeq r - \mathbf{r}\cdot\mathbf{r}'/r$$

Assume that the source is localized and the observation point is far away $(r >> r')$

observation point

$$\mathbf{A}\left(\mathbf{r},t\right),\ \Phi(\mathbf{r},t)$$



$$\mathbf{J}\left(\mathbf{r}',t_r\right)\quad \rho\left(\mathbf{r}',t_r\right)$$

localized charge/current source

$$\hat{\Phi}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{Vol} d\tau' \frac{\hat{\rho}(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|} e^{ik\left|\mathbf{r}-\mathbf{r}'\right|}$$

$$\text{far zone } k\,r = 2\pi\frac{r}{\lambda} >> 1$$

$$\left|\mathbf{r}-\mathbf{r}'\right| \simeq r - \hat{\mathbf{n}}\cdot\mathbf{r}' \quad \text{for } r >> r'$$

$$\hat{\mathbf{n}} = \frac{\mathbf{r}}{\left|\mathbf{r}\right|} \text{ unit vector}$$

Far Field Potentials

$$\text{Using } \left|\mathbf{r}-\mathbf{r}'\right| \simeq r - \hat{\mathbf{n}}\cdot\mathbf{r}'$$

$$\hat{\mathbf{A}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{Vol} d\tau' \frac{\hat{\mathbf{J}}(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|} e^{ik\left|\mathbf{r}-\mathbf{r}'\right|} \simeq \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

$$\hat{\Phi}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{Vol} d\tau' \frac{\hat{\rho}(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|} e^{ik\left|\mathbf{r}-\mathbf{r}'\right|} \simeq \frac{e^{ikr}}{4\pi\varepsilon_0 r} \int_{Vol} d\tau' \, \hat{\rho}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

$$\text{where } \mathbf{k} = k\,\hat{\mathbf{n}}$$

$$\text{for } r >> r' \;,\; \frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|} \simeq \frac{1}{r} \;\text{ and }\; k\left|\mathbf{r}-\mathbf{r}'\right| \simeq k\,r - k\,\hat{\mathbf{n}}\cdot\mathbf{r}' \;\text{ in exponent}$$

$$\hat{\mathbf{n}} = \mathbf{r} / r$$

$$\left|\mathbf{r}-\mathbf{r}'\right| = \sqrt{\left(\mathbf{r}-\mathbf{r}'\right)\cdot\left(\mathbf{r}-\mathbf{r}'\right)} = \sqrt{r^2 + r'^2 - 2\mathbf{r}\cdot\mathbf{r}'} \simeq r - \mathbf{r}\cdot\mathbf{r}' / r$$

Really small

Somewhat small

Calculating Fields from Potentials

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

$$\nabla = \hat{\mathbf{n}} \frac{\partial}{\partial r} \rightarrow i\,\mathbf{k} \qquad \hat{\mathbf{B}}(\mathbf{r}) = \nabla \times \hat{\mathbf{A}}(\mathbf{r}) \qquad \hat{\mathbf{B}}(\mathbf{r}) = i\,\mathbf{k} \times \hat{\mathbf{A}}(\mathbf{r})$$

$$\hat{\mathbf{B}}(\mathbf{r}) = i \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \, \mathbf{k} \times \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

$\hat{\mathbf{B}}(\mathbf{r})$ is transvere to $\hat{\mathbf{J}}$, $\mathbf{k} = k\hat{\mathbf{n}}$ and $\hat{\mathbf{E}}(\mathbf{r})$

Ampere's law

$$\nabla \times \mathbf{H} = \cancel{\mathbf{J}} + \partial \mathbf{D} / \partial t \qquad \hat{\mathbf{E}}(\mathbf{r}) = -\frac{1}{\varepsilon_0 \omega} \mathbf{k} \times \hat{\mathbf{H}}(\mathbf{r}) \qquad \mu_0 \hat{\mathbf{H}}(\mathbf{r}) = \hat{\mathbf{B}}(\mathbf{r})$$

$$\hat{\mathbf{E}}(\mathbf{r}) = -\frac{i}{\varepsilon_0 \mu_0 \omega} \mathbf{k} \times \left(\mathbf{k} \times \hat{\mathbf{A}}(\mathbf{r}) \right) = -i \frac{e^{ikr}}{4\pi r} \frac{1}{\varepsilon_0 \omega} \int_{Vol} d\tau' \, \mathbf{k} \times \left(\mathbf{k} \times \hat{\mathbf{J}}(\mathbf{r}') \right) e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

Simplify expressions for the fields

In the far zone $k\,r = 2\pi\,r\,/\,\lambda >> 1$

Define the Fourier transform of the current density

$$\hat{\mathbf{C}}(\mathbf{k}) = \int_{Vol} d\tau'\,\hat{\mathbf{J}}(\mathbf{r}')\,e^{-i\,\mathbf{k}\cdot\mathbf{r}'} \qquad d\tau' = dx'\,dy'\,dz'$$

The fields in terms of the $F-T$ of the current density are

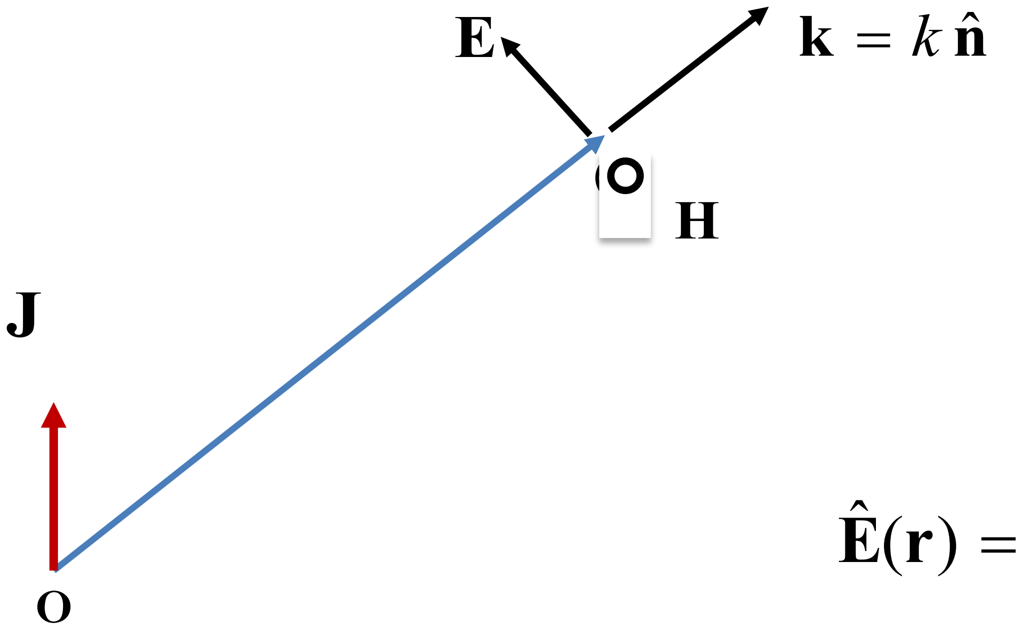
$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi\,r} e^{i\,k\,r}\,\hat{\mathbf{C}}(\mathbf{k}) \qquad \hat{\mathbf{B}}(\mathbf{r}) = i\,\frac{\mu_0}{4\pi\,r} e^{i\,k\,r}\,\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})$$

$$\hat{\mathbf{E}}(\mathbf{r}) = -\,i\,\frac{e^{i\,k\,r}}{4\,\pi\,r}\frac{1}{\varepsilon_0\,\omega}\mathbf{k}\times\left(\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})\right)$$

Radiation

In the far field zone

Direction of energy flow,
Poynting’s vector



$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr}\,\hat{\mathbf{C}}(\mathbf{k})$$

$$\hat{\mathbf{B}}(\mathbf{r}) = i\,\frac{\mu_0}{4\pi r} e^{ikr}\,\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})$$

$$\hat{\mathbf{E}}(\mathbf{r}) = -\,i\,\frac{e^{ikr}}{4\pi r}\frac{1}{\varepsilon_0\,\omega}\mathbf{k}\times\Big(\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})\Big)$$

$$\hat{\mathbf{C}}(\mathbf{k}) = \int_{Vol} d\tau'\,\hat{\mathbf{J}}(\mathbf{r}')\,e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

$\hat{\mathbf{B}}(\mathbf{r})$ is transverse to $\hat{\mathbf{J}}$,

$\mathbf{k} = k\,\hat{\mathbf{n}}$ and $\hat{\mathbf{E}}$

Radiation

Radiation intensity [W/m^2]
(average over time of Poynting's vector)

$$\langle \mathbf{S} \rangle = \frac{1}{2} \mathrm{Re}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}^* \right)$$

Radiation intensity is $\left| \langle \mathbf{S} \rangle \right|$

The average is over a wave period

$$\hat{\mathbf{E}}(\mathbf{r}) = -\, i \frac{e^{ikr}}{4\pi r} \frac{1}{\varepsilon_0 \omega} \mathbf{k} \times \left(\mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right)$$

$$\hat{\mathbf{H}}(\mathbf{r}) = i \frac{1}{4\pi r} e^{ikr} \, \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k})$$

$$\langle \mathbf{S} \rangle = \frac{1}{2} \mathrm{Re}\left(-\, i \frac{e^{ikr}}{4\pi r} \frac{1}{\varepsilon_0 \omega} \left(\mathbf{k} \times \left(\mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right) \right) \times \frac{-i}{4\pi r} e^{-ikr} \left(\mathbf{k} \times \hat{\mathbf{C}}^*(\mathbf{k}) \right) \right)$$

$$\langle \mathbf{S} \rangle = -\, \frac{1}{2} \left(\frac{1}{4\pi r} \right)^2 \frac{1}{\varepsilon_0 \omega} \mathrm{Re}\left(\left(\mathbf{k} \times \left(\mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right) \right) \times \mathbf{k} \times \hat{\mathbf{C}}^*(\mathbf{k}) \right)$$

Radiated Power Flux

using the vector identity $(\mathbf{b}\times\mathbf{c})\times\mathbf{a} = \mathbf{c}(\mathbf{a}\cdot\mathbf{b}) - \mathbf{b}(\mathbf{a}\cdot\mathbf{c})$

$$\left(\mathbf{k}\times\left(\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})\right)\right)\times\left(\mathbf{k}\times\hat{\mathbf{C}}^*(\mathbf{k})\right) = -\left|\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})\right|^2\mathbf{k}$$

$$\langle\mathbf{S}\rangle = \frac{\mathbf{k}}{32\pi^2\varepsilon_0\,\omega}\frac{\left|\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})\right|^2}{r^2}$$

The power flux falls off like $1/r^2$ and is in the direction of $\mathbf{k}=k\,\hat{\mathbf{n}}$

$$\boxed{\langle\mathbf{S}\rangle = \frac{\hat{\mathbf{n}}Z_0}{32\pi^2}\frac{\left|\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})\right|^2}{r^2}}$$

$Z_0 = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = \dfrac{1}{c\,\varepsilon_0} = 377\,\Omega$ impedance of vacuum $\qquad c = \dfrac{1}{\sqrt{\varepsilon_0\,\mu_0}}$

Total Radiated Power

$$\boxed{P_T = \oint_S \langle \mathbf{S} \rangle \cdot d\mathbf{a} = \oint_S \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle \, da}$$

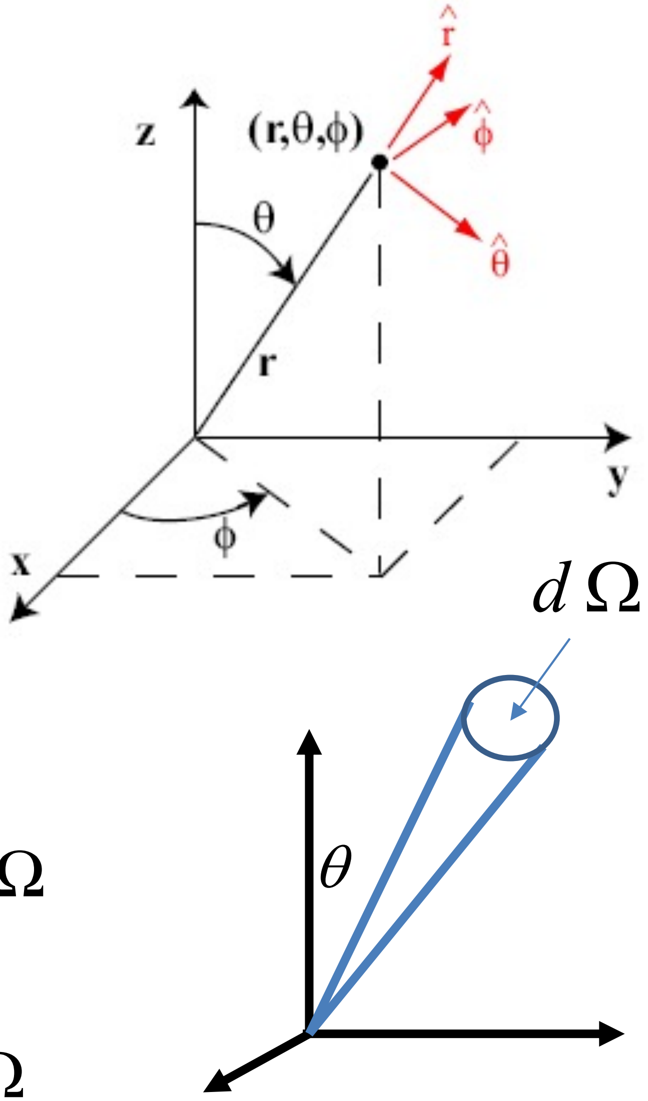
In spherical coordinates da = $r^2 \, d\Omega$

where $d\Omega = \sin\theta \, d\theta \, d\varphi$ is the solid angle

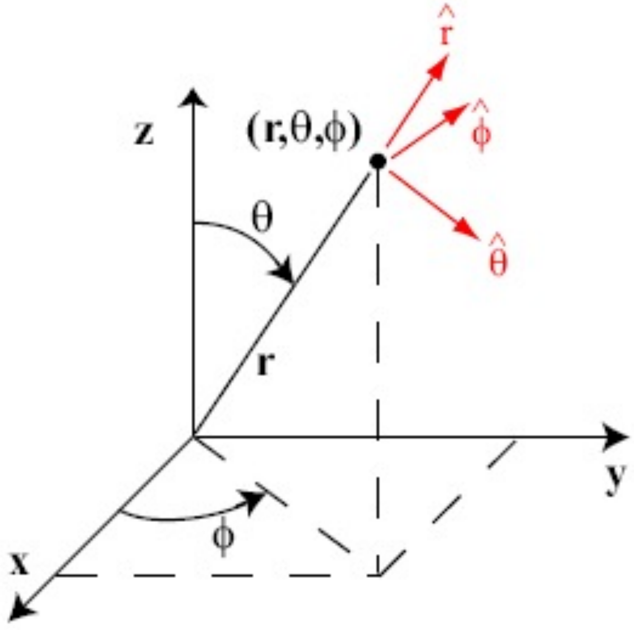
$$P_T = \oint_S \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle r^2 \, d\Omega = \oint_S \frac{dP_T}{d\Omega} d\Omega$$

Power radiated into the solid angle $d\Omega$

$$\boxed{\frac{dP_T}{d\Omega} = \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle r^2 = \frac{Z_0}{32\pi^2} \left| \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right|^2}$$

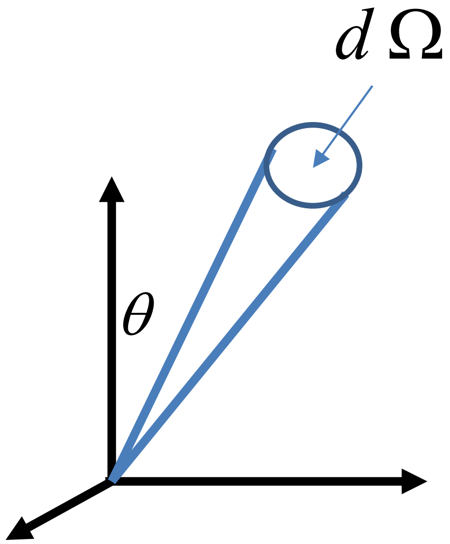


Summary



Define the Fourier transform of the current de

$$\hat{\mathbf{C}}(\mathbf{k}) = \int_{Vol} d\tau'\, \hat{\mathbf{J}}(\mathbf{r}')\, e^{-i\,\mathbf{k}\cdot\mathbf{r}'} \qquad d\tau' = dx'\,dy'\,dz'$$

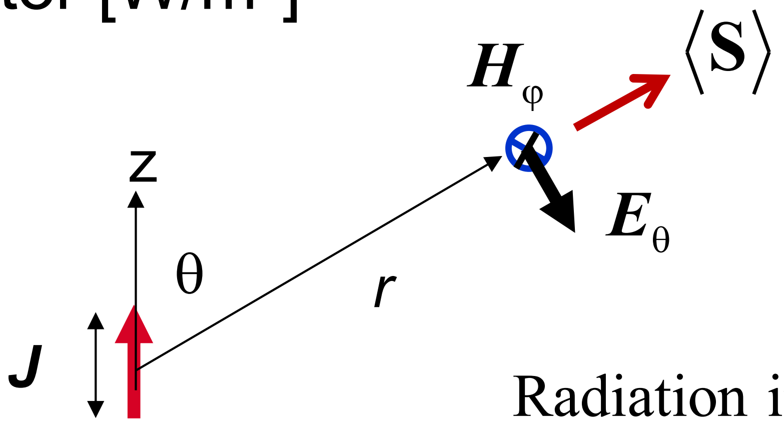


Power radiated into the solid angle $d\Omega$

$$\boxed{\frac{dP_T}{d\Omega} = \hat{\mathbf{n}}\cdot\langle\mathbf{S}\rangle\,\mathrm{r}^2 = \frac{Z_0}{32\,\pi^2}\left|\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})\right|^2}$$

Electric Dipole Radiation (Source Small)

Poynting vector [W/m^2]



$$\langle \mathbf{S} \rangle = \frac{1}{2} \mathrm{Re}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}^* \right)$$

Radiation intensity is $\left| \langle \mathbf{S} \rangle \right|$

The dipole moment is $\hat{\mathbf{p}} = \int_{Vol} \mathbf{r}' \, \hat{\rho}(\mathbf{r}') \, d\tau'$

The charge density is $\rho(\mathbf{r}, t) = \mathrm{Re}\left[\hat{\rho}(\mathbf{r}) \, e^{-i\omega t} \right]$

Conservation of charge, $\partial \rho / \partial t + \nabla \cdot \mathbf{J} = 0$, gives $\nabla \cdot \hat{\mathbf{J}}(\mathbf{r}) = i\,\omega\,\hat{\rho}(\mathbf{r})$

Electric Dipole Radiation

The dipole moment is

$$\hat{\mathbf{p}} = \int_{Vol} \mathbf{r}' \hat{\rho}(\mathbf{r}') d\tau' = -\frac{i}{\omega} \int_{Vol} \mathbf{r}' \nabla' \cdot \hat{\mathbf{J}}(\mathbf{r}') d\tau'$$

$$\left(-\frac{i}{\omega} \int_{Vol} \mathbf{r}' \nabla' \cdot \hat{\mathbf{J}}(\mathbf{r}') d\tau' \right)_j = -\frac{i}{\omega} \int_{Vol} r'_j \sum_{j'} \frac{\partial}{\partial r'_{j'}} J_{j'}(\mathbf{r}') d\tau' = \frac{i}{\omega} \int_{Vol} \sum_{j'} J_{j'}(\mathbf{r}') \frac{\partial}{\partial r'_{j'}} r'_j \, d\tau'$$

$$\frac{\partial}{\partial r'_{j'}} r'_j = \begin{cases} 1, j = j' \\ 0, j \neq j' \end{cases} \qquad \frac{i}{\omega} \int_{Vol} J_j(\mathbf{r}') d\tau' = \left(\frac{i}{\omega} \int_{Vol} \hat{\mathbf{J}}(\mathbf{r}') d\tau' \right)_j$$

$$\hat{\mathbf{p}} = \frac{i}{\omega} \int_{Vol} \hat{\mathbf{J}}(\mathbf{r}') d\tau'$$

Electric Dipole Radiation

Power radiated into the solid angle $d\Omega$ $\quad (da = r^2 d\Omega)$

$$\frac{dP_T}{d\Omega} = \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle \mathrm{r}^2 = \frac{Z_0}{32\pi^2} \left| \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right|^2 \qquad P_T = \oint_S \frac{dP_T}{d\Omega} d\Omega$$

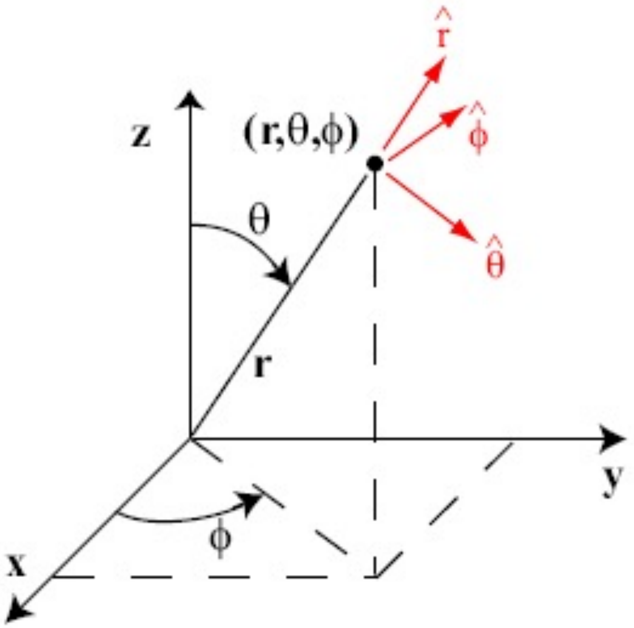
where $\hat{\mathbf{C}}(\mathbf{k}) = \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'}$ is F-T of the current density

If the wavelength is large compared to the dimensions of the dipole

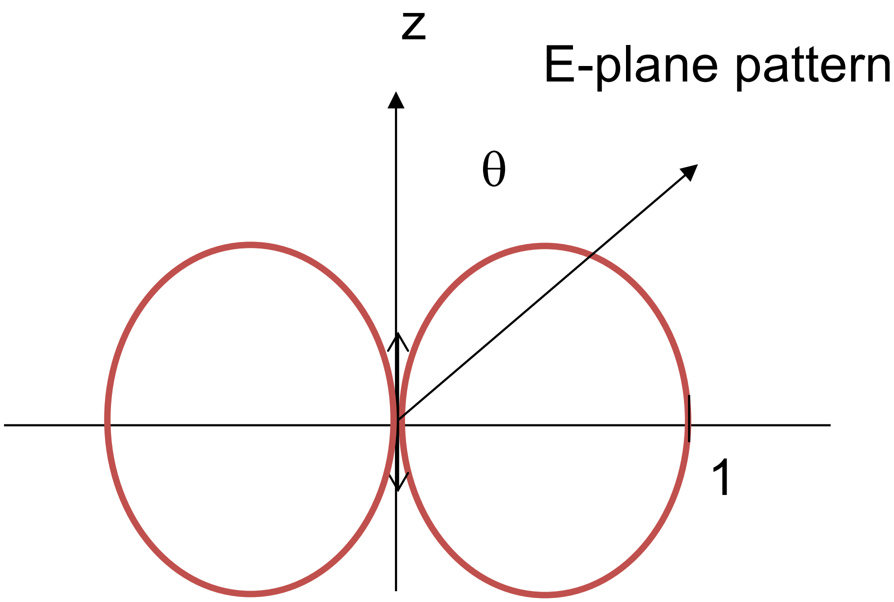
$|\mathbf{k} \cdot \mathbf{r}'| << 1, \quad k = 2\pi / \lambda$ $\qquad \hat{\mathbf{C}}(\mathbf{k}) \simeq \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') \qquad \hat{\mathbf{C}}(\mathbf{k}) \simeq -i\omega \hat{\mathbf{p}}$

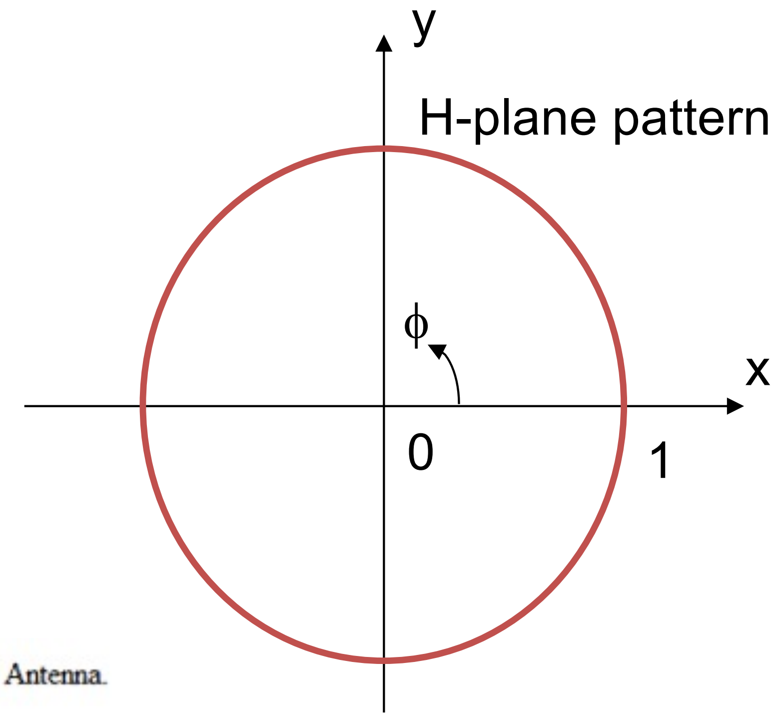
$$\boxed{\frac{dP_T}{d\Omega} = \frac{Z_0}{32\pi^2} \frac{\omega^4}{c^2} |\hat{\mathbf{n}} \times \hat{\mathbf{p}}|^2 = \frac{Z_0}{32\pi^2} \frac{\omega^4}{c^2} |\hat{\mathbf{p}}|^2 \sin^2\theta \ \sim \ \omega^4}$$

$$\frac{dP_T}{d\Omega} = \frac{Z_0}{32\pi^2}\frac{\omega^4}{c^2}\left|\hat{\mathbf{n}}\times\hat{\mathbf{p}}\right|^2 = \frac{Z_0}{32\pi^2}\frac{\omega^4}{c^2}\left|\hat{\mathbf{p}}\right|^2 \sin^2\theta \;\sim\; \omega^4$$

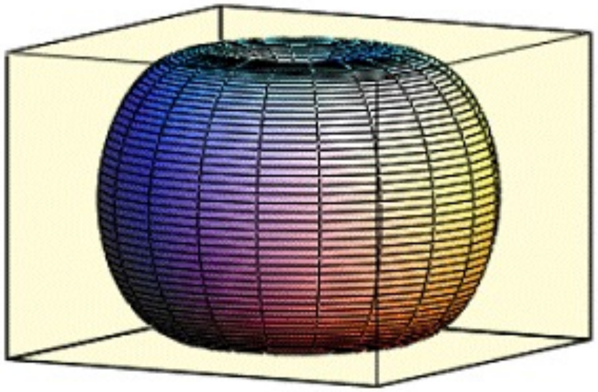


Antenna Pattern of a Hertzian Dipole





Radiation Pattern of the Dipole Antenna.



For a dipole of length
$L = 0.01\ (\lambda_0)$.

Higher Order Moments of the Fields

Include both electric and magnetic dipole contributions

In the far field zone

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

Taylor expand

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') \left(1 - ik\,\hat{\mathbf{n}}\cdot\mathbf{r}' - \frac{k^2}{2}(\mathbf{n}\cdot\mathbf{r}')^2 + \ldots \right)$$

terms fall off rapidly

First term

$$\int_{Vol} \hat{\mathbf{J}}(\mathbf{r}') \, d\tau' = -i\omega\hat{\mathbf{p}}$$

Second term $\int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}')\left(-ik\,\hat{\mathbf{n}}\cdot\mathbf{r}'\right)$

Electric and Magnetic Dipole Radiation

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'} \simeq \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}')(1 - i\,k\,\hat{\mathbf{n}}\cdot\mathbf{r}')$$

$$\boxed{\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \left\{ \overset{\text{electric dipole}}{-i\omega\hat{\mathbf{p}}} \;-\; \overset{\text{magnetic dipole}}{i\,k\,\hat{\mathbf{m}}\times\hat{\mathbf{n}}} \right\}}$$

$$\hat{\mathbf{m}} = \frac{1}{2} \int_{Vol} d\tau' \, \mathbf{r}' \times \hat{\mathbf{J}}(\mathbf{r}')$$

See Eqs. (20.169, 20.170) of text

$$\hat{\mathbf{p}} \approx q\mathbf{d}$$

$$\hat{\mathbf{m}} \approx q\mathbf{d}\times\dot{\mathbf{d}}$$

The F-T of the current density including both the **electric** and **magnetic dipole** contributions is

$$\omega\left|\hat{\mathbf{p}}\right| / \left|k\hat{\mathbf{m}}\right| \sim \left|\dot{\mathbf{d}}\right| / c$$

$$\hat{\mathbf{C}}(\mathbf{k}) = \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'} \simeq -i\omega(\hat{\mathbf{p}} + \hat{\mathbf{m}}\times\hat{\mathbf{n}} / c)$$

Relativistic effect

Electric and Magnetic Dipole Radiation

Power radiated into the solid angle $d\Omega$

$$\frac{dP_T}{d\Omega} = \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle \mathrm{r}^2 = \frac{Z_0}{32\pi^2} \left| \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right|^2$$

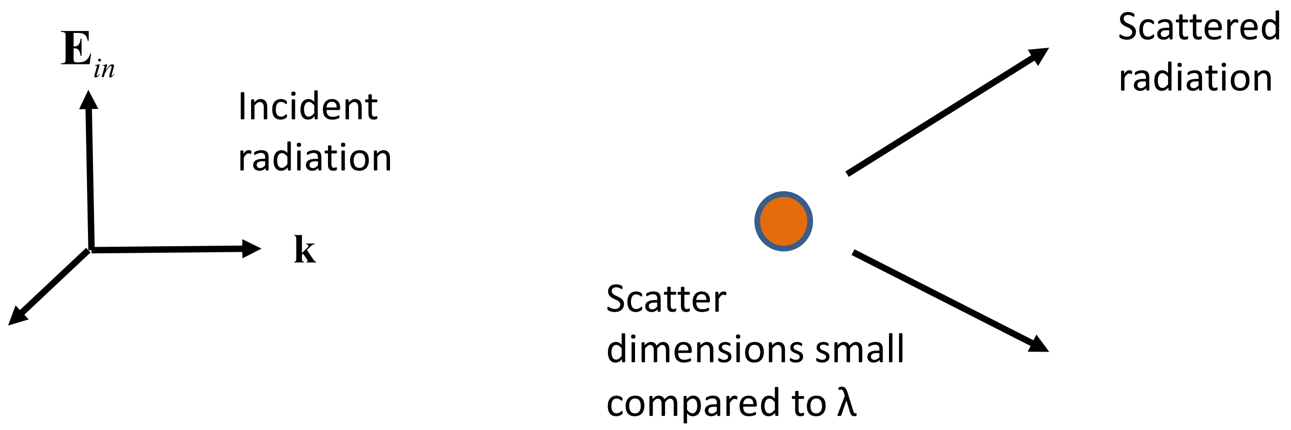
$$\boxed{\frac{dP_T}{d\Omega} = \frac{Z_0}{32\pi^2} \frac{\omega^4}{c^2} \left| \hat{\mathbf{n}} \times (\hat{\mathbf{p}} + \hat{\mathbf{m}} \times \hat{\mathbf{n}} / c) \right|^2}$$

Far field zone

$\hat{\mathbf{p}}$ and $\hat{\mathbf{m}}$ are the electric and magnetic dipole moments

Shorter wavelengths scatter more (blue sky)

Scattering at Long Wavelengths



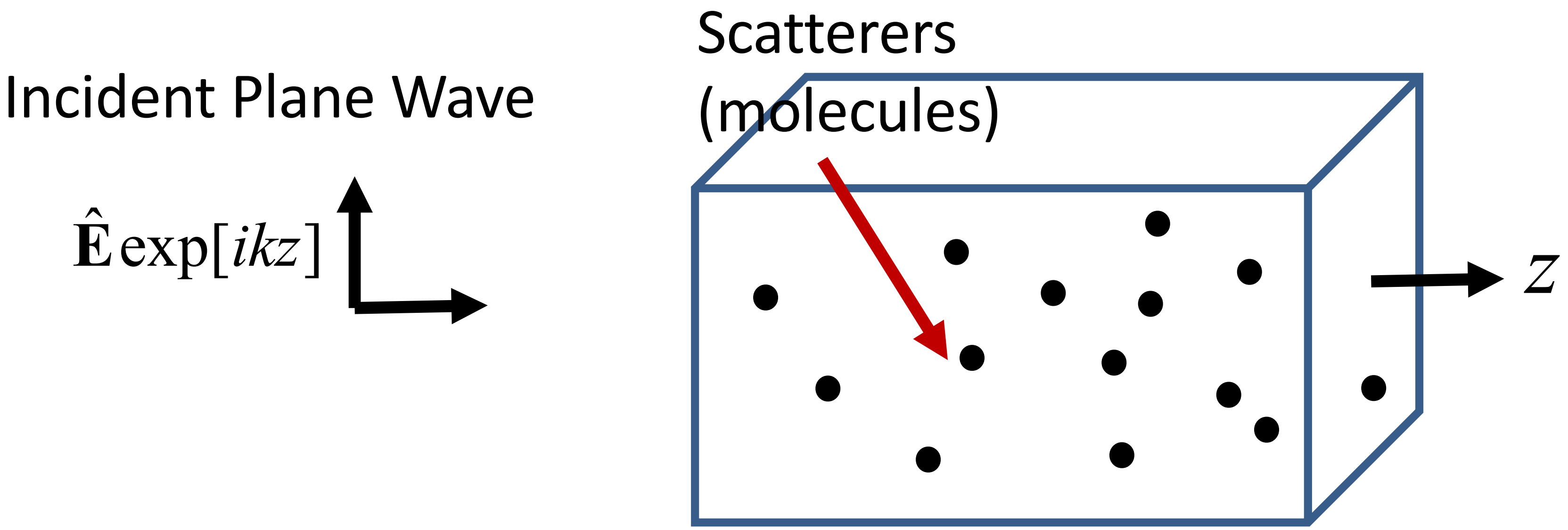
- The incident radiation induces an oscillating electric and magnetic dipole moment in the scatter

- The induced dipole moments radiate (scattered radiation)

- The scattered radiation is a function of the polarization and direction of both incident and scattered radiation

- If the wavelength is large compared to the size of the scatter the induced electric and magnetic dipole moments are sufficient to describe the scattered radiation   (opposite case is called Mie scattering)

Coherent vs Incoherent Scattering



Incident Plane Wave

$\hat{\mathbf{E}}\exp[ikz]$

Scatterers (molecules)

Amplitude due to ensemble of spatially distributed scatterers

$$\hat{\mathbf{C}}(\mathbf{k}) = \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'} \simeq -i\omega \sum_i \hat{\mathbf{p}}_i e^{-i\mathbf{k}\cdot\mathbf{r}_i}$$

Dipole moment proportional to local electric field

$$\hat{\mathbf{p}}_i = \gamma \hat{\mathbf{E}} \exp[ikz_i]$$

Radiated Power

$$\frac{dP_T}{d\Omega} = \frac{Z_0}{32\pi^2}\frac{\omega^4}{c^2}\left|\hat{\mathbf{n}}\times\gamma\hat{\mathbf{E}}\sum_i \exp[ikz_i - i\mathbf{k}\cdot\mathbf{r}_i]\right|^2$$

$$\frac{dP_T}{d\Omega} = \frac{dP_{T1}}{d\Omega} f(\mathbf{k},N)$$

Radiation due to single dipole

Form factor

$$\frac{dP_{T1}}{d\Omega} = \frac{Z_0}{32\pi^2}\frac{\omega^4}{c^2}\left|\hat{\mathbf{n}}\times\gamma\hat{\mathbf{E}}\right|^2$$

$$f = \left|\sum_i \exp[ikz_i - i\mathbf{k}\cdot\mathbf{r}_i]\right|^2$$

Three cases

$$f = \left| \sum_i \exp[ikz_i - i\mathbf{k}\cdot\mathbf{r}_i] \right|^2$$

Dipoles localized to a volume smaller than a wavelength

Dipoles distributed randomly in a volume larger than a wavelength

Dipoles in an ordered array

Cases

$$f = \left| \sum_i \exp[ikz_i - i\mathbf{k} \cdot \mathbf{r}_i] \right|^2$$

Dipoles localized to a volume smaller than a wavelength - coherent

$$f = \left| \sum_i \quad \right|^2 = N^2$$

Dipoles distributed randomly in a volume larger than a wavelength - incoherent

$$f = \left| \sum_{i,j} \exp[ik\left(z_i - z_j\right) - i\mathbf{k} \cdot \left(\mathbf{r}_i - \mathbf{r}_j\right)] \right|$$

Terms with i an j different average to zero. Only j=i survive

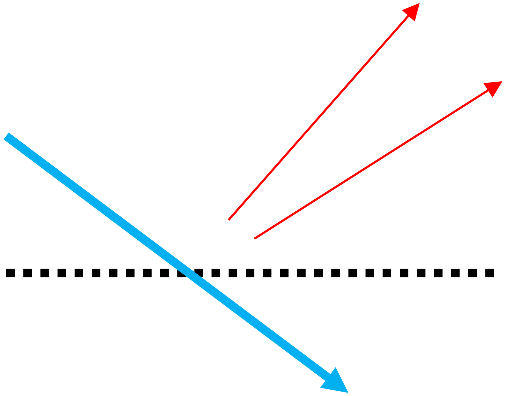
$$f = \left| \sum_i \quad \right| = N$$

Ordered array 1D

$$f = \left| \sum_i \exp[ikz_i - i\mathbf{k}\cdot\mathbf{r}_i] \right|^2$$

$$\mathbf{r}_i = \mathbf{d}i$$

$$f = \left| \sum_i \exp[i\left(kd\cos\theta - \mathbf{k}\cdot\mathbf{d}\right)i] \right|^2$$

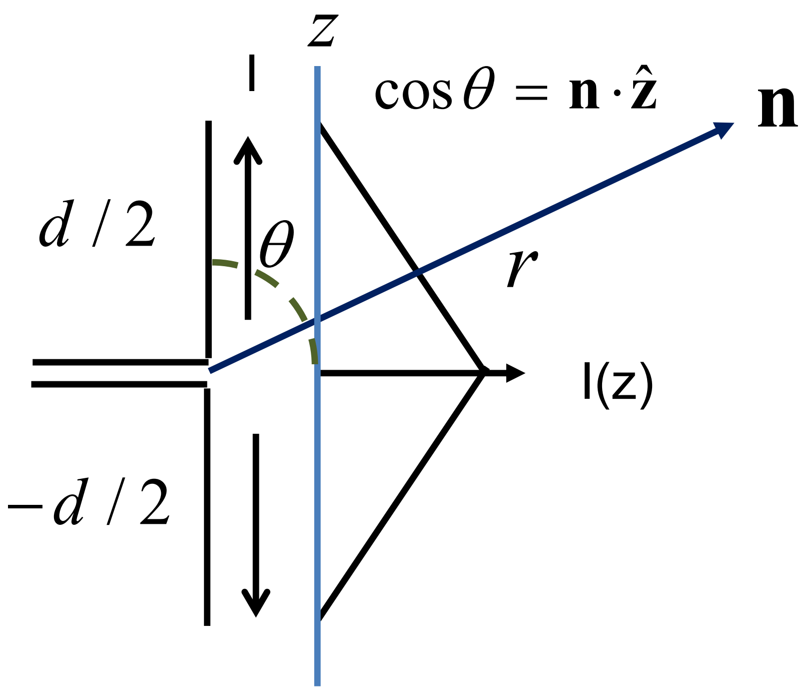


kd>>1

f peaks when

$$\left(kd\cos\theta - \mathbf{k}\cdot\mathbf{d}\right) = 2\pi n$$

Center Fed Linear Antenna



In short antennas current varies ~ linearly with z

Current density $\hat{\mathbf{J}}(\mathbf{r}) = I_0\,\delta(x)\delta(y)(1-2\frac{|z|}{d})\;\hat{\mathbf{z}}$ $\qquad |z| \le \frac{d}{2}$

for $r >> r'$

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'} = \frac{\mu_0}{4\pi r} e^{ikr} I_0\,\hat{\mathbf{z}} \int_{-d/2}^{d/2} dz'(1-2\frac{|z'|}{d}) e^{-ikz'\cos\theta}$$

Center Fed Linear Antenna

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq I_0\,\hat{\mathbf{z}} \int_{-d/2}^{d/2} dz'\,(1-2\frac{|z'|}{d})\,e^{-ikz'\cos\theta}$$

use Euler's equ.

$$= I_0\,\hat{\mathbf{z}} \int_{-d/2}^{d/2} dz' \left(1-2\frac{|z'|}{d}\right)\Big(\overset{\text{even}}{\cos(kz'\cos\theta)} - \overset{\text{odd}}{\cancel{i\sin(kz'\cos\theta)}}\Big)$$

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq 2I_0\,\hat{\mathbf{z}} \int_{0}^{d/2} dz' \left(1-2\frac{z'}{d}\right)\cos(kz'\cos\theta)$$

Center Fed Linear Antenna

To carry out the integration, let $\rho' = k\,z'\cos\theta$ and $\rho_0 = \dfrac{k\,d\cos\theta}{2}$

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq 2\,I_0\,\hat{\mathbf{z}}\frac{d}{\rho_0}\int_0^{\rho_0} d\rho'\left(1-\frac{\rho'}{\rho_0}\right)\cos\rho'$$

using $\int x\cos x\,dx = \cos x + x\sin x$

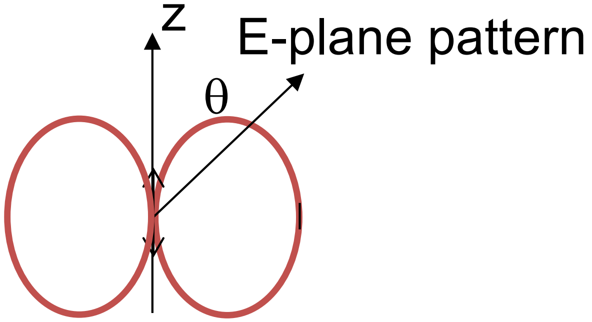
$$\hat{\mathbf{A}}(\mathbf{r}) \simeq I_0\,\hat{\mathbf{z}}\frac{d}{\rho_0^2}(1-\cos\rho_0) \quad \text{where } \rho_0 = \frac{k\,d\cos\theta}{2} = \frac{\pi\,d\cos\theta}{\lambda}$$

Antenna in the Dipoles Limit

In the dipole limit $\lambda >> d \quad (\rho_0 << 1)$

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} I_0 \frac{d}{2} \hat{\mathbf{z}} \qquad \hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \hat{\mathbf{C}}(\mathbf{k})$$

hence, $\mathbf{C}(\mathbf{k}) = I_0 \frac{d}{2} \hat{\mathbf{z}}$



Power radiated into the solid angle $d\Omega$

$$\frac{dP_T}{d\Omega} = \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle \mathrm{r}^2 = \frac{Z_0}{32\pi^2} \left| \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right|^2 = \frac{Z_0}{32\pi^2} k^2 \frac{I_0^2 d^2}{4} \sin^2 \theta$$

Total Power Radiated and Radiation Resistance

The total power radiated is $P_T = \oint\limits_S \dfrac{dP_T}{d\Omega} d\Omega$

where $d\Omega = \sin\theta \, d\theta \, d\varphi$ is the solid angle

$$\frac{dP_T}{d\Omega} = \frac{Z_0}{32\pi^2} k^2 \frac{I_0^2 d^2}{4} \sin^2\theta \qquad \int_0^{2\pi} d\varphi \int_0^{\pi} \sin^3\theta \, d\theta = 2\pi \frac{4}{3}$$

$$\boxed{P_T = \frac{Z_0}{48\pi} k^2 d^2 I_0^2 = \frac{1}{2} Z_{rad} I_0^2}$$

where the radiation resistance is $Z_{rad} = \dfrac{Z_0}{24\pi} k^2 d^2 \ [\Omega]$