Lecture 15

Periodic Structures

Filters
Gratings
Slow Wave Structures
- particle accelerators
- Cherenkov microwave generators

Metamaterials

Floquet Theory

Floquet Theory

$$E(z,t) = \mathrm{Re}\left\{\hat{E}(z)e^{-i\omega t}\right\}$$

Time harmonic, spatially dependent field

$$\frac{\partial^2}{\partial z^2}\hat{E}(z) + \frac{\omega^2}{c^2}\varepsilon_{rel}(z)\hat{E}(z) = 0$$

Inhomogeneous relative dielectric

$$\varepsilon_{rel}(z) = \varepsilon_{rel}(z+L)$$

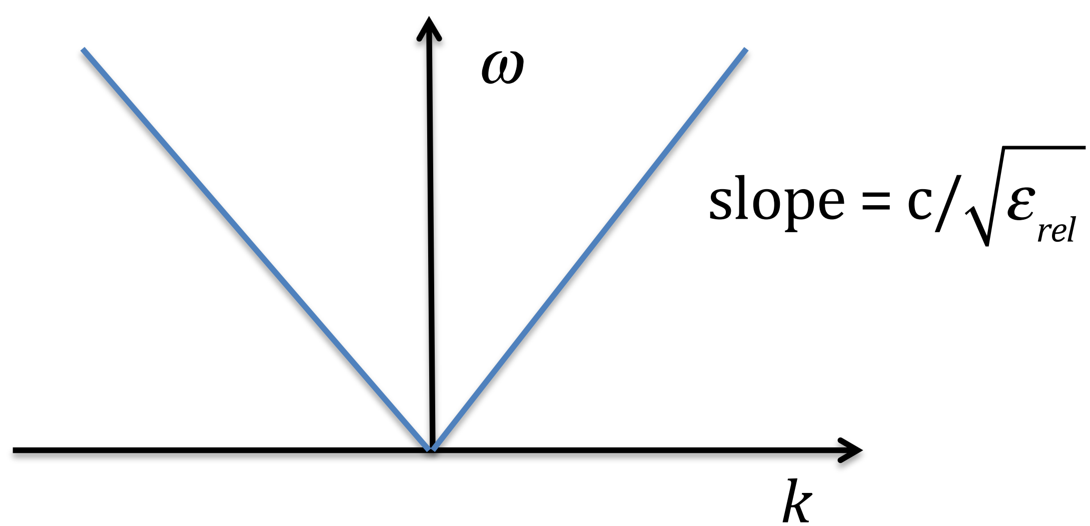
Dielectric is spatially periodic

Special case - homogeneous

$$\partial\varepsilon_{rel} / \partial z = 0$$

$$\hat{E}(z) = \hat{E}_0 \exp(ikz)$$

$$\omega(k) = \pm kc / \sqrt{\varepsilon_{rel}}$$



Spatially Varying Case

$$E(z,t)=\mathrm{Re}\left\{\hat{E}(z)e^{-i\omega t}\right\}$$

$$\frac{\partial^2}{\partial z^2}\hat{E}(z)+\frac{\omega^2}{c^2}\varepsilon_{rel}(z)\hat{E}(z)=0$$

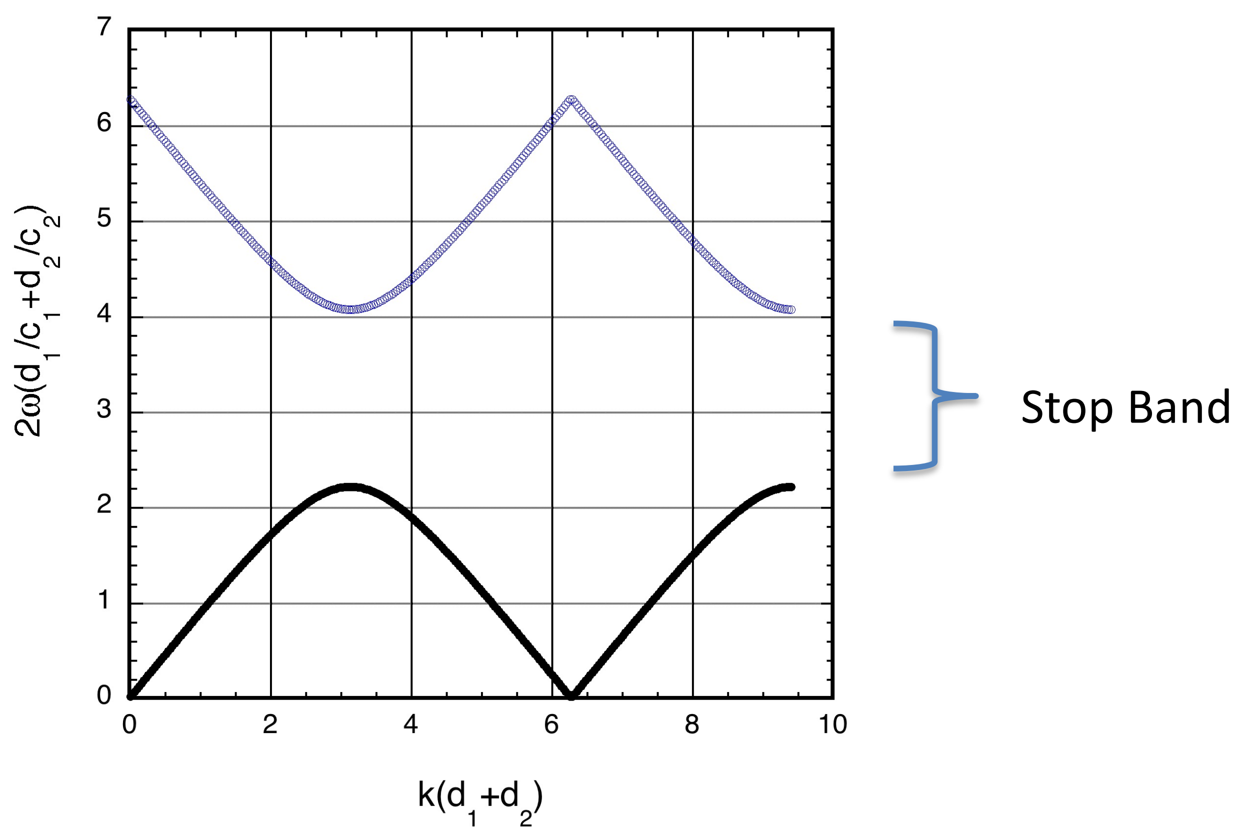
$$\varepsilon_{rel}(z)=\varepsilon_{rel}(z+L)$$

$$\hat{E}(z)=\hat{E}_0(k,z)\exp\left(ikz\right)$$

$$\hat{E}_0(k,z)=\hat{E}_0(k,z+L)$$

$$\omega(k)=\omega(k+k_0)$$

$$k_0=2\pi/L$$



Smith Island Cake

The Smith Island Cake is the official dessert of the State of Maryland. It consists of alternating layers of two dielectric materials as pictured at right. Suppose the dielectric constants and the thicknesses of the two alternating layers are $\varepsilon_1, \varepsilon_2$ and d_1, d_2, respectively. In this sense the cake is a metamaterial.

yumsugar.com

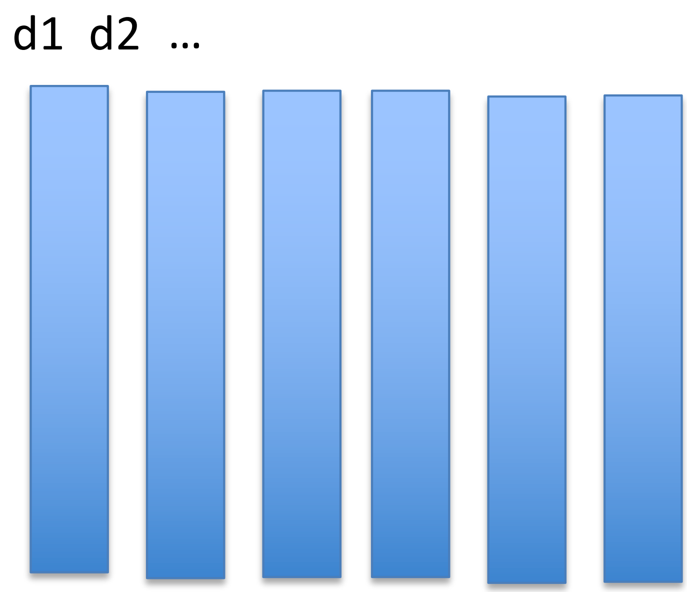
$$E(z) = \left[E_+ e^{ikz} + E_- e^{-ikz} \right]$$

$$H(z) = Z^{-1} \left[E_+ e^{ikz} - E_- e^{-ikz} \right]$$

$$Z = \sqrt{\mu_0 / \varepsilon_{1,2}}$$

$$E(0) = E_+ + E_-$$

$$H(0) = Z^{-1} \left[E_+ - E_- \right] \qquad k_{1,2} = \omega \sqrt{\varepsilon_{1,2} \mu_0}$$



$$E(z) = \left[E_+ e^{ikz} + E_- e^{-ikz} \right]$$

$$H(z) = Z^{-1} \left[E_+ e^{ikz} - E_- e^{-ikz} \right]$$

$$E(0) = E_+ + E_-$$

$$H(0) = Z^{-1} \left[E_+ - E_- \right]$$

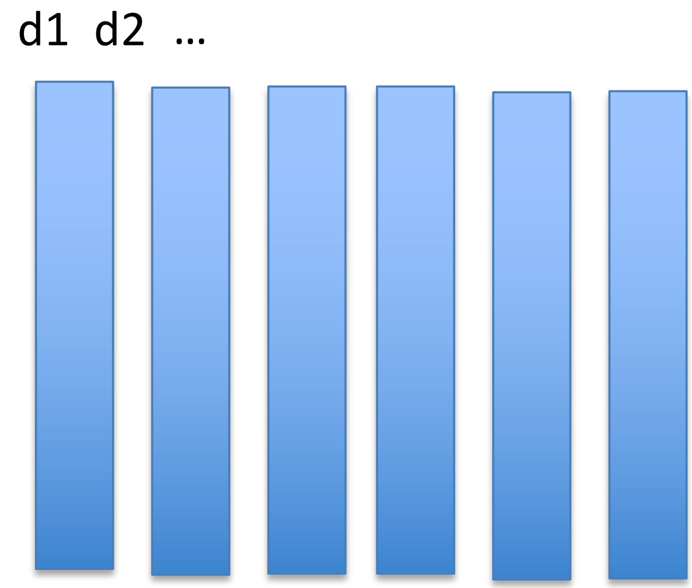
Solve for E+/-

$$E_+ = \frac{1}{2} \left[E(0) + ZH(0) \right]$$

$$E_+ = \frac{1}{2} \left[E(0) - ZH(0) \right]$$

$$Z = \sqrt{\mu_0 / \varepsilon}$$

$$\theta = kd = \omega \sqrt{\varepsilon \mu_0} d$$



Find fields at z=d

$$\begin{pmatrix} E(d) \\ H(d) \end{pmatrix} = \begin{bmatrix} \cos\theta & iZ\sin\theta \\ iZ^{-1}\sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} E(0) \\ H(0) \end{pmatrix}$$

Smith Island Cake

$$\begin{pmatrix} E(d) \\ H(d) \end{pmatrix} = \begin{bmatrix} \cos\theta & iZ\sin\theta \\ iZ^{-1}\sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} E(0) \\ H(0) \end{pmatrix}$$

$$Z = \sqrt{\mu_0/\varepsilon}$$

$$\theta = kd = \omega\sqrt{\varepsilon\mu_0}\,d$$

Two layers of different material

$$\begin{pmatrix} E(d_2+d_1) \\ H(d_2+d_1) \end{pmatrix} = \begin{bmatrix} \cos\theta_1 & iZ_1\sin\theta_1 \\ iZ_1^{-1}\sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & iZ_2\sin\theta_2 \\ iZ_2^{-1}\sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{pmatrix} E(0) \\ H(0) \end{pmatrix}$$

$$\begin{pmatrix} E(d_2+d_1) \\ H(d_2+d_1) \end{pmatrix} = \underline{\underline{M}} \begin{pmatrix} E(0) \\ H(0) \end{pmatrix}$$

After Multiple layers

$$\begin{pmatrix} E(n(d_2+d_1)) \\ H(n(d_2+d_1)) \end{pmatrix} = \left(\underline{\underline{M}}\right)^n \begin{pmatrix} E(0) \\ H(0) \end{pmatrix}$$

Find eigenfunctions and eigenvalues of M

$$\begin{pmatrix} E(z+d_1+d_2) \\ H(z+d_1+d_2) \end{pmatrix} = \underline{\underline{M}} \begin{pmatrix} E(z) \\ H(z) \end{pmatrix} = \lambda \begin{pmatrix} E(z) \\ H(z) \end{pmatrix}$$

$$\underline{\underline{M}} = \begin{bmatrix} \cos\theta_1\cos\theta_2 - \dfrac{Z_2}{Z_1}\sin\theta_1\sin\theta_2 & i\left(Z_1\sin\theta_1\cos\theta_2 + Z_2\cos\theta_1\sin\theta_2\right) \\ i\left(Z^{-1}_1\sin\theta_1\cos\theta_2 + Z_2^{-1}\cos\theta_1\sin\theta_2\right) & \cos\theta_1\cos\theta_2 - \dfrac{Z_1}{Z_2}\sin\theta_1\sin\theta_2 \end{bmatrix}$$

$$\det\left[\underline{\underline{M}} - \lambda \underline{\underline{1}}\right] = 0$$

$$\lambda^2 + b\lambda + 1 = 0$$

$$b = \left(\frac{Z_2}{Z_1} + \frac{Z_1}{Z_2}\right)\sin\theta_1 \sin\theta_2 - 2\cos\theta_1 \cos\theta_2$$

$$\lambda = \lambda_\pm = \frac{-b \pm \sqrt{b^2 - 4}}{2}$$

$$b = -2\cos\left(\theta_1 + \theta_2\right) + \Delta \sin\theta_1 \sin\theta_2$$

$$\Delta = \frac{\left(Z_1 - Z_2\right)^2}{Z_1 Z_2}$$

$$\lambda_+ \lambda_- = 1$$

$$\lambda_\pm = \exp(\pm i\phi) \quad \text{or} \quad \lambda_\pm - \text{real}$$

Fields advance by phase on each layer
Or decay exponentially

$$\cos\phi = \cos(\theta_1 + \theta_2) - \frac{\Delta}{2}\sin(\theta_1)\sin(\theta_2)$$

$$\Delta = \frac{\left(Z_1 - Z_2\right)^2}{Z_1 Z_2}$$

Special case: $\theta_1 = \theta_2$

$$\cos(\theta_1 + \theta_2) = \frac{\cos\phi + \Delta/4}{1 + \Delta/4}$$

$$\theta_1 + \theta_2 = \frac{\omega}{c}\left(d_1\sqrt{\varepsilon_1} + d_2\sqrt{\varepsilon_2}\right)$$

Solutions

$$\begin{pmatrix} E(z+d_1+d_2) \\ H(z+d_1+d_2) \end{pmatrix} = \lambda \begin{pmatrix} E(z) \\ H(z) \end{pmatrix}$$

$$\lambda = e^{i\phi}$$

$$\theta = kd = \omega\sqrt{\varepsilon\mu_0}\,d$$

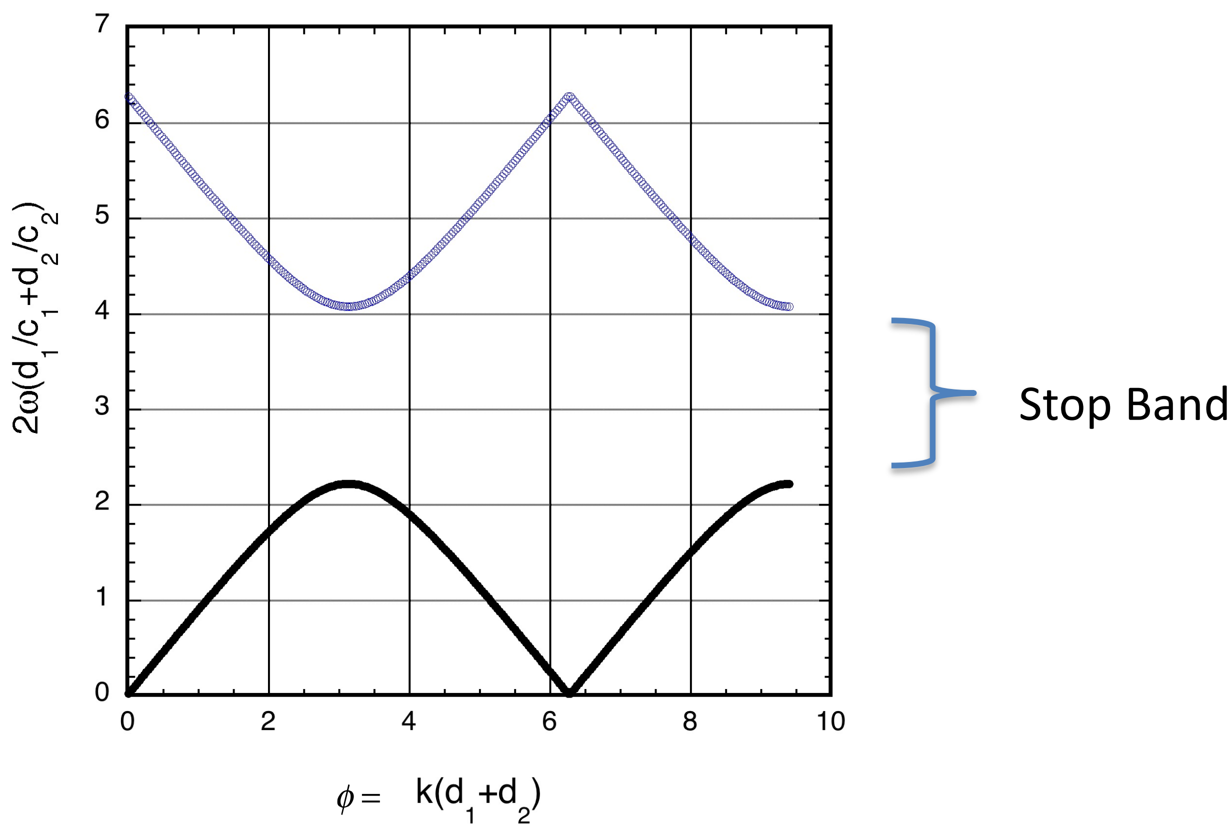
$$\cos\phi = \cos(\theta_1+\theta_2) - \frac{\Delta}{2}\sin(\theta_1)\sin(\theta_2)$$

$$\Delta = \frac{\left(Z_1 - Z_2\right)^2}{Z_1 Z_2}$$

Special case: $\theta_1 = \theta_2$

$$\cos(\theta_1+\theta_2) = \frac{\cos\phi + \Delta/4}{1+\Delta/4}$$

$$\theta_1+\theta_2 = \frac{\omega}{c}\left(d_1\sqrt{\varepsilon_1} + d_2\sqrt{\varepsilon_2}\right)$$



Continuous Variations

Mathieu Equation

$$\left[\frac{d^2}{dz^2} + \frac{\omega^2}{c^2}(1+\delta\cos k_0 z)\right]\hat{E}(z) = 0 \qquad k_0 = \frac{2\pi}{L}$$

periodic

Write $\hat{E}(z) = e^{ikz}\,\hat{E}_0(z,k)$

$$\hat{E}_0 = \sum_{n=-\infty}^{\infty} \hat{E}_n \exp(i k_n z) \qquad k_n = n k_0$$

Fourier Series

Solution by Fourier Series

$$\left[\frac{d^2}{dz^2} + \frac{\omega^2}{c^2}(1 + \delta \cos k_0 z) \right] \hat{E}(z) = 0$$

$$\int_0^L \frac{dz}{L} \exp(-i(k+nk_0)z) \cdot \{\text{Mathieu Equation}\} = 0$$

$$\left[-(k+nk_0)^2 + \frac{\omega^2}{c^2} \right] E_n + \frac{\omega^2}{c^2} \delta \sum_m \int_0^L \frac{dz}{L} e^{-ink_0 z} \cos k_0 z \, E_m e^{ik_m z} = 0$$

note

$$\int_0^L \frac{dz}{L} \cos k_0 z \exp[ik_0 z (m-n)] = \begin{cases} \frac{1}{2} & \text{if } m = n \pm 1 \\ 0 & \text{otherwize} \end{cases}$$

$$\left[\frac{\omega^2}{c^2} - (k+nk_0)^2\right] E_n + \frac{\omega^2}{c^2}\frac{\delta}{2}\left[E_{n+1} + E_{n-1}\right] = 0$$

Note if $\omega(k)$ is a solution with E_n

Then $k \to k+k_0$ $n \to n-1$ is also a solution

$$\omega(k) = \omega(k+k_0)$$

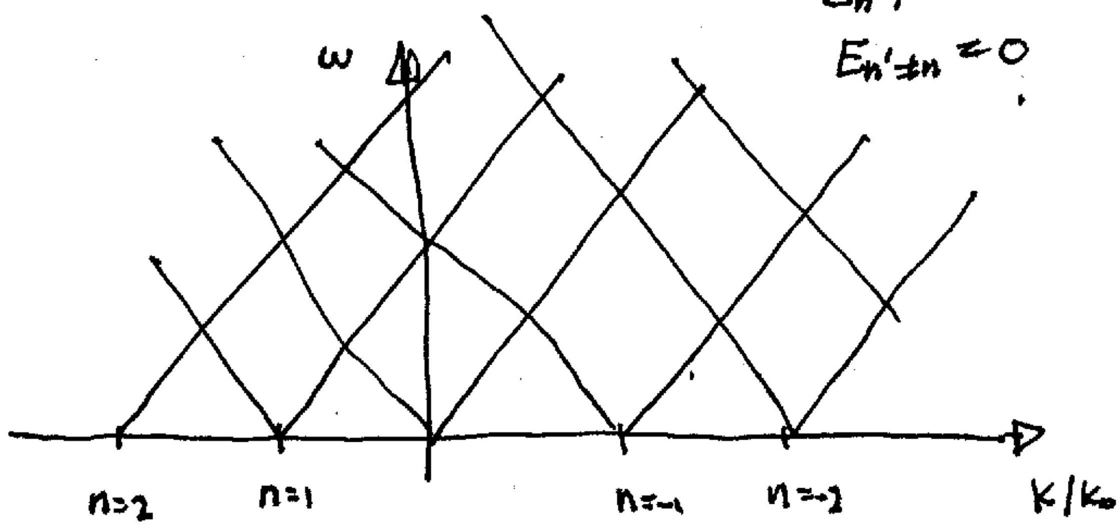
$$\left[\frac{\omega^2}{c^2} - (k+nk_0)^2\right] E_n + \frac{\omega^2}{c^2}\frac{\delta}{2}\left[E_{n+1} + E_{n-1}\right] = 0$$

Approximat solutions for $|\delta| \ll 1$

For each n $\frac{\omega^2}{c^2} = (k+nk_0)^2$ is a solution

$E_n \neq 0$

$E_{n'\neq n} = 0$



pattern repeat with period k_0

Solution breaks down where lines cross

$n=0$

$$\left[\frac{\omega^2}{c^2} - k^2\right] E_0 + \frac{\omega^2}{c^2}\frac{\delta}{2} E_{-1} = 0$$

$n=-1$

$$\left[\frac{\omega^2}{c^2} - (k-k_0)^2\right] E_{-1} + \frac{\omega^2}{c^2}\frac{\delta}{2} E_0 = 0$$

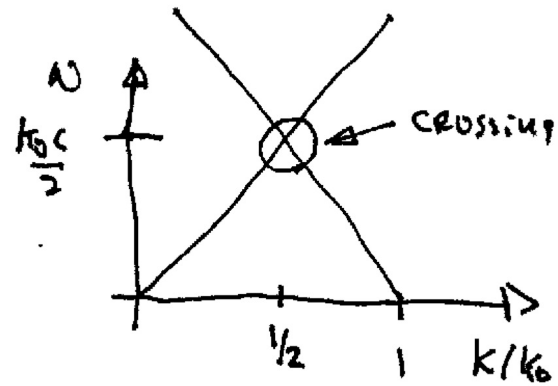
Combine

$$\left[\frac{\omega^2}{c^2} - k^2\right]\left[\frac{\omega^2}{c^2} - (k-k_0)^2\right] = \left(\frac{\omega^2}{c^2}\frac{\delta}{2}\right)^2$$

Combine

$$\left[\frac{\omega^2}{c^2} - k^2\right]\left[\frac{\omega^2}{c^2} - (k-k_0)^2\right] = \left(\frac{\omega^2}{c^2}\frac{\delta}{2}\right)^2$$

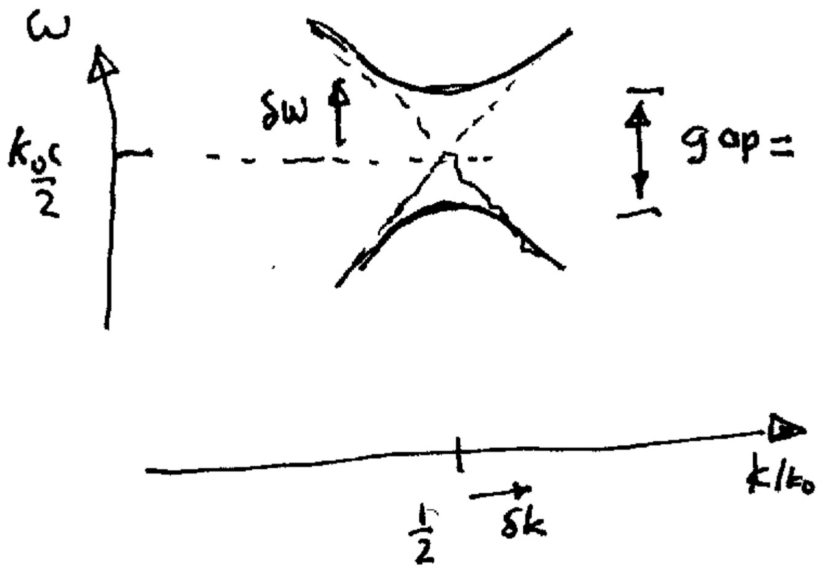
Let $\omega = \frac{k_0}{2}c + \delta\omega$ $\qquad k = \frac{k_0}{2} + \delta k$



$$\left[\frac{k_0 c\,\delta\omega}{c^2} - k_0\delta k\right]\left[\frac{k_0 c\,\delta\omega}{c^2} + k_0\delta k\right] = \frac{\delta^2}{2^2}\left(\frac{k_0 c}{c\,2}\right)^4$$

$$\left[\frac{\delta\omega}{c} - \delta k\right]\left[\frac{\delta\omega}{c} + \delta k\right] = \frac{\delta^2}{64}\,k_0^2$$

$$\left(\frac{\delta\omega}{c}\right)^2 = \delta k^2 + \frac{k_0^2}{64}\delta^2$$

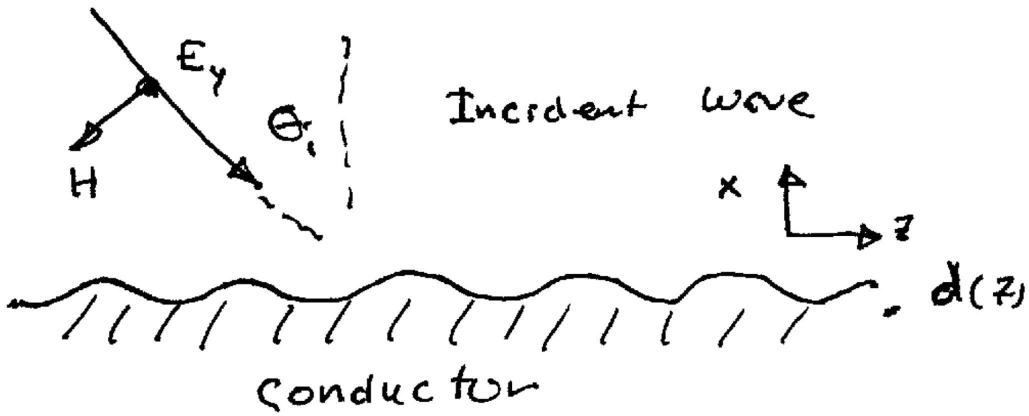


$$\text{gap} = \frac{2 c k_0 \delta}{8} = \frac{\delta}{4} k_0 c$$

Avoided Crossing

Gap is like band gap for electrons in a crystal structure

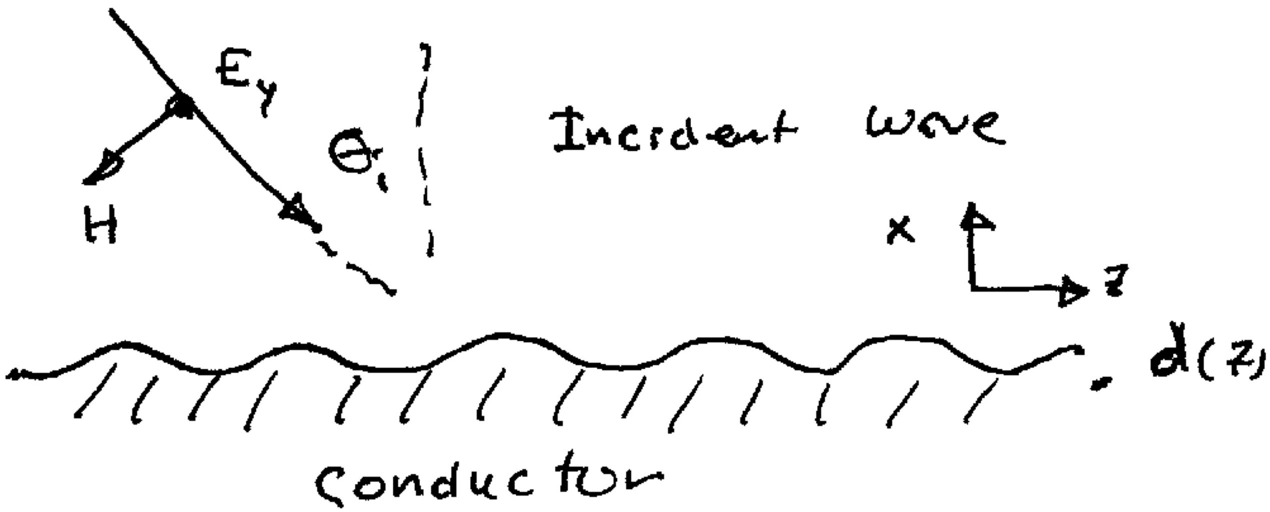
Grating



$$\nabla^2 \hat{E}(x,z) + \frac{\omega^2}{c^2}\hat{E}(x,z) = 0$$

Boundary Condition $\hat{E}(x=d(z), z) = 0$

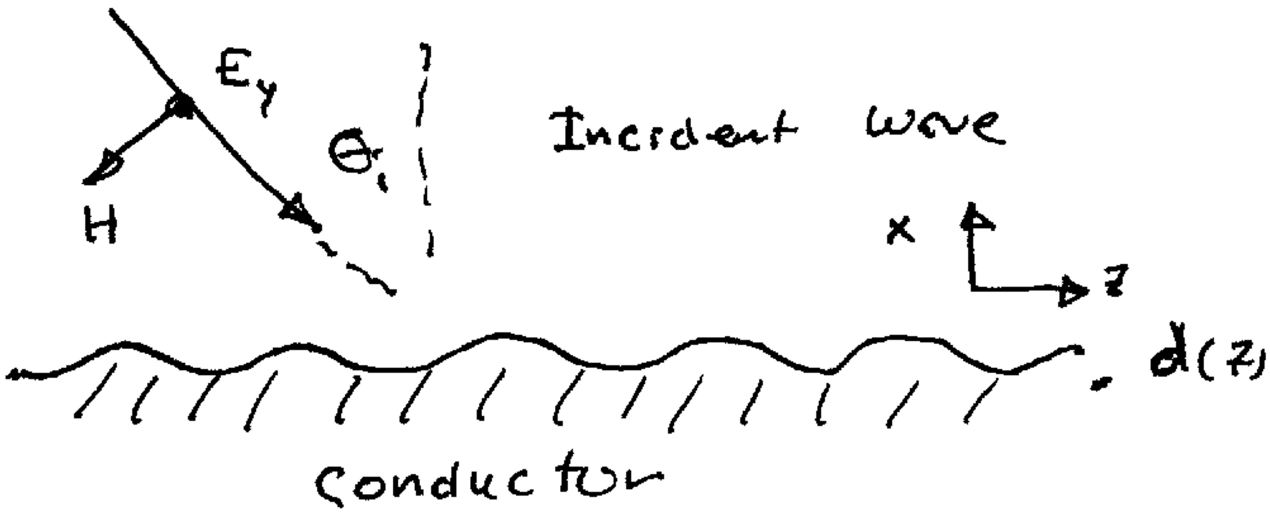
$$d(z) = \sum_m d_m e^{imk_0 z}$$



Incident Wave

$$\hat{E} = \hat{E}_{inc} \exp[i k_z z - i k_x x]$$

$$k_z = \frac{\omega}{c} \sin\theta_i \qquad k_x = \frac{\omega}{c} \cos\theta_i$$



Reflected Waves

$$\hat{E}_{ref} = \sum_m \hat{E}_m \exp[i k_z z + i m k_0 z + i k_{xm} x]$$

$$k_{xm} = \begin{cases} \sqrt{\frac{\omega^2}{c^2} - (k_z + m k_0)^2} & |k_z + m k_0| < \omega/c \\ +i\sqrt{(k_z + m k_0)^2 - \frac{\omega^2}{c^2}} & |k_z + m k_0| > \omega/c \end{cases}$$

Boundary Condition

$$e^{ik_z z}\left[\hat{E}_{inc}\exp[-ik_x d(z)] + \sum_m \hat{E}_m \exp(imk_o z + ik_{xm} d(z))\right] = 0$$

Specialize to small $d(z)$

Specularly reflected wave ($m=0$)

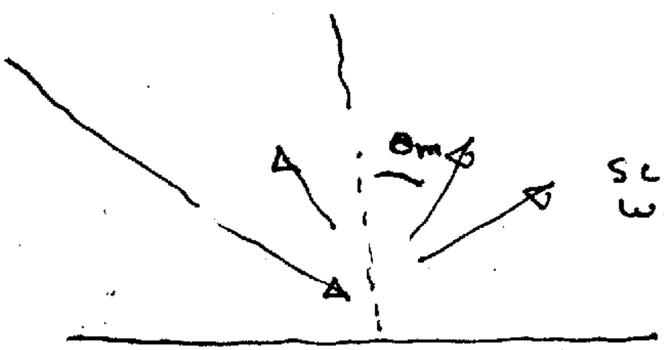
$$\hat{E}_0 = -\hat{E}_{inc}$$

OTHER waves

substitute $d(z) = \sum_m d_m e^{ik_m z}$

$$\hat{E}_{inc} \left\{ (1 - ik_x d) - (1 + ik_x d) + \sum_{m \neq 0} E_m \exp(imk_0 z) \right\} = 0$$

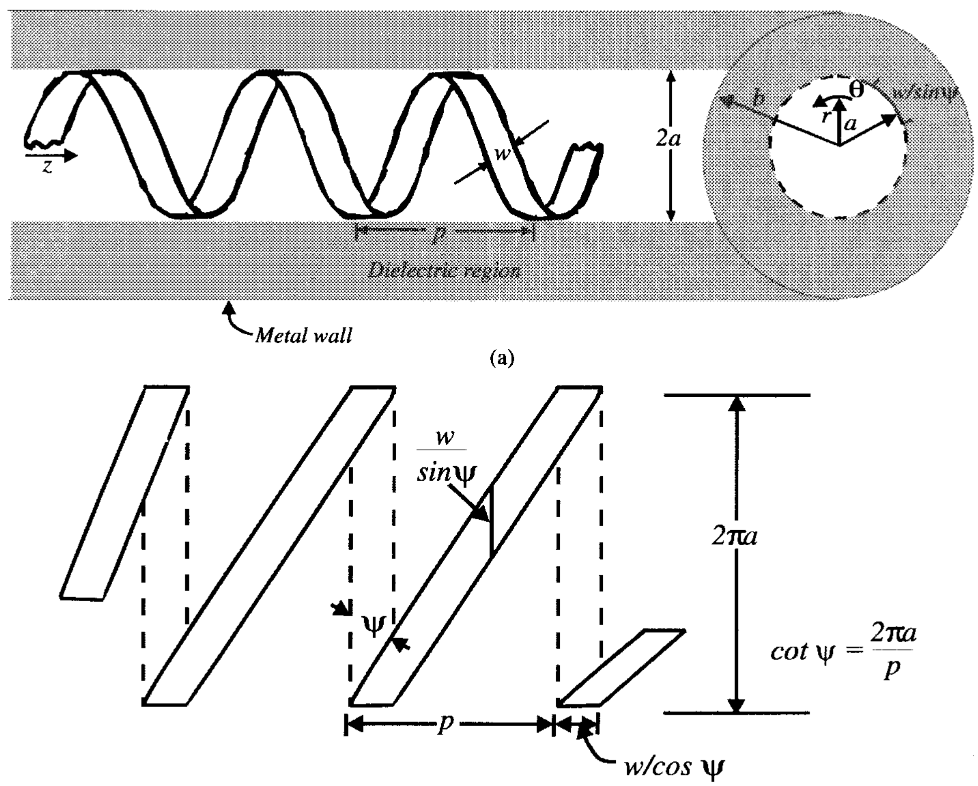
$$E_m = 2ik_x d_m \hat{E}_{inc}$$



$$\frac{\omega}{c} \cos\theta_m = \sqrt{\frac{\omega^2}{c^2} - \left(\frac{\omega}{c}\sin\theta_i + mk_0\right)^2}$$

$$\cos\theta_m = \sqrt{1 - (\sin\theta_i + mk_0c/\omega)^2}$$

Tape Helix

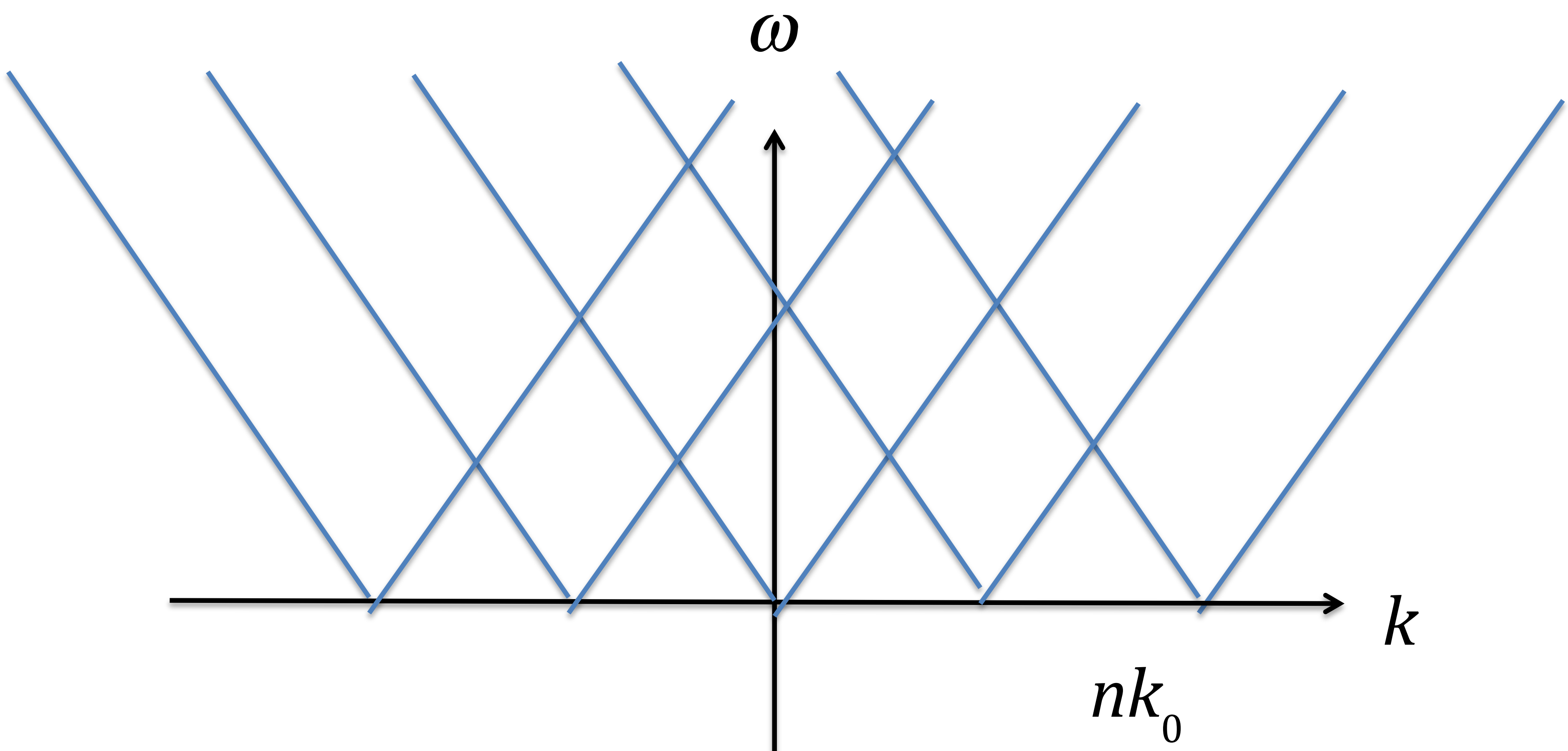


Approximate solution

$$\omega = k v_p$$

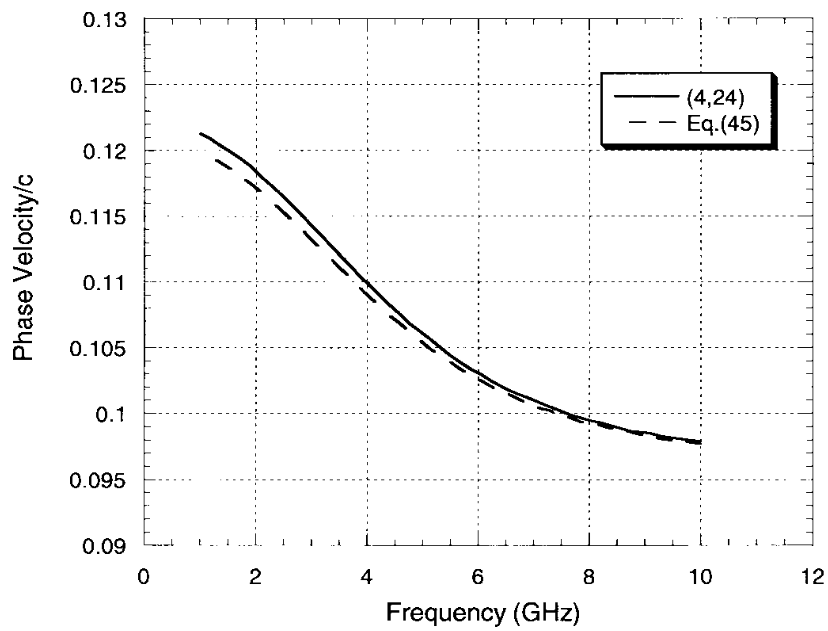
$$v_p = c \frac{p}{\sqrt{p^2 + \left(2\pi r\right)^2}}$$

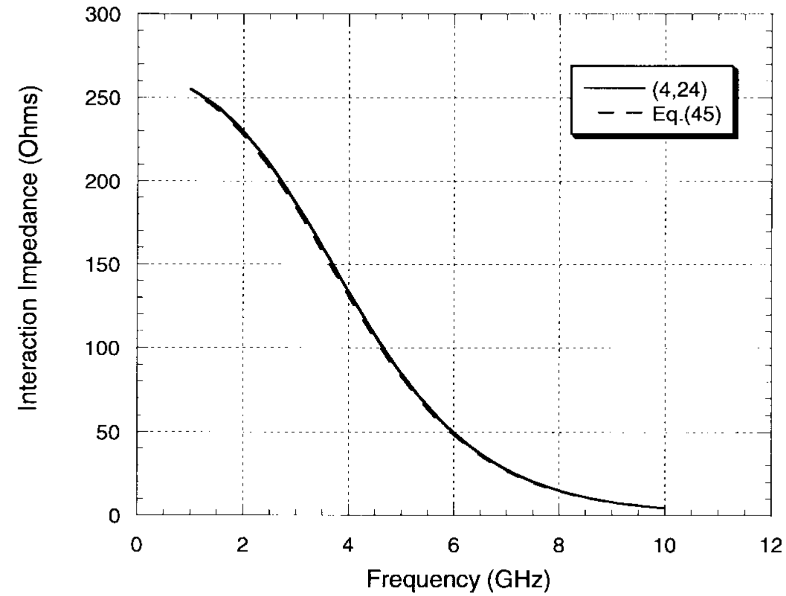
Crossings – not gaps



Consequence of helical symmetry

$$k_0 = 2\pi / p$$

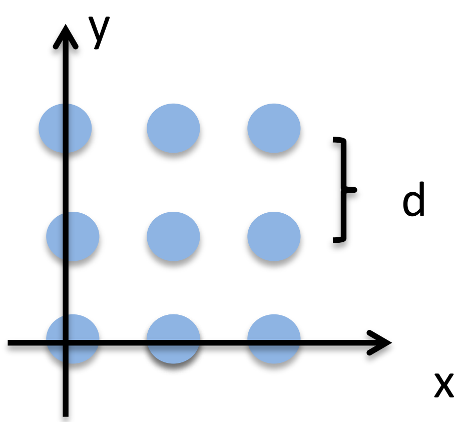




$$Z_{Pierce} = \frac{\left|\bar{E}_z\right|^2}{2k_z^2 P}$$

Pierce: Vacuum electronics pioneer
Pulse code modulation
First communications satellite
Bohlen-Pierce musical scale
Coined name “Transistor”

Higher Dimensions



$$\nabla^2 E(x,y)+\frac{\omega^2}{c^2}\left(1+\chi(x,y)\right)E(x,y)=0$$

$$\chi(x,y)=\chi(x+d,y)=\chi(x,y+d)$$

$$E(x,y)=\sum_{m,n}\bar{E}_{m,n}\exp\left[i\left(k_x+nk_o\right)x+i\left(k_y+mk_o\right)\right]$$

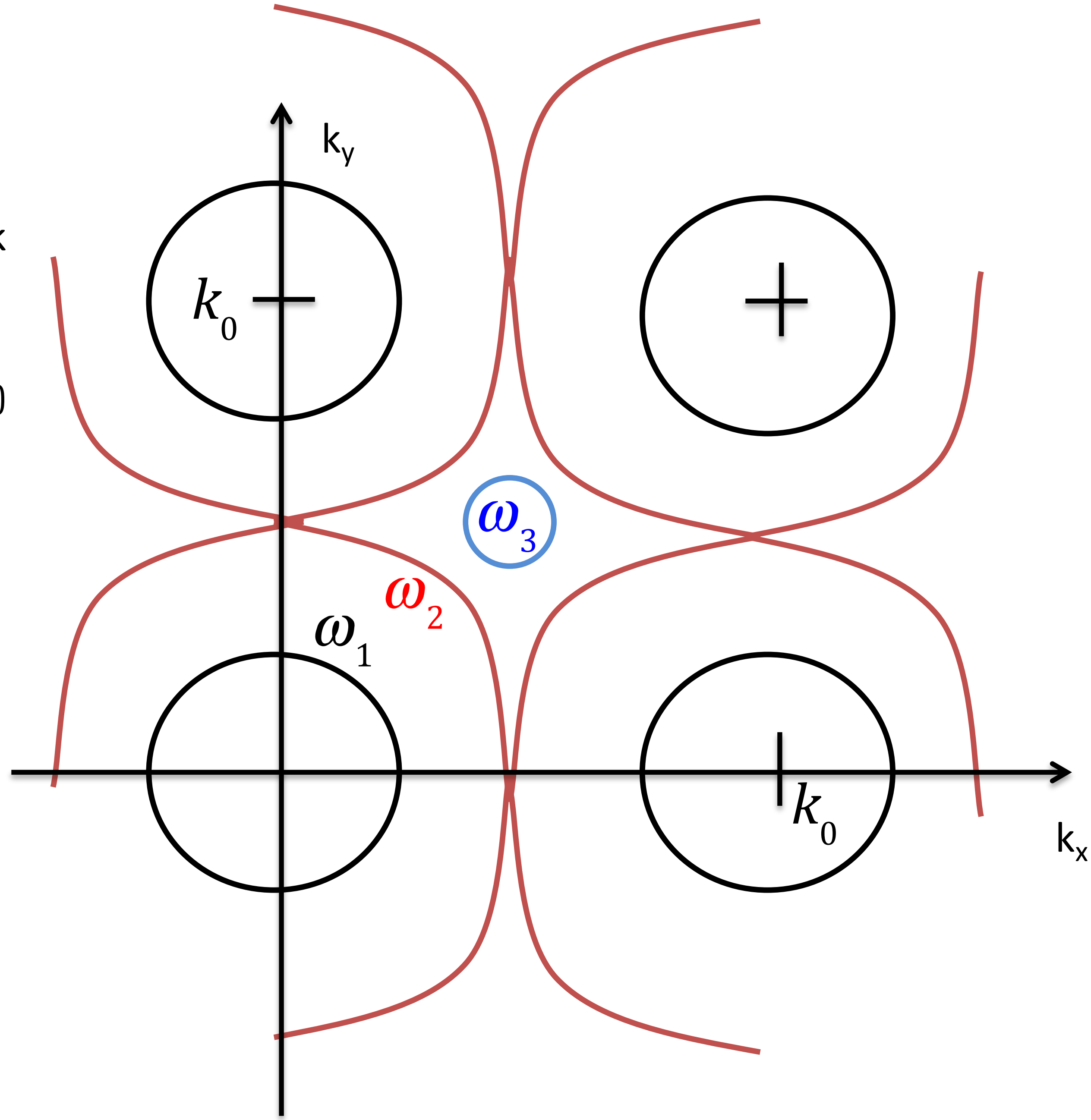
$$\omega(k_x,k_y)=\omega(k_x+qk_0,k_y+pk_0)$$

$$k_0=2\pi/d$$

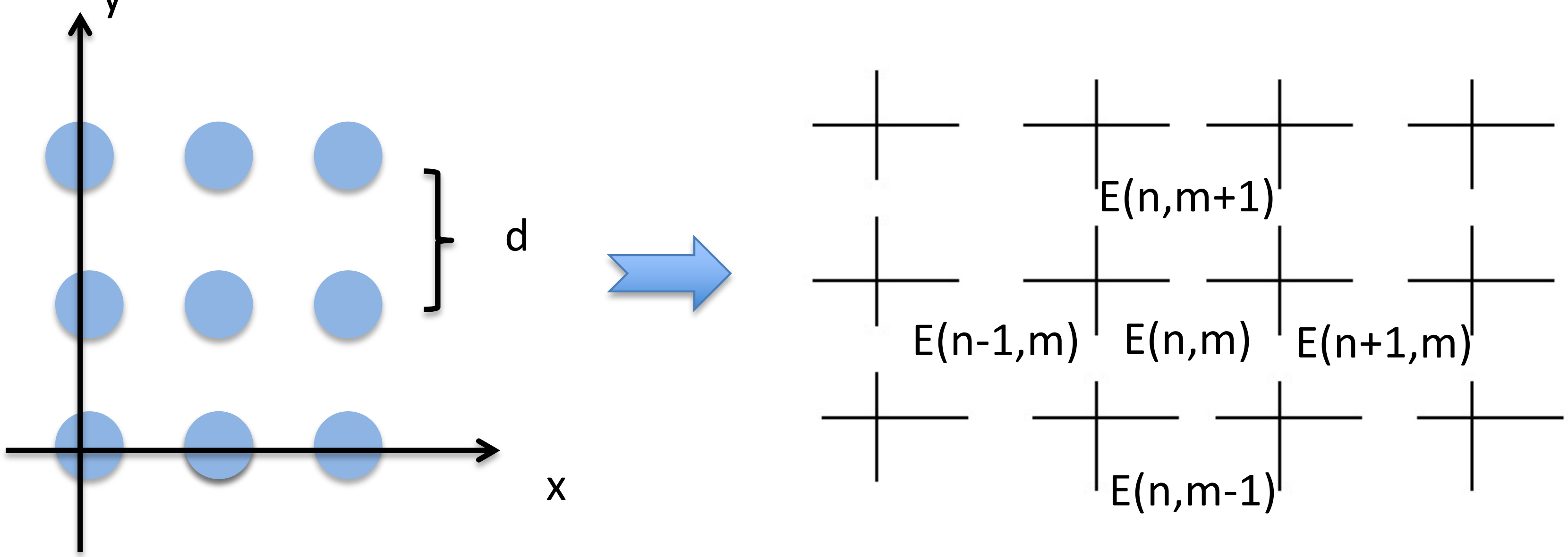
Level curves of frequency in the k plane

$\omega(k_x,k_y)=\omega(k_x+qk_0,k_y+pk_0)$

$k_0=2\pi/d$



Creation of Stop Band



$$\left[\omega^2-\omega_c^2\right]E(n,m)=\frac{\delta}{2}\omega_c^2\left[E(n+1,m)+E(n-1,m)+E(n,m+1)+E(n,m-1)\right]$$

$$E(n,m)=E(0,0)\exp\left[i(k_x dn+k_y dm)\right]$$

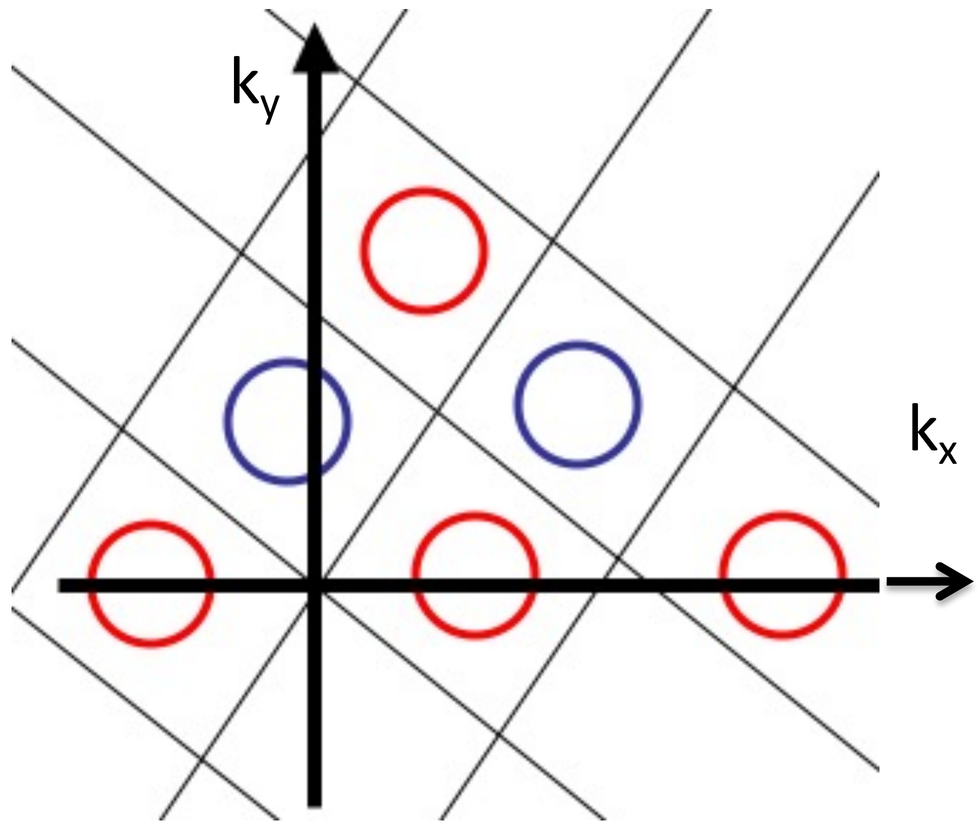
$$\left[\omega^2-\omega_c^2\right]=\delta\omega_c^2\left[\cos(k_x d)+\cos(k_y d)\right]=\delta\omega_c^2\cos\left[(k_x-k_y)d\right]\cos\left[(k_x+k_y)d\right]$$

$$\left[\omega^2-\omega_c^2\right]E(n,m)=\frac{\delta}{2}\omega_c^2\left[E(n+1,m)+E(n-1,m)+E(n,m+1)+E(n,m-1)\right]$$

$$E(n,m)=E(0,0)\exp\left[i(k_x dn+k_y dm)\right]$$

$$\left[\omega^2-\omega_c^2\right]=\delta\omega_c^2\left[\cos(k_x d)+\cos(k_y d)\right]$$

$$=\delta\omega_c^2\cos\left[(k_x-k_y)d\right]\cos\left[(k_x+k_y)d\right]$$



Individual cavities have a set of modes, $\omega_c^2 = \omega_{c1}^2, \omega_{c2}^2, \omega_{c3}^2$....

If the spacing between modes is greater that the frequency shift induced by coupling

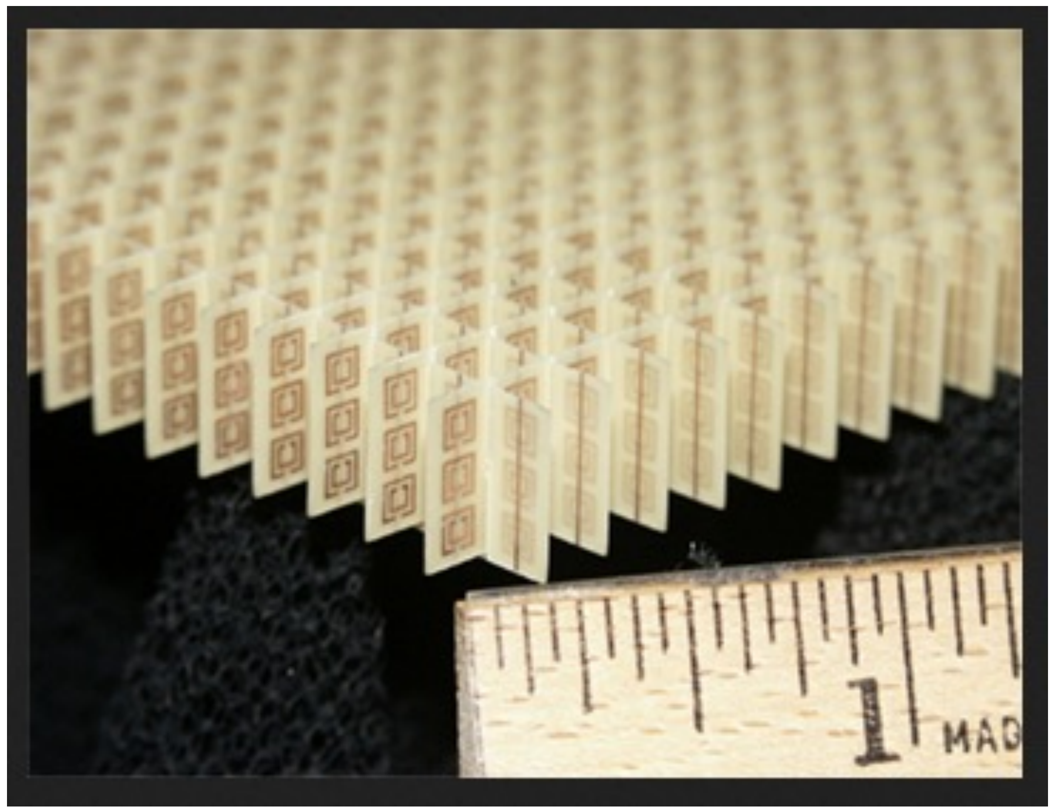
$$\left|\omega_{cp}^2 - \omega_{cp+1}^2\right| < \delta\omega_c^2$$

then gaps in the spectrum with no propagating modes appear.

Metamaterials

Metamaterials are periodic structures that have engineered properties in the long wave length limit,

kd<<1



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Negative epsilon and negative mu

In a restricted range of frequencies the effective constituitive parameters may be negative.

If both are positive or both are negative waves propagate.

$$k^2 = \omega^2 \varepsilon \mu > 0$$

If both are negative waves satisfy the left hand rule.

$$\mathbf{k} \times \mathbf{E} = \omega \mu \mathbf{H}$$

For ε<0 or μ<0 they must be functions of frequency.

Media are passive, stored energy is positive.

$$\mathrm{U}_E = \frac{1}{2}\frac{\partial}{\partial\omega}\big(\omega\varepsilon(\omega)\big)|E|^2 > 0, \qquad \frac{\partial}{\partial\omega}\big(\omega\varepsilon(\omega)\big) = \varepsilon(\omega) + \omega\frac{\partial}{\partial\omega}\varepsilon(\omega) > 0$$

If both ε<0 and μ<0 group and phase velocities are opposite

If both ε<0 and μ<0 group and phase velocities are opposite

$$\frac{1}{v_g} = \frac{\partial}{\partial\omega}k = \frac{\partial}{\partial\omega}\left(\omega\sqrt{\varepsilon\mu}\right) = \sqrt{\varepsilon\mu} + \frac{\omega}{2\sqrt{\varepsilon\mu}}\frac{\partial}{\partial\omega}\left(\varepsilon\mu\right)$$

$$\frac{1}{v_g} = \frac{1}{2\sqrt{\varepsilon\mu}}\left[\mu\frac{\partial}{\partial\omega}\left(\omega\varepsilon(\omega)\right) + \varepsilon\frac{\partial}{\partial\omega}\left(\omega\mu(\omega)\right)\right] < 0 \quad \text{if both } \varepsilon \text{ \& } \eta < 0$$

$$\frac{1}{v_p} = \frac{1}{\sqrt{\varepsilon\mu}}$$

Backward Waves