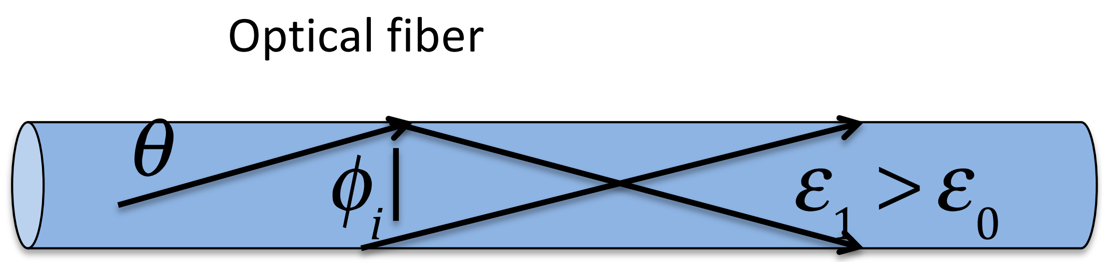
# Dielectric Waveguides

ENEE 681

Basic principles
Slab model
- two regions
- boundary conditions
- dispersion relation – odd and even modes

Cylindrical geometry

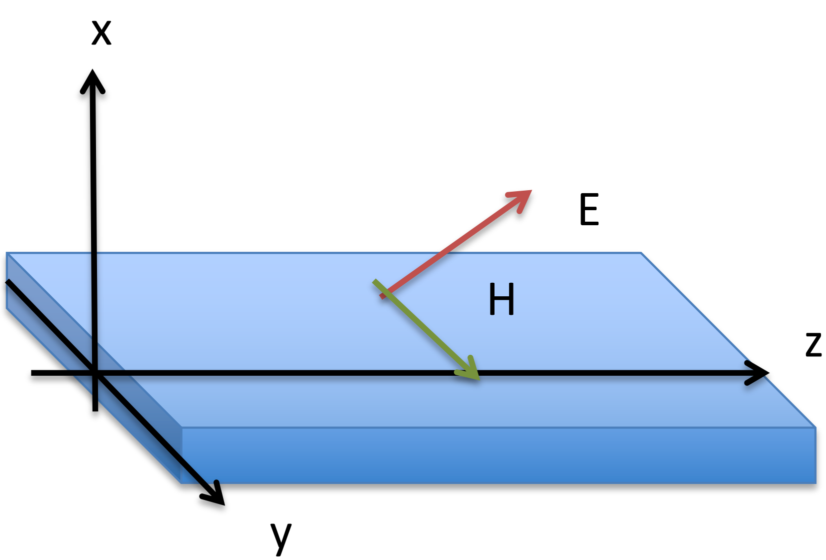
# Guiding by total internal reflection




Total reflection if

$$\sin\phi_i = \cos\theta > \sqrt{\varepsilon_1 / \varepsilon_0}$$

# Slab Model

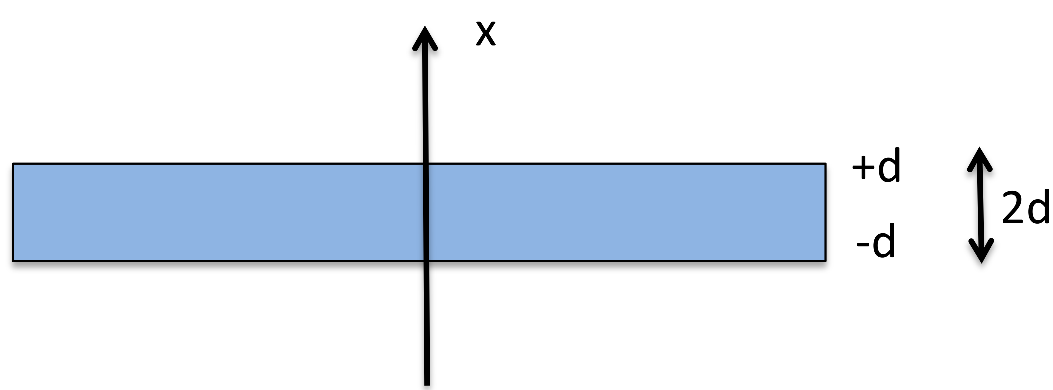



$$\varepsilon = \begin{cases} \varepsilon_1 & |x| < d \\ \varepsilon_0 & |x| > d \end{cases}$$

Look for solutions

$$\mathbf{E} = \mathrm{Re}\left\{(\hat{E}_x, 0, \hat{E}_z)\exp\left[i\left(k_z z - \omega t\right)\right]\right\}$$

$$\mathbf{H} = \mathrm{Re}\left\{(0, \hat{H}_y, 0)\exp\left[i\left(k_z z - \omega t\right)\right]\right\}$$



# Maxwell's Equations

$$\nabla \times \vec{\mathbf{E}} = -\mu \frac{\partial \vec{\mathbf{H}}}{\partial t}$$

$$\frac{\partial \hat{E}_z}{\partial y} - ik_z \hat{E}_y = i\omega\mu \hat{H}_x$$

$$ik_z \hat{E}_x - \frac{\partial \hat{E}_z}{\partial x} = i\omega\mu \hat{H}_y$$

$$\frac{\partial \hat{E}_y}{\partial x} - \frac{\partial \hat{E}_x}{\partial y} = i\omega\mu \hat{H}_z$$

$$\nabla \times \vec{\mathbf{H}} = \varepsilon \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

$$\frac{\partial \hat{H}_z}{\partial y} - ik_z \hat{H}_y = -i\omega\varepsilon \hat{E}_x$$

$$ik_z \hat{H}_x - \frac{\partial \hat{H}_z}{\partial x} = -i\omega\varepsilon \hat{E}_y$$

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$$\frac{\partial \hat{H}_z}{\partial y} - \boxed{ik_z \hat{H}_y = -i\omega\varepsilon \hat{E}_x}$$

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# Maxwell's Equations

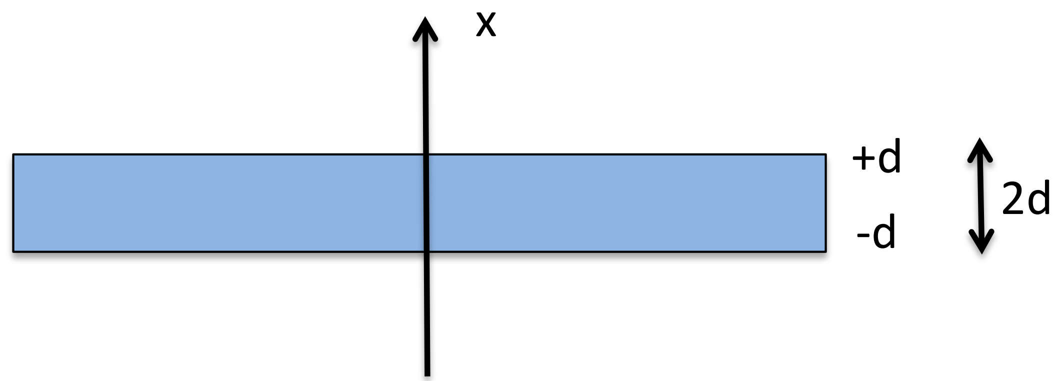
$$ik_z\hat{E}_x - \frac{\partial \hat{E}_z}{\partial x} = i\omega\mu\hat{H}_y$$

$$-ik_z\hat{H}_y = -i\omega\varepsilon\hat{E}_x$$

$$\frac{\partial \hat{H}_y}{\partial x} = -i\omega\varepsilon\hat{E}_z$$

$$\varepsilon = \begin{cases} \varepsilon_1 & |x| < d \\ \varepsilon_0 & |x| > d \end{cases}$$

What are the boundary conditions at x = +/- d?



# Maxwell's Equations

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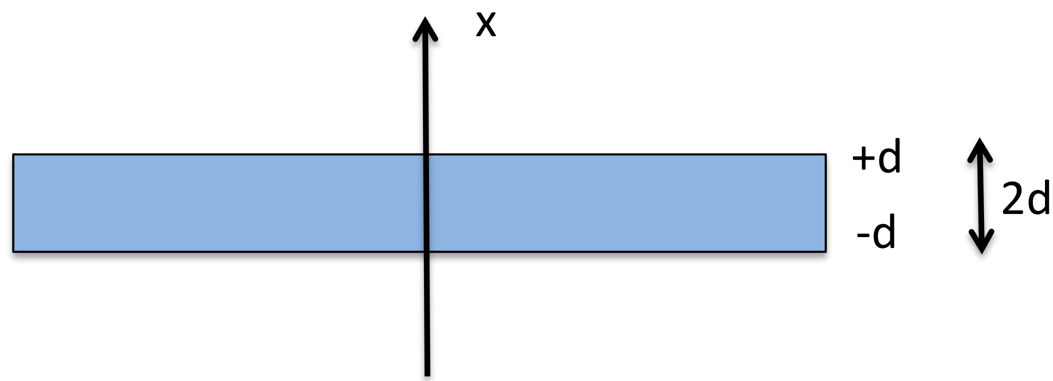



$$\left.\begin{matrix} \hat{E}_z \\ \hat{H}_y \end{matrix}\right\} \text{continuous at } x = \pm d$$

# Combine to a single equation

$$ik_z\hat{E}_x - \frac{\partial \hat{E}_z}{\partial x} = i\omega\mu\hat{H}_y$$

$$-ik_z\hat{H}_y = -i\omega\varepsilon\hat{E}_x$$

$$\frac{\partial \hat{H}_y}{\partial x} = -i\omega\varepsilon\hat{E}_z$$

$$k_z^2\hat{E}_x + ik_z\frac{\partial \hat{E}_z}{\partial x} = i\omega\mu\left(-ik_z\hat{H}_y\right) = \omega^2\varepsilon\mu\hat{E}_x$$

$$\hat{E}_x = \frac{ik_z}{\omega^2\varepsilon\mu - k_z^2}\frac{\partial \hat{E}_z}{\partial x}$$

$$\hat{H}_y = \frac{i\omega\varepsilon}{\omega^2\varepsilon\mu - k_z^2}\frac{\partial \hat{E}_z}{\partial x}$$

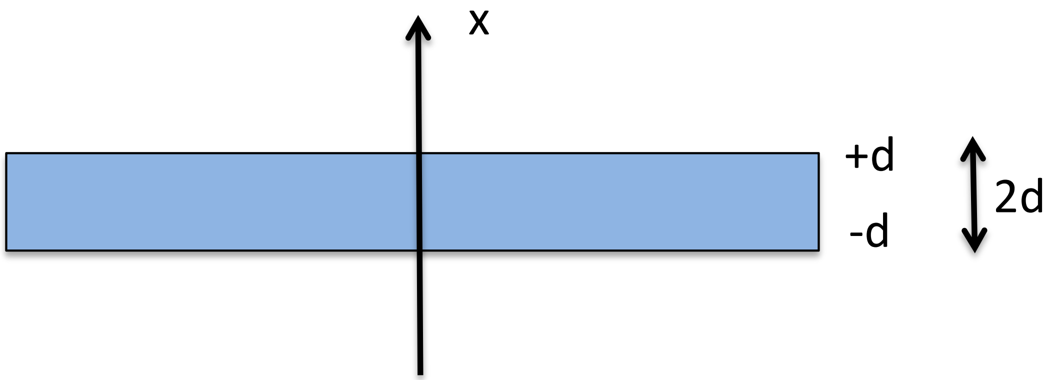
$$\frac{\partial \hat{H}_y}{\partial x} = -i\omega\varepsilon\hat{E}_z$$

$$\frac{\partial}{\partial x}\frac{\varepsilon}{\omega^2\varepsilon\mu - k_z^2}\frac{\partial \hat{E}_z}{\partial x} = -\varepsilon\hat{E}_z$$

If $\varepsilon$ is constant $\quad \frac{\partial}{\partial x}\frac{\partial \hat{E}_z}{\partial x} + \left(\omega^2\varepsilon\mu - k_z^2\right)\hat{E}_z = 0$

# Solve in two regions

If $\varepsilon$ is constant $\quad \dfrac{\partial}{\partial x}\dfrac{\partial \hat{E}_z}{\partial x} + \left(\omega^2 \varepsilon \mu - k_z^2\right)\hat{E}_z = 0$



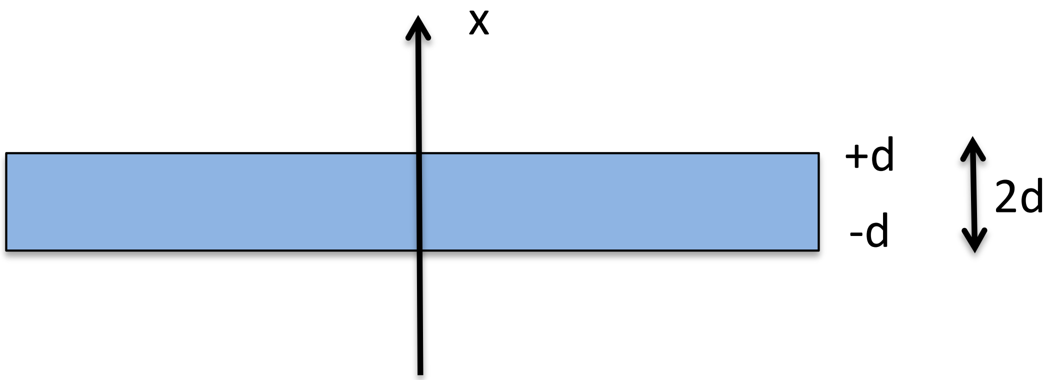

$$\varepsilon = \begin{cases} \varepsilon_1 & |x| < d, \quad \omega^2 \varepsilon_1 \mu - k_z^2 > 0 \\ \varepsilon_0 & |x| > d, \quad \omega^2 \varepsilon_0 \mu - k_z^2 < 0 \end{cases}$$

For x>d, $\varepsilon = \varepsilon_0 \qquad \hat{E}_z = A \exp(-\kappa x) \quad \kappa = \sqrt{k_z^2 - \omega^2 \varepsilon_0 \mu}$

Evanescent (Spatially decaying) field

# Solve in two regions

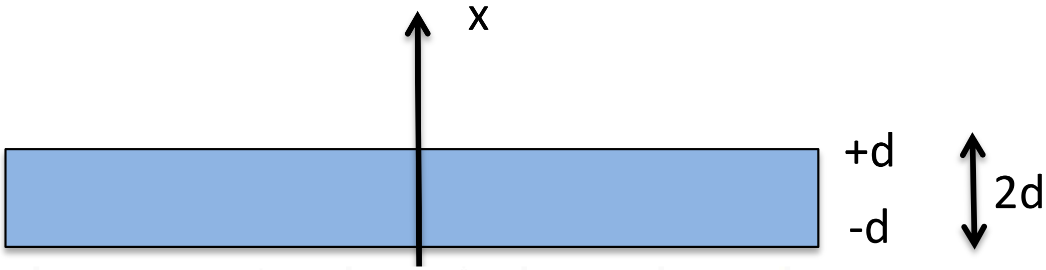
If $\varepsilon$ is constant $\quad \dfrac{\partial}{\partial x}\dfrac{\partial \hat{E}_z}{\partial x}+\left(\omega^2\varepsilon\mu-k_z^2\right)\hat{E}_z=0$




For $|x|<d$, $\quad \varepsilon=\varepsilon_1 \qquad \hat{E}_z=B\sin\left(k_x x\right)+C\cos\left(k_x x\right) \quad k_x=\sqrt{\omega^2\varepsilon_1\mu-k_z^2}$

Spatially oscillating field

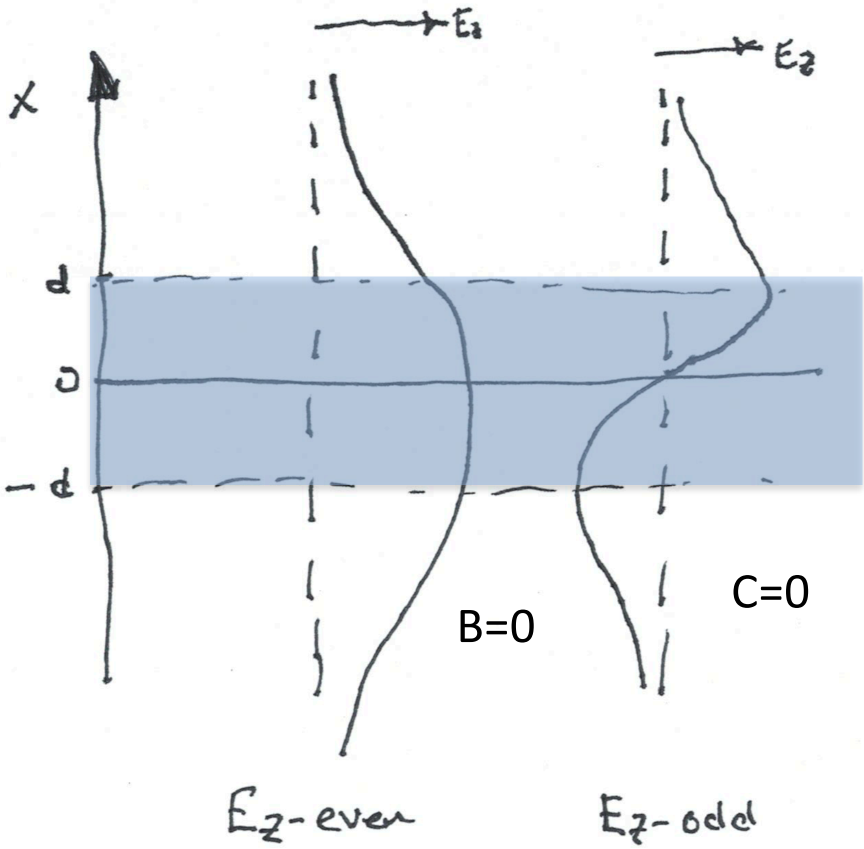
# Odd and Even solutions




For $|x|<d, \quad \varepsilon=\varepsilon_1$

$$\hat{E}_z = B\sin(k_x x) + C\cos(k_x x)$$

$$k_x = \sqrt{\omega^2\varepsilon_1\mu - k_z^2}$$




Either: B=0, $C \neq 0$

or $B \neq 0$, $C = 0$

$$\hat{E}_x = \frac{ik_z}{\omega^2\varepsilon\mu - k_z^2}\frac{\partial \hat{E}_z}{\partial x}$$

$$\hat{H}_y = \frac{i\omega\varepsilon}{\omega^2\varepsilon\mu - k_z^2}\frac{\partial \hat{E}_z}{\partial x}$$

# $E_z$ –even ($E_x$ – odd) solution

For $|x|<d$, $\hat{E}_z = C\cos(k_x x) \quad k_x = \sqrt{\omega^2 \varepsilon_1 \mu - k_z^2}$

For $x>d$, $\hat{E}_z = A\exp(-\kappa x) \quad \kappa = \sqrt{k_z^2 - \omega^2 \varepsilon_0 \mu}$

$$\hat{E}_x = \frac{ik_z}{\omega^2 \varepsilon\mu - k_z^2}\frac{\partial \hat{E}_z}{\partial x}$$

$$\hat{H}_y = \frac{i\omega\varepsilon}{\omega^2 \varepsilon\mu - k_z^2}\frac{\partial \hat{E}_z}{\partial x}$$

At x=d $\quad \hat{E}_z = C\cos(k_x d) = A\exp(-\kappa d)$

$$\hat{H}_y = \frac{i\omega\varepsilon_1}{k_x^2}\frac{\partial}{\partial x} C\cos(k_x x)\Big|_{x=d} = -\frac{i\omega\varepsilon_1}{k_x} C\sin(k_x d)$$

$$\hat{H}_y = \frac{i\omega\varepsilon_0}{-\kappa^2}\frac{\partial}{\partial x} A\exp(-\kappa x)\Big|_{x=d} = \frac{i\omega\varepsilon_0}{\kappa} A\exp(-\kappa d)$$

# Dispersion Relation

$$C\cos\left(k_x d\right) = A\exp\left(-\kappa d\right) \qquad -\frac{\varepsilon_1}{k_x}C\sin\left(k_x d\right) = \frac{\varepsilon_0}{\kappa}A\exp\left(-\kappa d\right)$$

Even $E_z$

$$-\frac{\varepsilon_1}{\varepsilon_0}\kappa d = k_x d\cot\left(k_x d\right)$$

Odd $E_z$

$$\frac{\varepsilon_1}{\varepsilon_0}\kappa d = k_x d\tan\left(k_x d\right)$$

$$\kappa d = d\sqrt{k_z^2 - \omega^2\varepsilon_0\mu} = \sqrt{\omega^2\left(\varepsilon_1 - \varepsilon_0\right)\mu d^2 - k_x^2 d^2}$$

$$\kappa d = \sqrt{\omega^2\left(\varepsilon_1 - \varepsilon_0\right)\mu d^2 - k_x^2 d^2}$$

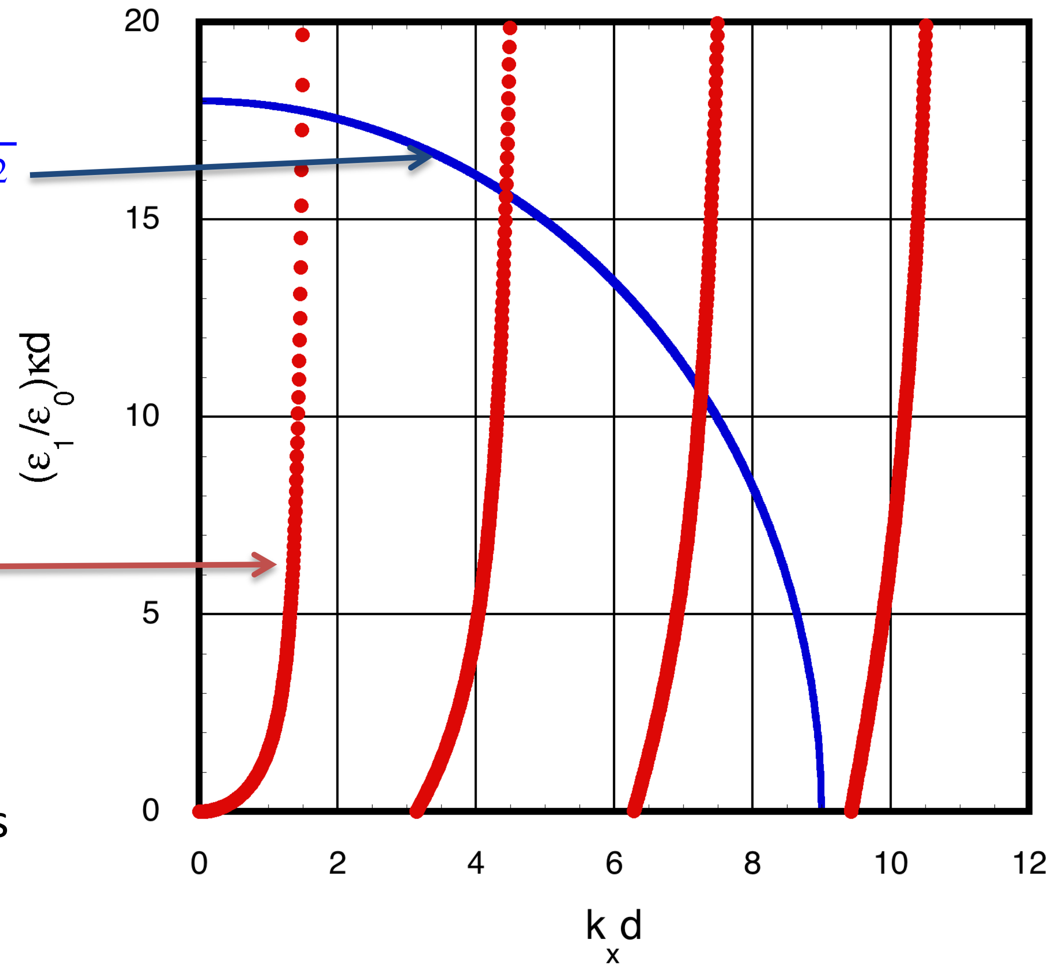
# Plot both sides versus $k_x d$

Odd $E_z$

$$\frac{\varepsilon_1}{\varepsilon_0}\kappa d = k_x d \tan\left(k_x d\right)$$

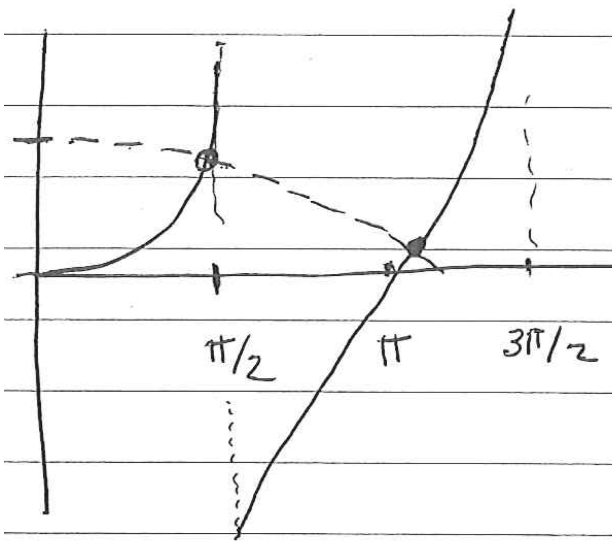
$$\frac{\varepsilon_1}{\varepsilon_0}\kappa d = \frac{\varepsilon_1}{\varepsilon_0}\sqrt{\omega^2\left(\varepsilon_1 - \varepsilon_0\right)\mu d^2 - k_x^2 d^2}$$

$$\frac{\varepsilon_1}{\varepsilon_0}\kappa d = k_x d \tan\left(k_x d\right)$$



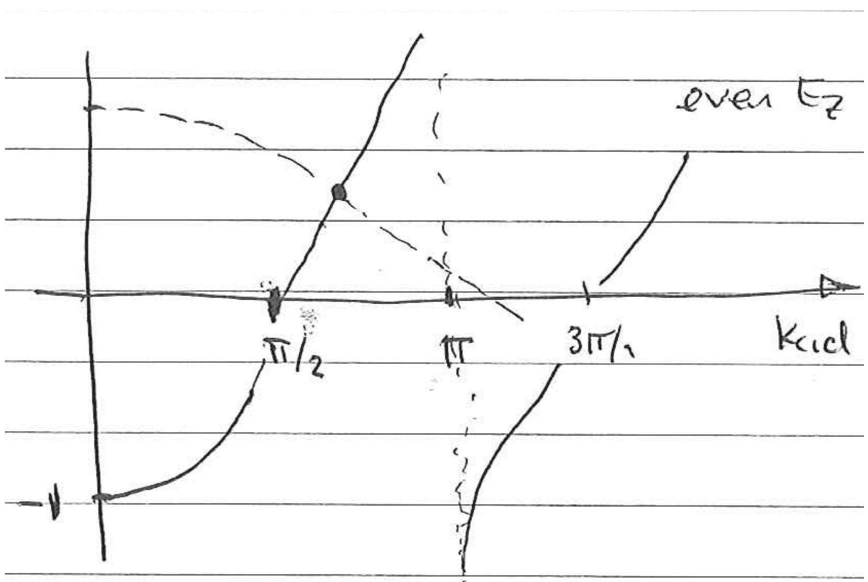

In this case three solutions – modes
There will always be at least one.

# Comments




Odd Ez –Even Ex

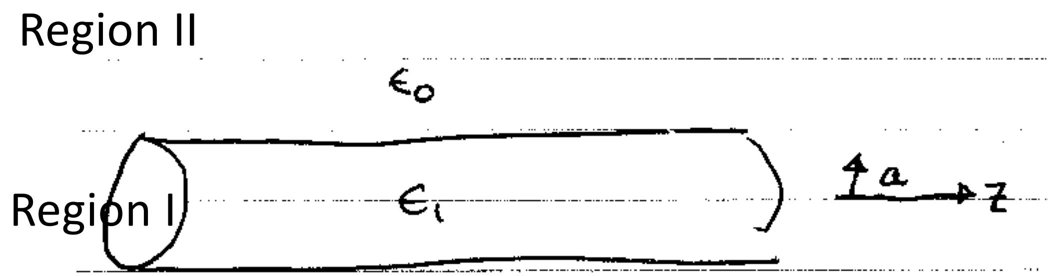
Always at least one solution
For small diameter d fields outside dielectric
Requires cladding




Even Ez –Odd Ex

Possibly no solutions

# Cylindrical Dielectrics




$$\text{Field} \sim \{\hat{\underline{E}}, \hat{\underline{H}}(r,\theta)\} \exp[ikz - i\omega t]$$

Express transverse components in terms of axial components

$$\hat{\underline{E}}_\perp = \frac{1}{k_\perp^2}\left[ik_z \nabla_\perp \hat{E}_z - i\omega\mu\, \underline{e}_z \times \nabla_\perp \hat{H}_z\right\}$$

$$\hat{\underline{H}}_\perp = \frac{1}{k_\perp^2}\left[ik_z \nabla_\perp \hat{H}_z + i\omega\epsilon\, \underline{e}_z \times \nabla_\perp \hat{E}_z\right]$$

$k_\perp^2 = \omega^2\epsilon\mu - k_z^2$ – different in each region

# Equation for Axial Components

$$\nabla_\perp^2 \hat{E}_z + k_\perp^2 \hat{E}_z = 0 \qquad \nabla_\perp^2 \hat{H}_z + k_\perp^2 \hat{H}_z = 0$$

$$k_\perp^2 = \omega^2 \epsilon \mu - k_z^2 \quad \text{— different in each region}$$

Region I: $k_{\perp 1}^2 = \omega^2 \epsilon_1 \mu - k_z^2$ >0

Region II $k_{\perp 2}^2 = \omega^2 \epsilon_0 \mu - k_z^2 = -\gamma_2^2 < 0$

$$\hat{E}_z = A_1 \exp(i l \theta) J_l(k_{\perp 1} r) \qquad \hat{E}_z(r,\theta) = A_2 e^{il\theta} K_l(\gamma_2 r)$$

$l$ = integer

$$\hat{H}_z = B_1 \exp(i l \theta) J_l(k_{\perp 1} r) \qquad \hat{H}_z(r,\theta) = B_2 e^{il\theta} K_l(\gamma_2 r)$$

Ordinary Bessel functions
Regular at origin

Modified Bessel functions
Goes to zero at infinity

# Unknowns and Boundary Conditions

$$\hat{E}_z = A_1 \exp(il\theta) J_l(k_{\perp 1} r) \qquad \hat{E}_z(r,\theta) = A_2 e^{il\theta} K_l(\gamma_2 r)$$

$$\hat{H}_z = B_1 \exp(il\theta) J_l(k_{\perp 1} r) \qquad \hat{H}_z(r,\theta) = B_2 e^{il\theta} K_l(\gamma_2 r)$$

Unkowns $A_1, A_2, B_1, B_2$ & $\omega$ for given $k_z$

Boundary Conditions: at $r = a$

Tangential components of E and H continuous

$E_z, E_\theta, H_z, H_\theta$ — continuous at $r = a$

$H_z, E_z$ — continuous

$$A_1 J_l(k_{\perp 1} a) = A_2 K_l(\gamma_2 a)$$

$$B_1 J_l(k_{\perp 1} a) = B_2 K_l(\gamma_2 a)$$

# Continuity of Transverse Components

Transverse components in terms of axial components

$$\hat{E}_\perp = \frac{1}{k_\perp^2}\left[ i k_z \nabla_\perp \hat{E}_z - i\omega\mu \, \mathbf{e}_z \times \nabla_\perp \hat{H}_z \right\}$$

$$\hat{H}_\perp = \frac{1}{k_\perp^2}\left[ i k_z \nabla_\perp \hat{H}_z + i\omega\epsilon \, \mathbf{e}_z \times \nabla_\perp \hat{E}_z \right]$$

Region I

$$\hat{E}_\theta(a) = \frac{1}{k_{\perp 1}^2}\left\{ i k_z \left(\frac{i\ell}{a}\right) A_1 J_\ell(k_{\perp 1} a) - i\omega\mu B_1 k_{\perp 1} J'_\ell(k_{\perp 1} a) \right\}$$

$$H_\theta(r=a) = \frac{1}{k_{\perp 1}^2}\left\{ i k_z \left(\frac{i\ell}{a}\right) B_1 J_\ell(k_{\perp 1} a) + i\omega\epsilon k_{\perp 1} A_1 J'_\ell(k_{\perp 1} a) \right\}$$

Region II

$$\hat{E}_\theta(a) = -\frac{1}{\gamma_2^2}\left\{ i k_z \left(\frac{i\ell}{a}\right) A_2 K_\ell(\gamma_2 a) - i\omega\mu \gamma_2 B_2 K'_\ell(\gamma_2 a) \right\}$$

$$k_{\perp 2}^2 = -\gamma_2^2$$

$$H_\theta(a) = -\frac{1}{\gamma_2^2}\left\{ i k_z \left(\frac{i\ell}{a}\right) B_2 K_\ell(\gamma_2 a) + i\omega\epsilon \gamma_2 A_2 K'_\ell(\gamma_2 a) \right\}$$

# Solution

$$A_1 J_\ell(k_{\perp 1} a) = A_2 K_\ell(\gamma_2 a)$$

$$B_1 J_\ell(k_{\perp 1} a) = B_2 K_\ell(\gamma_2 a)$$

$$\hat{E}_\theta(a) = \frac{1}{k_{\perp 1}^2}\left\{ ik_z \left(\frac{i\ell}{a}\right) A_1 J_\ell(k_{\perp 1} a) - i\omega\mu B_1 k_{\perp 1} J_\ell'(k_{\perp 1} a)\right\}$$

$$H_\theta(r=a) = \frac{1}{k_{\perp 1}^2}\left\{ ik_z \left(\frac{i\ell}{a}\right) B_1 J_\ell(k_{\perp 1} a) + i\omega\epsilon k_{\perp 1} A_1 J_\ell'(k_{\perp 1} a)\right\}$$

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$$H_\theta(a) = -\frac{1}{\gamma_2^2}\left\{ ik_z \left(\frac{i\ell}{a}\right) B_2 K_\ell(\gamma_2 a) + i\omega\epsilon \gamma_2 A_2 K_\ell'(\gamma_2 a)\right\}$$

↓

Replace $A_2 = A_1 J_\ell / K_\ell$ $\quad B_2 = B_1 J_\ell / K_\ell$

$$\left[\begin{array}{c} 2\times 2 \\ \underline{\underline{M}} \end{array}\right]\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = 0$$

solution $\det(\underline{\underline{M}}) = 0$ determines $\omega(k)$