ENEE 681

Lectures 10-11

Guided waves

Why guided waves?

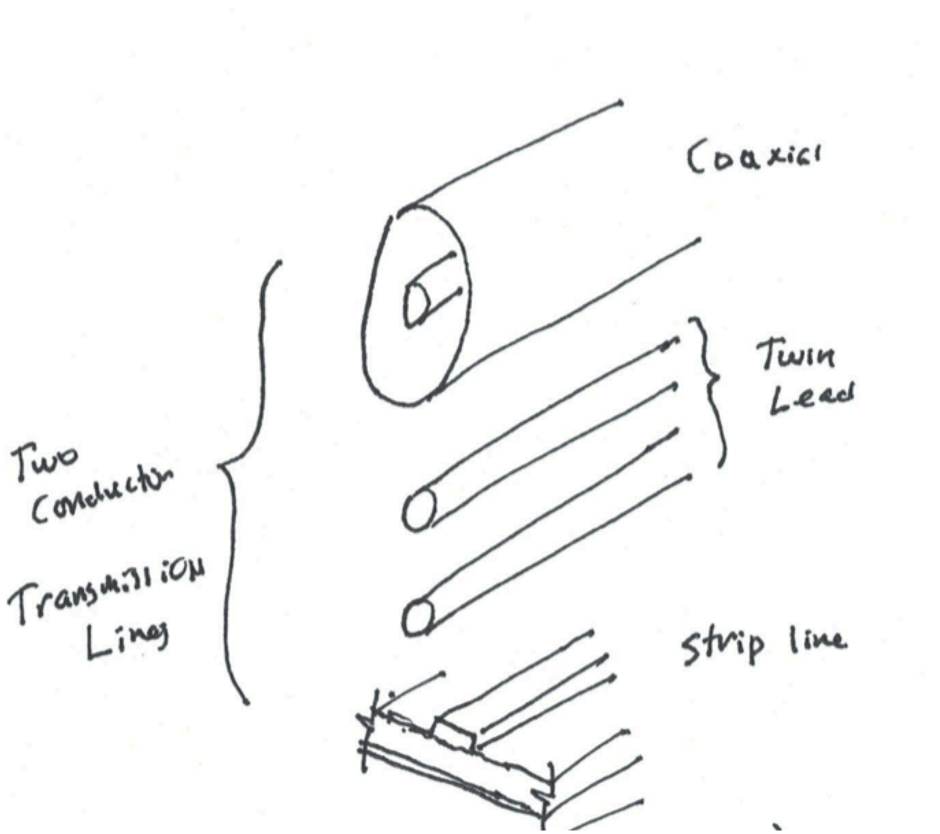
To overcome reduction of power due to diffraction

In the far field power density - $1/r^2$

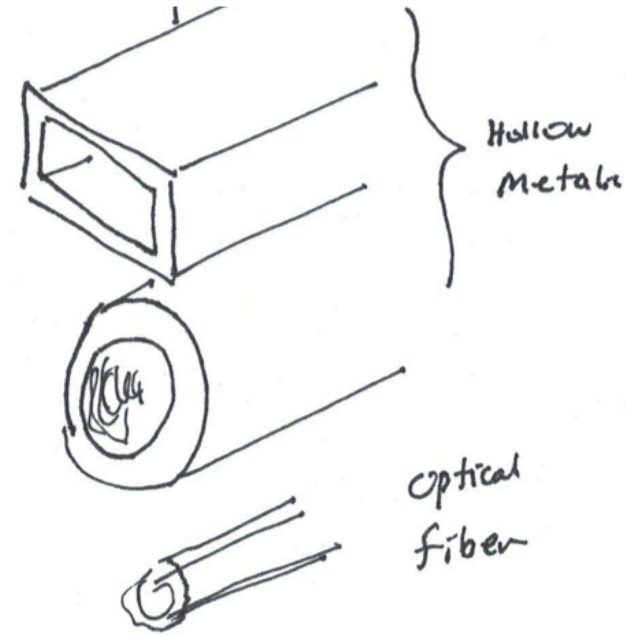
To avoid cross talk or interference between signals.

Desired waves are confined and other waves are shielded out.

Types of guiding structures

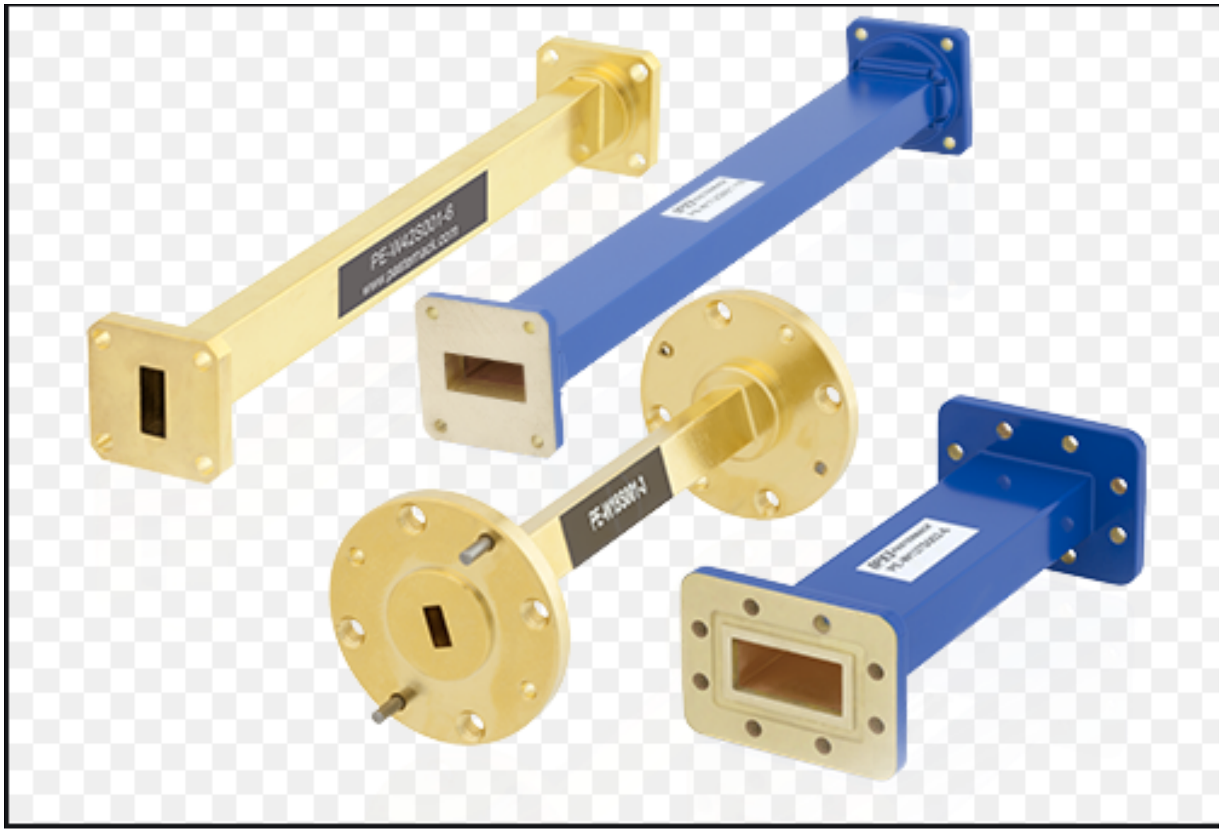


Transmission Lines



Single volume waveguides

Examples



Pasternak Enterprizes
https://www.pasternack.com/

A bundle of optical fibers

Wikipedia

General Features

Waveguides have propagation "modes".

- Modes have fields with specific structure in the transverse direction
- Mode structure is determined by the cross section shape of the WG and material in the case of optical fibers
- The dependence of fields along the axis of the guide is harmonic.

$$\mathbf{E} = \mathrm{Re}\left\{\hat{\mathbf{E}}(x,y)\exp\left[i\left(k_z z - \omega t\right)\right]\right\}$$

- The dispersion relation gives frequency as a function of axial wave number

$$\omega(k_z)$$

- There is an impedance associated with each mode that relates the electric to magnetic components of the wave. It can be used to calculate reflection coefficients

Relation between WG modes and plane waves

Start with the dispersion relation for plane waves in infinite homogeneous space.

$$\omega^2 \varepsilon\mu = \frac{\omega^2}{v^2} = k^2 = k_x^2 + k_y^2 + k_z^2$$

Call $k_\perp^2 = k_x^2 + k_y^2$

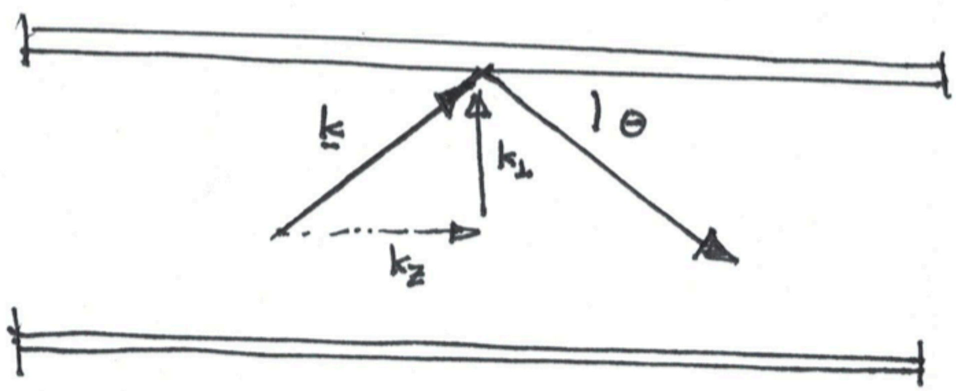
$$\frac{\omega^2}{v^2} = k_\perp^2 + k_z^2$$

A mode has a fixed value of $k_\perp^2$ determined by the size and shape of the WG cross section.

A mode will propagate, k_z – real, if $\omega^2 > k_\perp^2 v^2 \equiv \omega_c^2$

Cut off frequency

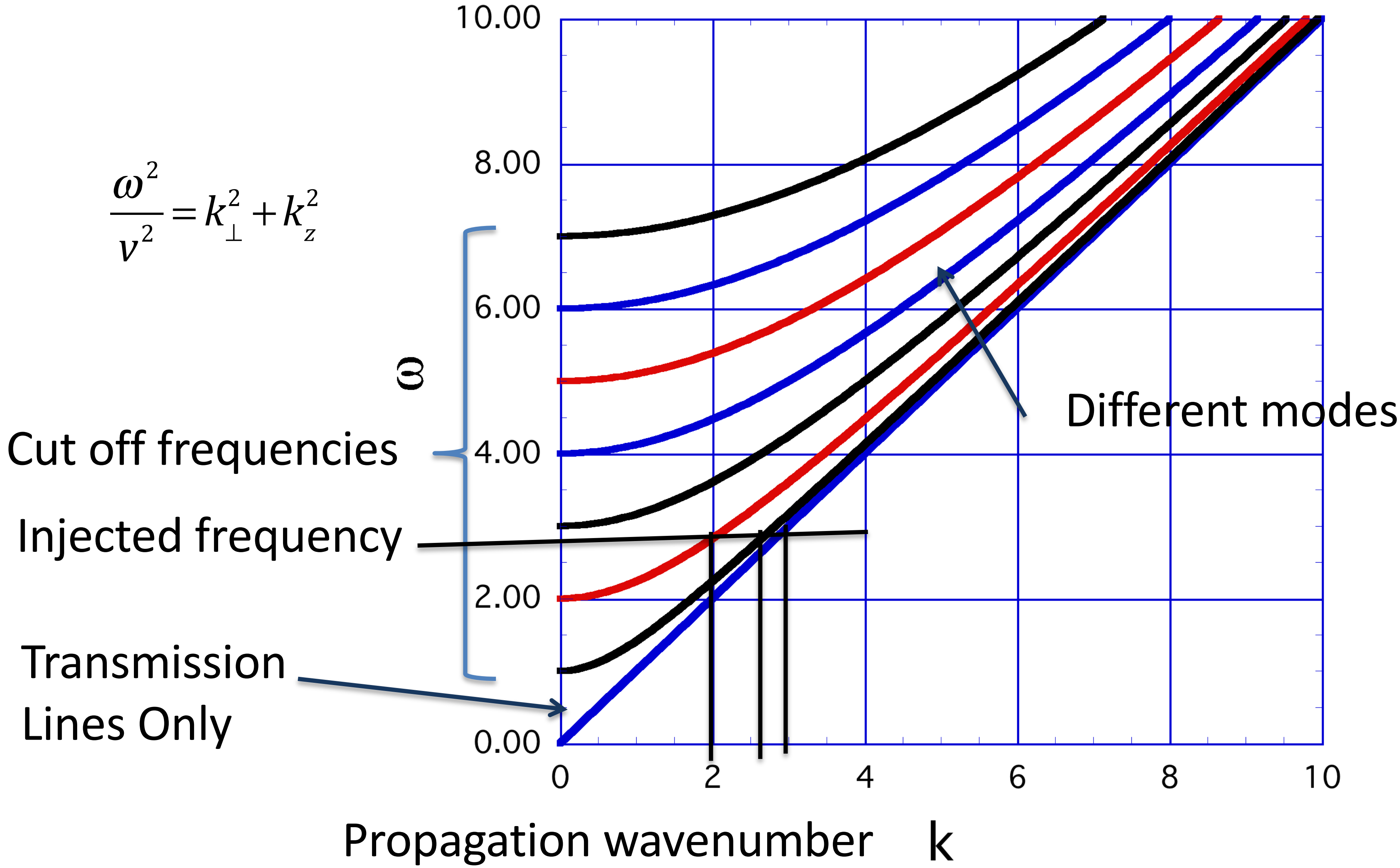
Plane waves in a WG



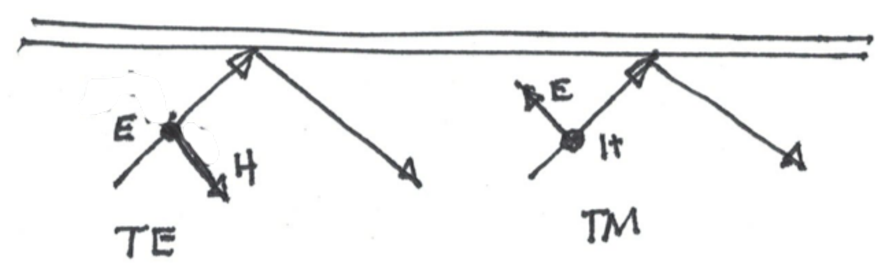
$$\frac{\omega^2}{v^2} = k_\perp^2 + k_z^2$$

WG Dispersion Relations

$$\frac{\omega^2}{v^2} = k_\perp^2 + k_z^2$$



Polarizations



TE – Transverse Electric

$\mathbf{E}=(E_x, E_y, E_z=0)$
$\mathbf{H}=(H_x, H_y, H_z)$

TM – Transverse Magnetic

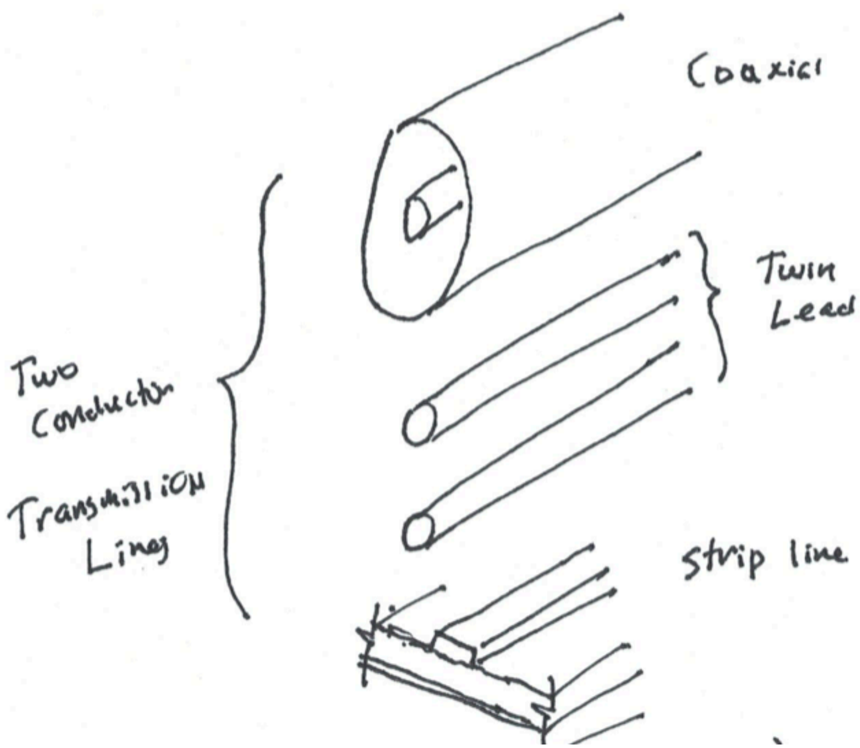
$\mathbf{E}=(E_x, E_y, E_z)$
$\mathbf{H}=(H_x, H_y, H_z=0)$

TEM – Transverse Electric and Magnetic

$\mathbf{E}=(E_x, E_y, E_z=0)$
$\mathbf{H}=(H_x, H_y, H_z=0)$

Transmission Lines

Two Conductors



Characterization

L' Inductance per unit length

C' Capacitance per unit length

$$v = \frac{1}{\sqrt{L'C'}}$$ Propagatein speed

$$Z_0 = \sqrt{\frac{L'}{C'}}$$ Characteristic Impedance

Compare with waves in free space

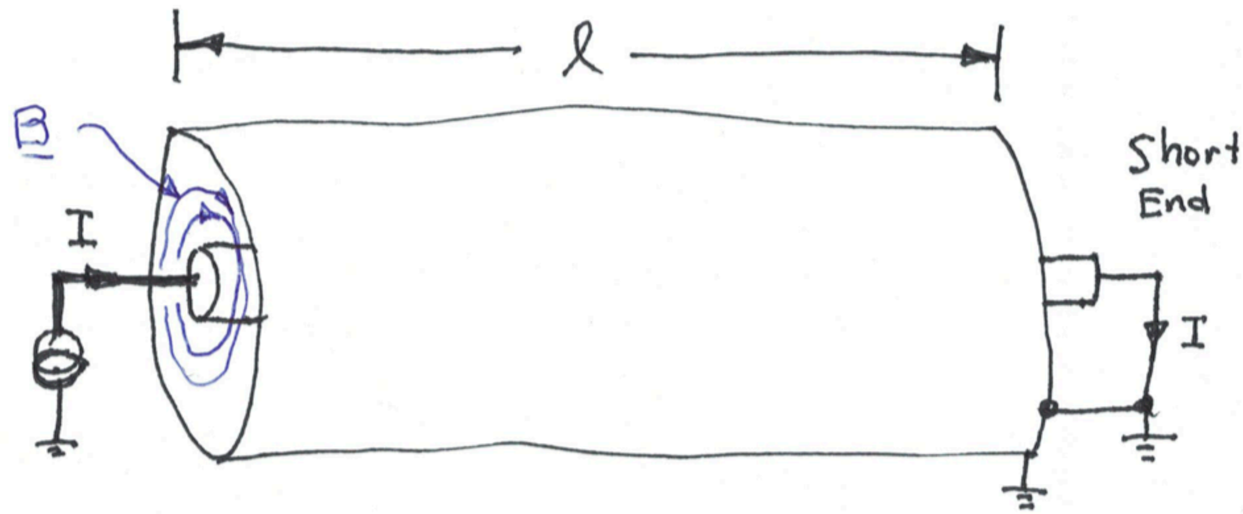
$$v = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

L’ and C’

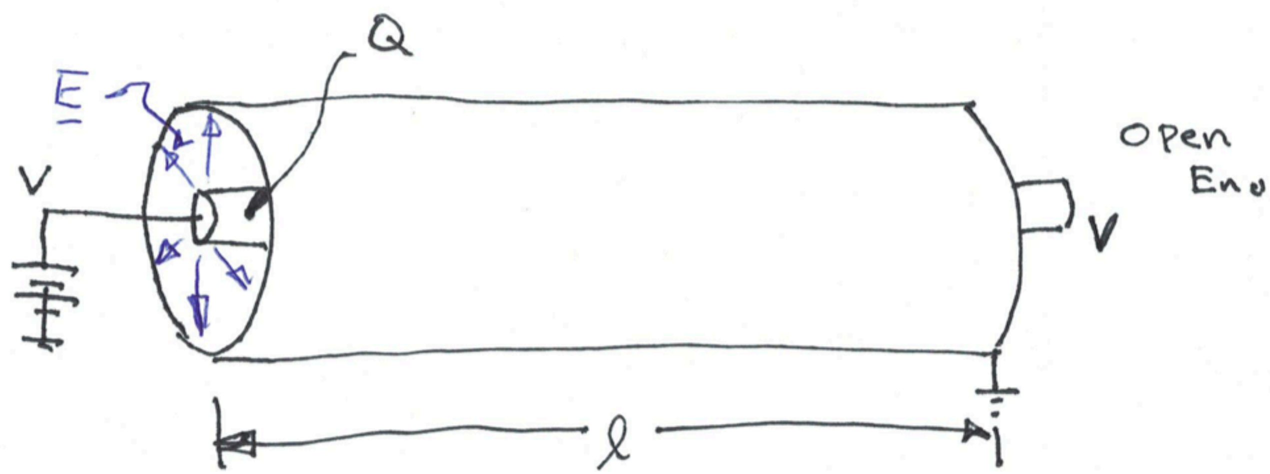
Short End

Inductance = $L'l$

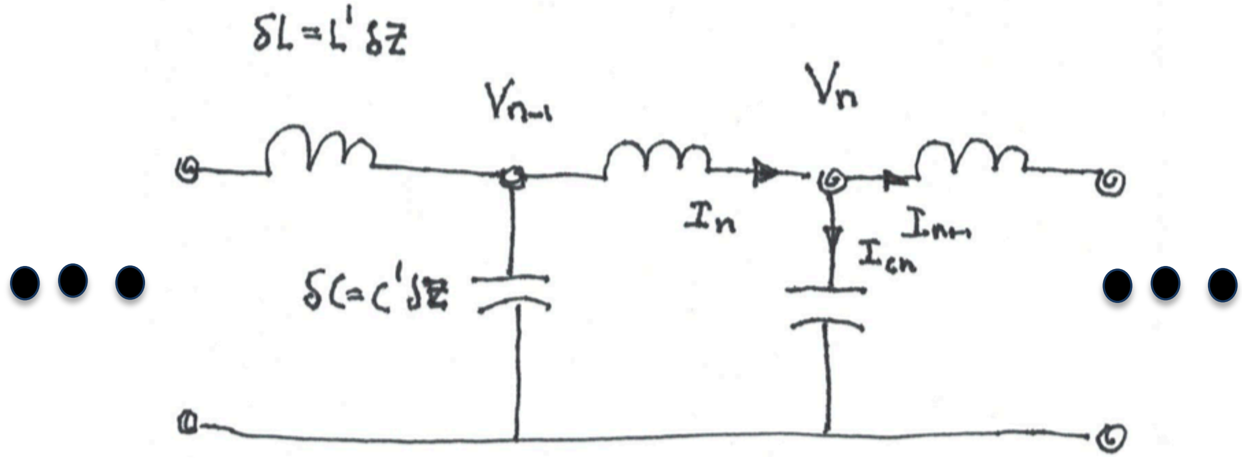


Open End

Capacitance =$C'l$



Circuit Model



KCL at node-n $I_n = I_{n+1} + I_{cn}$

$\delta C \frac{\partial V_n}{\partial t} = I_{cn} = I_n - I_{n+1}$

Inductor-n $V_{n-1} - V_n = \delta L \frac{\partial I_n}{\partial t}$

$$V(z-\delta z) - V(z) = \delta z L' \frac{\partial}{\partial t} I(z) \qquad \delta z C' \frac{\partial}{\partial t} V(z) = I(z) - I(z+\delta z)$$

$$-\frac{\partial}{\partial z} V(z) = L' \frac{\partial}{\partial t} I(z) \qquad C' \frac{\partial}{\partial t} V(z) = -\frac{\partial}{\partial z} I(z)$$

1D Wave Equation

$$-\frac{\partial}{\partial z}V(z)=L'\frac{\partial}{\partial t}I(z) \qquad C'\frac{\partial}{\partial t}V(z)=-\frac{\partial}{\partial z}I(z)$$

$$C'\frac{\partial^2}{\partial t^2}V(z,t)=-\frac{\partial}{\partial z}\frac{\partial}{\partial t}I(z)=\frac{1}{L'}\frac{\partial^2}{\partial z^2}V(z,t) \longrightarrow C'L'\frac{\partial^2}{\partial t^2}V(z,t)=\frac{\partial^2}{\partial z^2}V(z,t) \qquad v^2=\frac{1}{C'L'}$$

Solution:

$$V(z,t)=V_+(z-vt)+V_-(z+vt)$$

$$I(z,t)=\frac{1}{Z_0}\left(V_+(z-vt)-V_-(z+vt)\right)$$

Impedance:

$$Z_0=\sqrt{\frac{L'}{C'}}$$

Power in z-direction

$$P=V(z,t)I(z,t)=\frac{1}{Z_0}\left(\left|V_+(z-vt)\right|^2-\left|V_-(z+vt)\right|^2\right)$$

Problem

$$V(z,t)=V_+(z-vt)+V_-(z+vt)$$

$$I(z,t)=\frac{1}{Z_0}\left(V_+(z-vt)-V_-(z+vt)\right)$$

Impedance:

$$Z_0=\sqrt{\frac{L'}{C'}}$$

A 50 Ohm transmission line is terminated with a 32 pf capacitor.

What is the reflection coefficient for a 100 MHz signal?

Repeat for 10 MHz

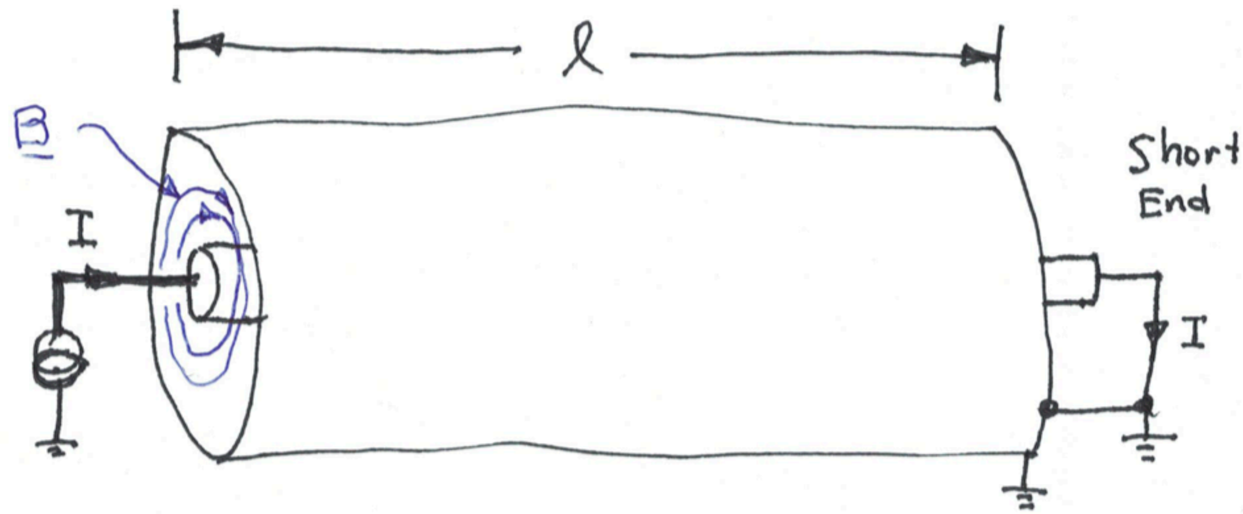
L' and C' for coaxial line

$$B_\theta(r) = \frac{\mu I}{2\pi r}$$

$$\psi = \int_a^b dr \int_0^l dz \frac{\mu I}{2\pi r} = \frac{\mu I l}{2\pi} \ln\left(\frac{b}{a}\right)$$

Inductance/length

$$L' = \psi / Il = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right)$$

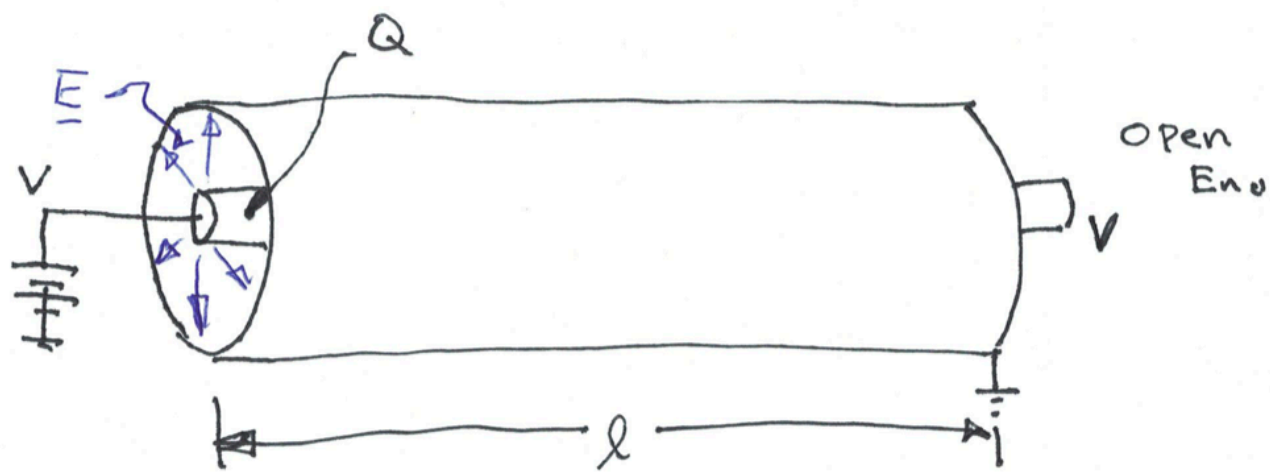


$$E_r(r) = \frac{Q/l}{2\pi\varepsilon r}$$

$$V = \int_a^b dr E_r = \frac{Q/l}{2\pi\varepsilon} \ln\left(\frac{b}{a}\right)$$

Capacitance/length

$$C' = \frac{Q/l}{V} = 2\pi\varepsilon / \ln\left(\frac{b}{a}\right)$$



Coaxial Transmission Line

Inductance/length

$$L' = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right)$$

Capacitance/length

$$C' = 2\pi\varepsilon / \ln\left(\frac{b}{a}\right)$$

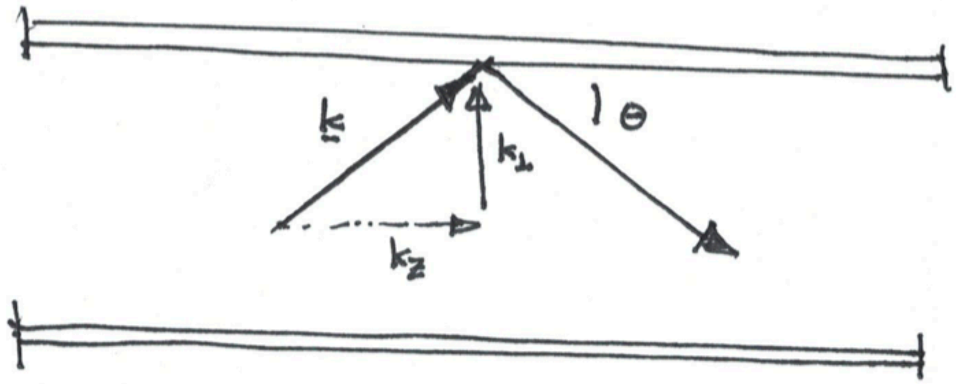
Propagation Speed

$$v^2 = 1/\left(L'C'\right) = 1/\left(\mu\varepsilon\right)$$

Characteristic Impedance

$$Z_0 = \sqrt{L'/C'} = \sqrt{\frac{\mu}{\varepsilon}}\left[\frac{1}{2\pi}\ln\left(\frac{b}{a}\right)\right]$$

Metal Waveguides



$$\frac{k_z}{k} = \cos\theta$$

$$k_z^2 = \frac{\omega^2}{v^2} - k_\perp^2 = \frac{\omega^2 - \omega_c^2}{v^2}$$

Cut-off freq.

$$k^2 = \frac{\omega^2}{v^2}$$

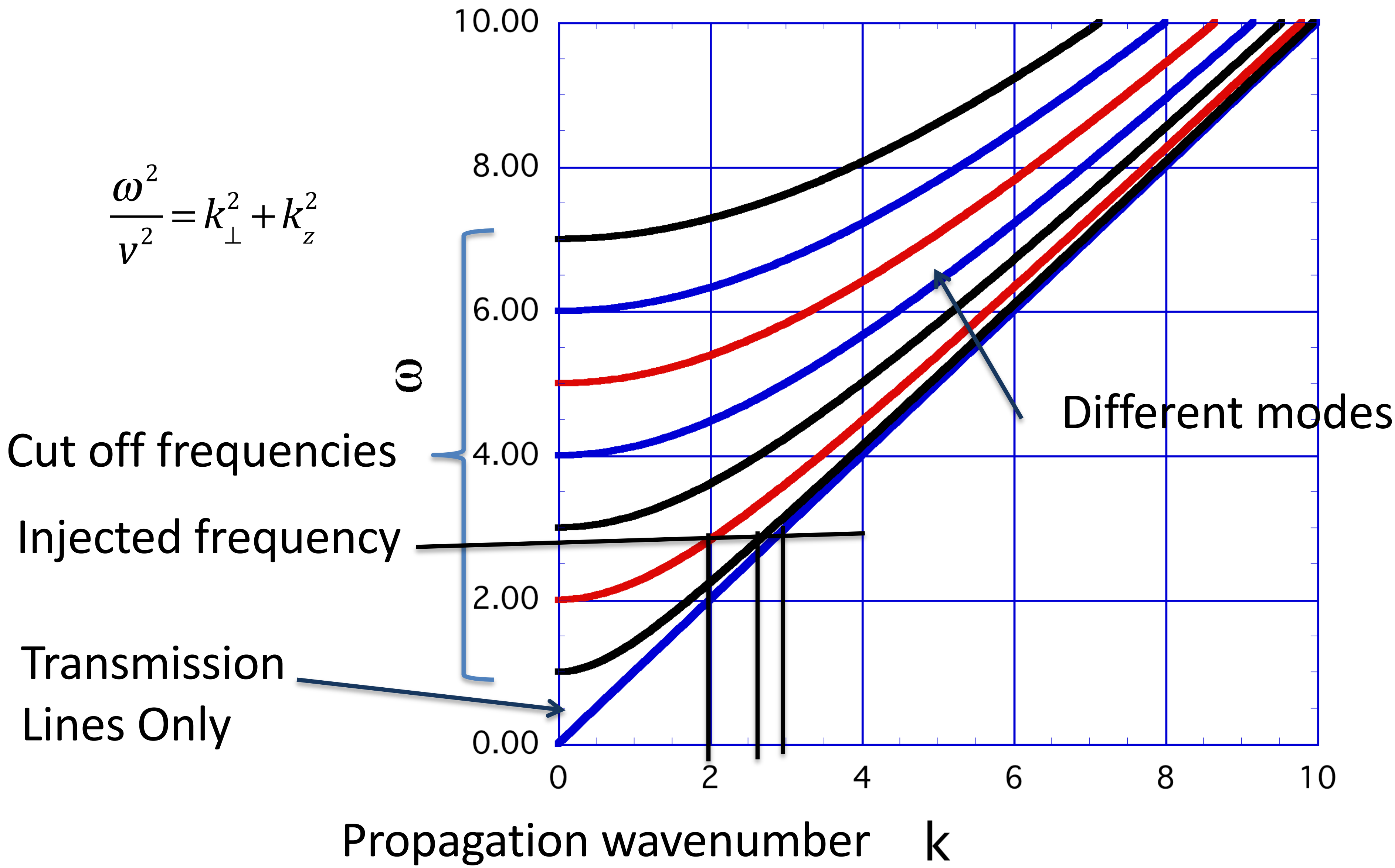
$$\frac{k_z}{k} = \left(1 - \frac{\omega_c^2}{\omega^2}\right)^{1/2} = \cos\theta$$

$$\mathbf{E} = \mathrm{Re}\left\{\hat{\mathbf{E}}(x,y)\exp\left[i\left(k_z z - \omega t\right)\right]\right\}$$

$$\mathbf{H} = \mathrm{Re}\left\{\hat{\mathbf{H}}(x,y)\exp\left[i\left(k_z z - \omega t\right)\right]\right\}$$

WG Dispersion Relations

$$\frac{\omega^2}{v^2} = k_\perp^2 + k_z^2$$



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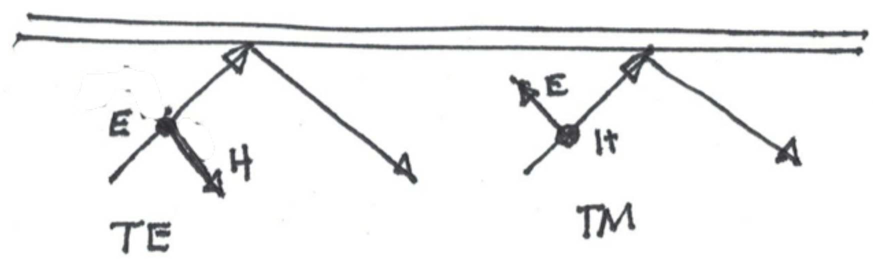

TE – Transverse Electric

$\mathbf{E}=(E_x, E_y, E_z=0)$
$\mathbf{H}=(H_x, H_y, H_z)$

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$\mathbf{H}=(H_x, H_y, H_z=0)$

Procedure

1. Write down Maxwell's equations in component form.
2. Introduce phasor representation.
3. Solve for transverse field components in terms of axial (z) components.

$$\hat{\mathbf{E}}_\perp = \frac{i}{(\omega/v)^2 - k_z^2}\left[k_z \nabla_\perp \hat{E}_z - \omega\mu\hat{\mathbf{z}} \times \nabla_\perp \hat{H}_z\right]$$

$$\hat{\mathbf{H}}_\perp = \frac{i}{(\omega/v)^2 - k_z^2}\left[k_z \nabla_\perp \hat{H}_z + \omega\varepsilon\hat{\mathbf{z}} \times \nabla_\perp \hat{E}_z\right]$$

4. Obtain wave equations for axial components.

$$\frac{\partial^2 \hat{H}_z}{\partial x^2} + \frac{\partial^2 \hat{H}_z}{\partial y^2} = -\left((\omega/v)^2 - k_z^2\right)\hat{H}_z \qquad \frac{\partial^2 \hat{E}_z}{\partial x^2} + \frac{\partial^2 \hat{E}_z}{\partial y^2} = -\left((\omega/v)^2 - k_z^2\right)\hat{E}_z$$

5. Satisfy BC's on metal wall. $\hat{E}_z\Big|_{wall} = 0, \quad \mathbf{n}\cdot\nabla_\perp \hat{H}_z\Big|_{wall} = 0$

Cartesian Components of ME

$$\mathbf{E} = \mathrm{Re}\left\{\hat{\mathbf{E}}(x,y)\exp\left[i\left(k_z z - \omega t\right)\right]\right\}$$

$$\mathbf{H} = \mathrm{Re}\left\{\hat{\mathbf{H}}(x,y)\exp\left[i\left(k_z z - \omega t\right)\right]\right\}$$

$$\frac{\partial}{\partial t} \Rightarrow -i\omega, \quad \frac{\partial}{\partial z} \Rightarrow ik_z$$

Phasors

$$\nabla \times \vec{\mathbf{E}} = -\mu \frac{\partial \vec{\mathbf{H}}}{\partial t}$$

$$\frac{\partial \hat{E}_z}{\partial y} - ik_z \hat{E}_y = i\omega\mu \hat{H}_x$$

$$ik_z \hat{E}_x - \frac{\partial \hat{E}_z}{\partial x} = i\omega\mu \hat{H}_y$$

$$\frac{\partial \hat{E}_y}{\partial x} - \frac{\partial \hat{E}_x}{\partial y} = i\omega\mu \hat{H}_z$$

$$\nabla \times \vec{\mathbf{H}} = \varepsilon \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

$$\frac{\partial \hat{H}_z}{\partial y} - ik_z \hat{H}_y = -i\omega\varepsilon \hat{E}_x$$

$$ik_z \hat{H}_x - \frac{\partial \hat{H}_z}{\partial x} = -i\omega\varepsilon \hat{E}_y$$

$$\frac{\partial \hat{H}_y}{\partial x} - \frac{\partial \hat{H}_x}{\partial y} = -i\omega\varepsilon \hat{E}_z$$

Solve for Transverse Field Components

$$\nabla \times \vec{\mathbf{E}} = -\mu \frac{\partial \vec{\mathbf{H}}}{\partial t} \qquad \nabla \times \vec{\mathbf{H}} = \varepsilon \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

$$\frac{\partial \hat{E}_z}{\partial y} - ik_z \hat{E}_y = i\omega\mu \hat{H}_x \qquad \frac{\partial \hat{H}_z}{\partial y} - ik_z \hat{H}_y = -i\omega\varepsilon \hat{E}_x$$

$$ik_z \hat{E}_x - \frac{\partial \hat{E}_z}{\partial x} = i\omega\mu \hat{H}_y \qquad ik_z \hat{H}_x - \frac{\partial \hat{H}_z}{\partial x} = -i\omega\varepsilon \hat{E}_y$$

$$\frac{\partial \hat{E}_y}{\partial x} - \frac{\partial \hat{E}_x}{\partial y} = i\omega\mu \hat{H}_z \qquad \frac{\partial \hat{H}_y}{\partial x} - \frac{\partial \hat{H}_x}{\partial y} = -i\omega\varepsilon \hat{E}_z$$

Solve algebraic equations for E_x, E_y, H_x, H_y in terms of derivatives of E_z and H_z

Transverse Fields

$$\hat{E}_x = \frac{i}{(\omega / v)^2 - k_z^2}\left[k_z \frac{\partial \hat{E}_z}{\partial x} + \omega\mu \frac{\partial \hat{H}_z}{\partial y} \right]$$

$$\hat{E}_y = \frac{i}{(\omega / v)^2 - k_z^2}\left[k_z \frac{\partial \hat{E}_z}{\partial y} - \omega\mu \frac{\partial \hat{H}_z}{\partial x} \right]$$

$$\hat{H}_x = \frac{i}{(\omega / v)^2 - k_z^2}\left[k_z \frac{\partial \hat{H}_z}{\partial x} - \omega\varepsilon \frac{\partial \hat{E}_z}{\partial y} \right]$$

$$\hat{H}_y = \frac{i}{(\omega / v)^2 - k_z^2}\left[k_z \frac{\partial \hat{H}_z}{\partial y} + \omega\varepsilon \frac{\partial \hat{E}_z}{\partial x} \right]$$

$$\hat{\mathbf{E}}_\perp = \frac{i}{(\omega / v)^2 - k_z^2}\left[k_z \nabla_\perp \hat{E}_z - \omega\mu \hat{\mathbf{z}} \times \nabla_\perp \hat{H}_z \right]$$

$$\hat{\mathbf{H}}_\perp = \frac{i}{(\omega / v)^2 - k_z^2}\left[k_z \nabla_\perp \hat{H}_z + \omega\varepsilon \hat{\mathbf{z}} \times \nabla_\perp \hat{E}_z \right]$$

Plug into remaining two components

$$\nabla \times \vec{\mathbf{E}} = -\mu \frac{\partial \vec{\mathbf{H}}}{\partial t}$$

$$\frac{\partial \hat{E}_z}{\partial y} - ik\hat{E}_y = i\omega\mu \hat{H}_x$$

$$ik\hat{E}_x - \frac{\partial \hat{E}_z}{\partial x} = i\omega\mu \hat{H}_y$$

$$\frac{\partial \hat{E}_y}{\partial x} - \frac{\partial \hat{E}_x}{\partial y} = i\omega\mu \hat{H}_z$$

$$\nabla \times \vec{\mathbf{H}} = \varepsilon \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

$$\frac{\partial \hat{H}_z}{\partial y} - ik\hat{H}_y = -i\omega\varepsilon \hat{E}_x$$

$$ik\hat{H}_x - \frac{\partial \hat{H}_z}{\partial x} = -i\omega\varepsilon \hat{E}_y$$

$$\frac{\partial \hat{H}_y}{\partial x} - \frac{\partial \hat{H}_x}{\partial y} = -i\omega\varepsilon \hat{E}_z$$

$$\frac{\partial \hat{E}_y}{\partial x} - \frac{\partial \hat{E}_x}{\partial y} = i\omega\mu \hat{H}_z \qquad \frac{\partial \hat{H}_y}{\partial x} - \frac{\partial \hat{H}_x}{\partial y} = -i\omega\varepsilon \hat{E}_z$$

$$\hat{E}_x = \frac{i}{(\omega / v)^2 - k_z^2}\left[k_z \frac{\partial \hat{E}_z}{\partial x} + \omega\mu \frac{\partial \hat{H}_z}{\partial y} \right] \qquad \hat{H}_x = \frac{i}{(\omega / v)^2 - k_z^2}\left[k_z \frac{\partial \hat{H}_z}{\partial x} - \omega\varepsilon \frac{\partial \hat{E}_z}{\partial y} \right]$$

$$\hat{E}_y = \frac{i}{(\omega / v)^2 - k_z^2}\left[k_z \frac{\partial \hat{E}_z}{\partial y} - \omega\mu \frac{\partial \hat{H}_z}{\partial x} \right] \qquad \hat{H}_y = \frac{i}{(\omega / v)^2 - k_z^2}\left[k_z \frac{\partial \hat{H}_z}{\partial y} + \omega\varepsilon \frac{\partial \hat{E}_z}{\partial x} \right]$$

$$\frac{-i\omega\mu}{(\omega / v)^2 - k_z^2}\left[\frac{\partial^2 \hat{H}_z}{\partial x^2} + \frac{\partial^2 \hat{H}_z}{\partial y^2} \right] = i\omega\mu \hat{H}_z \qquad \frac{i}{(\omega / v)^2 - k_z^2}\frac{\omega}{v^2}\left[\frac{\partial^2 \hat{E}_z}{\partial x^2} + \frac{\partial^2 \hat{E}_z}{\partial y^2} \right] = -i\omega\varepsilon \hat{E}_z$$

$$\frac{\partial^2 \hat{H}_z}{\partial x^2} + \frac{\partial^2 \hat{H}_z}{\partial y^2} = -\left((\omega / v)^2 - k_z^2 \right) \hat{H}_z \qquad \frac{\partial^2 \hat{E}_z}{\partial x^2} + \frac{\partial^2 \hat{E}_z}{\partial y^2} = -\left((\omega / v)^2 - k_z^2 \right) \hat{E}_z$$

Wave Equations

$$\frac{\partial^2 \hat{H}_z}{\partial x^2} + \frac{\partial^2 \hat{H}_z}{\partial y^2} = -\left((\omega / v)^2 - k_z^2 \right) \hat{H}_z \qquad \frac{\partial^2 \hat{E}_z}{\partial x^2} + \frac{\partial^2 \hat{E}_z}{\partial y^2} = -\left((\omega / v)^2 - k_z^2 \right) \hat{E}_z$$

It is to be expected that we arrive at scalar wave equations for the z-component of the electric and magnetic fields. The solutions in general are superpositions of plane waves. But, we haven’t accounted for the metal boundaries yet.

The tangential components of the electric field should vanish on the conducting boundary.

Boundary conditions

$$\frac{\partial^2 \hat{H}_z}{\partial x^2} + \frac{\partial^2 \hat{H}_z}{\partial y^2} = -\left((\omega / v)^2 - k_z^2\right)\hat{H}_z \qquad \frac{\partial^2 \hat{E}_z}{\partial x^2} + \frac{\partial^2 \hat{E}_z}{\partial y^2} = -\left((\omega / v)^2 - k_z^2\right)\hat{E}_z$$

Tangential E vanish on conductor

Choice #1 $\quad \hat{E}_z\Big|_B = 0, \qquad \hat{H}_z = 0$ everywhere

E_x, E_y normal to boundary

$$\hat{E}_x = \frac{i}{(\omega / v)^2 - k_z^2}\left[k\frac{\partial \hat{E}_z}{\partial x}\right]$$

$$\hat{E}_y = \frac{i}{(\omega / v)^2 - k_z^2}\left[k\frac{\partial \hat{E}_z}{\partial y}\right]$$

TM Modes

Boundary Conditions

$$\frac{\partial^2 \hat{H}_z}{\partial x^2} + \frac{\partial^2 \hat{H}_z}{\partial y^2} = -\left((\omega / v)^2 - k_z^2\right)\hat{H}_z \qquad \frac{\partial^2 \hat{E}_z}{\partial x^2} + \frac{\partial^2 \hat{E}_z}{\partial y^2} = -\left((\omega / v)^2 - k_z^2\right)\hat{E}_z$$

Tangential E vanish on conductor

Choice #2 $\quad \mathbf{n}_\perp \cdot \nabla \hat{H}_z \Big|_B = 0, \qquad \hat{E}_z = 0 \quad$ everywhere

Ex, Ey normal to boundary

$$\hat{E}_x = \frac{i}{(\omega / v)^2 - k_z^2}\left[+\omega\mu \frac{\partial \hat{H}_z}{\partial y}\right]$$

$$\hat{E}_y = \frac{i}{(\omega / v)^2 - k_z^2}\left[-\omega\mu \frac{\partial \hat{H}_z}{\partial x}\right]$$

TE Modes

TM Modes

$$\frac{\partial^2 \hat{E}_z}{\partial x^2} + \frac{\partial^2 \hat{E}_z}{\partial y^2} = -\left((\omega / v)^2 - k_z^2\right)\hat{E}_z \qquad \hat{E}_z\Big|_B = 0$$

This is an eigenvalue equation. It can only be satisfied for a discrete set of values,

$$(\omega / v)^2 - k_z^2 = k_{\perp n}^2, \qquad n = 1,2,3,\ldots$$

https://www.acs.psu.edu/drussell/Demos/MembraneCircle/Circle.html

Impedance

$$\hat{\mathbf{E}}_\perp = \frac{k_z}{\omega\varepsilon}\hat{\mathbf{H}}_\perp \times \hat{\mathbf{z}} = \frac{k_z v}{\omega}\sqrt{\frac{\mu}{\varepsilon}}\hat{\mathbf{H}}_\perp \times \hat{\mathbf{z}}$$

$$Z = \cos\theta\sqrt{\frac{\mu}{\varepsilon}}, \quad \cos\theta = \frac{k_z v}{\omega}$$

$$\hat{E}_x = \frac{i}{(\omega / v)^2 - k_z^2}\left[k\frac{\partial \hat{E}_z}{\partial x}\right]$$

$$\hat{E}_y = \frac{i}{(\omega / v)^2 - k_z^2}\left[k\frac{\partial \hat{E}_z}{\partial y}\right]$$

$$\hat{H}_x = \frac{i}{(\omega / v)^2 - k_z^2}\left[-\omega\varepsilon\frac{\partial \hat{E}_z}{\partial y}\right]$$

$$\hat{H}_y = \frac{i}{(\omega / v)^2 - k_z^2}\left[\omega\varepsilon\frac{\partial \hat{E}_z}{\partial x}\right]$$

TE Modes

$$\frac{\partial^2 \hat{H}_z}{\partial x^2} + \frac{\partial^2 \hat{H}_z}{\partial y^2} = -\left((\omega / v)^2 - k_z^2 \right) \hat{H}_z \qquad \mathbf{n}_\perp \cdot \nabla \hat{H}_z \Big|_B = 0$$

This is an eigenvalue equation. It can only be satisfied for a discrete set of values, in general different from TM

$$(\omega / v)^2 - k_z^2 = k_{\perp n}^2, \qquad n = 1,2,3,...$$

Impedance

$$\hat{\mathbf{E}}_\perp = \frac{\omega \mu}{k_z} \hat{\mathbf{H}}_\perp \times \hat{\mathbf{z}} = \frac{\omega}{k_z v} \sqrt{\frac{\mu}{\varepsilon}} \hat{\mathbf{H}}_\perp \times \hat{\mathbf{z}}$$

$$Z = \frac{1}{\cos\theta} \sqrt{\frac{\mu}{\varepsilon}}, \quad \cos\theta = \frac{k_z v}{\omega}$$

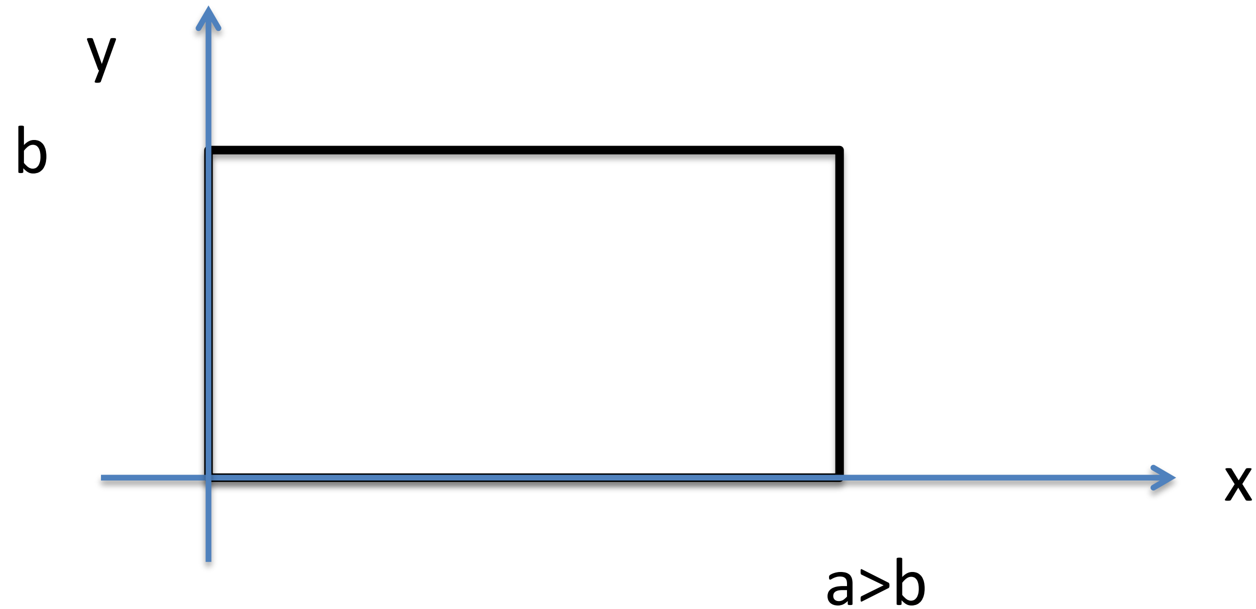
$$\hat{E}_x = \frac{i}{(\omega / v)^2 - k_z^2} \left[\omega \mu \frac{\partial \hat{H}_z}{\partial y} \right]$$

$$\hat{E}_y = \frac{i}{(\omega / v)^2 - k_z^2} \left[-\omega \mu \frac{\partial \hat{H}_z}{\partial x} \right]$$

$$\hat{H}_x = \frac{i}{(\omega / v)^2 - k_z^2} \left[k_z \frac{\partial \hat{H}_z}{\partial x} \right]$$

$$\hat{H}_y = \frac{i}{(\omega / v)^2 - k_z^2} \left[k_z \frac{\partial \hat{H}_z}{\partial y} \right]$$

Modes of a Rectangular WG



TM$_{nm}$: $\hat{E}_z = E_0 \sin(k_x x)\sin(k_y y) \quad k_x = \frac{n\pi}{a},\ \ k_y = \frac{m\pi}{b}: \ \ n,m = 1,2,3,....$

TE$_{nm}$: $\hat{H}_z = H_0 \cos(k_x x)\cos(k_y y) \quad k_x = \frac{n\pi}{a},\ \ k_y = \frac{m\pi}{b}: \ \ n,m = 0^*,1,2,3,...$

* one or the other, but not both

Cut-Off frequencies

$$\omega_{c,n,m} = v\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

TE_{10} Lowest Cut-off Frequency

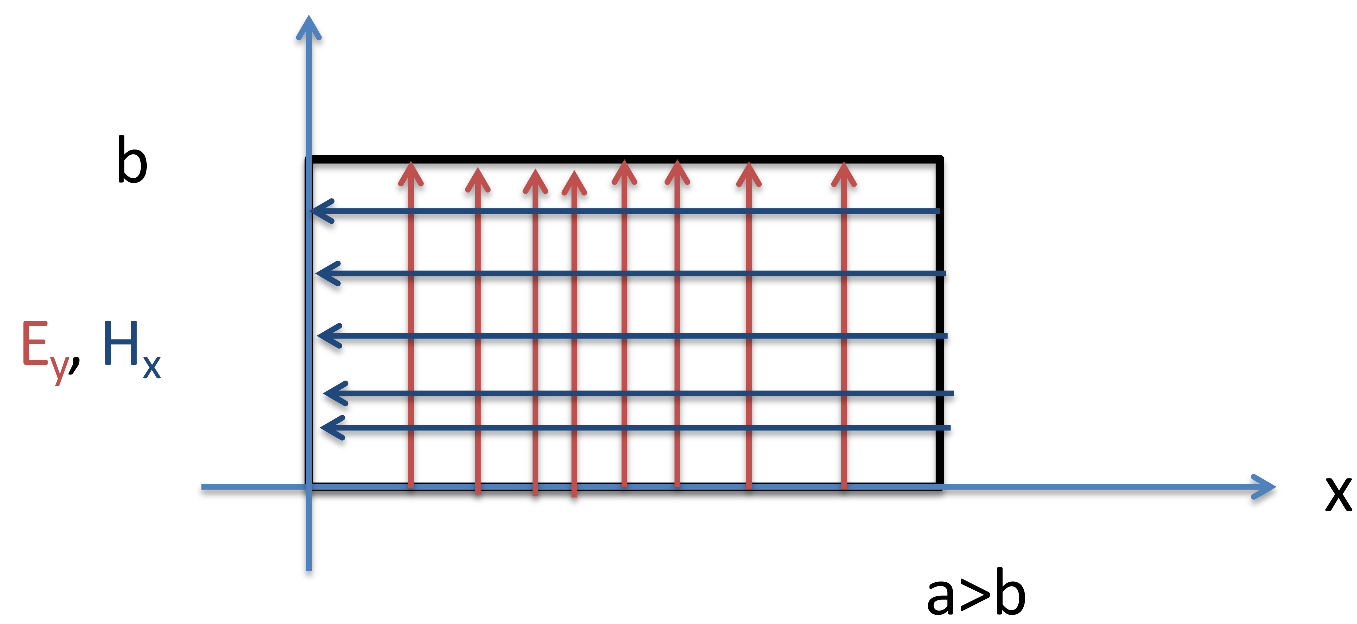
$$TE_{10}: \quad \hat{H}_z = H_0 \cos\left(\frac{\pi x}{a}\right)$$

$$\hat{E}_x = \frac{i}{(\omega/v)^2 - k_z^2}\left[\omega\mu \frac{\partial \hat{H}_z}{\partial y}\right] = 0$$

$$\hat{E}_y = \frac{\pi}{a}\frac{i\omega\mu H_0}{(\omega/v)^2 - k_z^2}\sin\left(\frac{\pi x}{a}\right)$$

Cut-Off frequencies

$$\omega_{c,10} = \frac{v\pi}{a}$$



Phase and Group Velocity

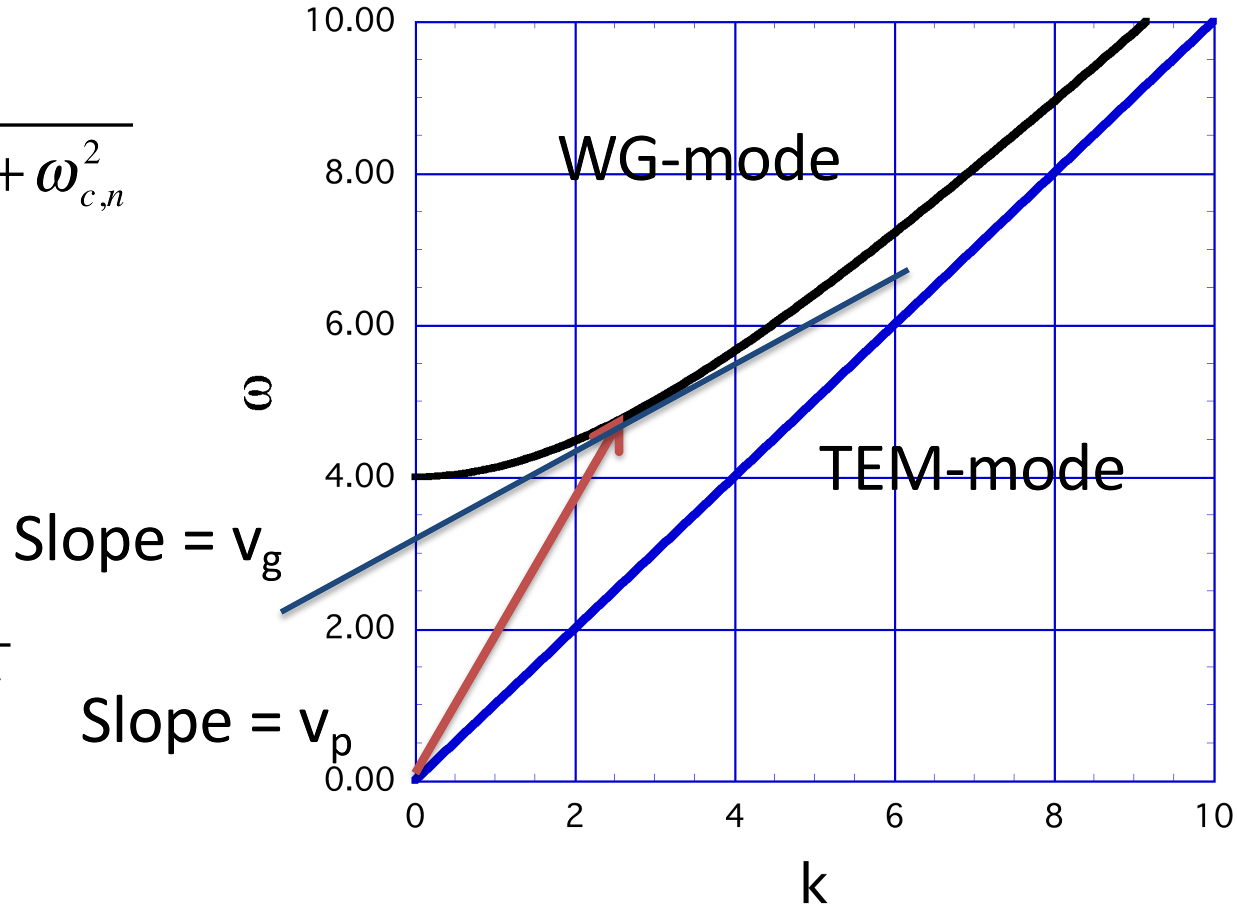
WG-mode

$$\omega = v\sqrt{k_z^2 + k_{\perp n}^2} = \sqrt{v^2 k_z^2 + \omega_{c,n}^2}$$

$$v_p = \frac{\omega}{k_z} = \frac{v}{\sqrt{1-\omega_{c,n}^2/\omega^2}}$$

$$v_g = \frac{\partial\omega}{\partial k_z} = v\sqrt{1-\omega_{c,n}^2/\omega^2}$$

$$v_g v_p = v^2$$

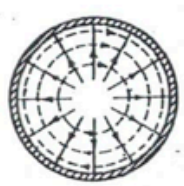
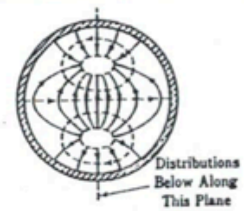
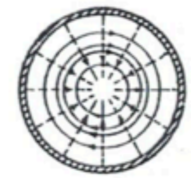
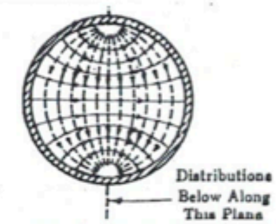
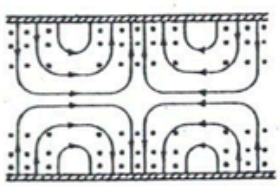
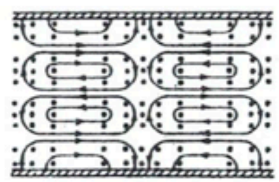
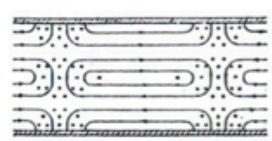
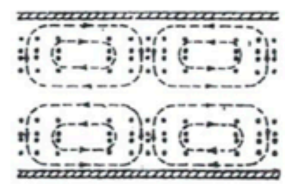
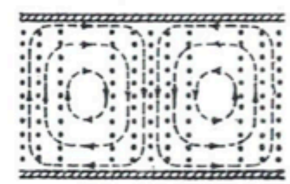


TEM-mode $v_p = v_g = v$

Circular Waveguides (RW&VD)

Table 8.9

Summary of wave types for circular guides[a]

Wave Type	TM_{01}	TM_{02}	TM_{11}	TE_{01}	TE_{11}
Field distributions in cross-sectional plane, at plane of maximum transverse fields					
Field distributions along guide					
Field components present	E_z, E_r, H_ϕ	E_z, E_r, H_ϕ	$E_z, E_r, E_\phi, H_r, H_\phi$	H_z, H_r, E_ϕ	$H_z, H_r, H_\phi, E_r, E_\phi$
p_{nl} or p'_{nl}	2.405	5.52	3.83	3.83	1.84
$(k_c)_{nl}$	$\frac{2.405}{a}$	$\frac{5.52}{a}$	$\frac{3.83}{a}$	$\frac{3.83}{a}$	$\frac{1.84}{a}$
$(\lambda_c)_{nl}$	$2.61a$	$1.14a$	$1.64a$	$1.64a$	$3.41a$
$(f_c)_{nl}$	$\frac{0.383}{a\sqrt{\mu\varepsilon}}$	$\frac{0.877}{a\sqrt{\mu\varepsilon}}$	$\frac{0.609}{a\sqrt{\mu\varepsilon}}$	$\frac{0.609}{a\sqrt{\mu\varepsilon}}$	$\frac{0.293}{a\sqrt{\mu\varepsilon}}$
Attenuation due to imperfect conductors	$\frac{R_s}{a\eta}\frac{1}{\sqrt{1-(f_c/f)^2}}$	$\frac{R_s}{a\eta}\frac{1}{\sqrt{1-(f_c/f)^2}}$	$\frac{R_s}{a\eta}\frac{1}{\sqrt{1-(f_c/f)^2}}$	$\frac{R_s}{a\eta}\frac{(f_c/f)^2}{\sqrt{1-(f_c/f)^2}}$	$\frac{R_s}{a\eta}\frac{1}{\sqrt{1-(f_c/f)^2}}\left[\left(\frac{f_c}{f}\right)^2+0.420\right]$

[a] Electric field lines are shown solid and magnetic field lines are dashed.

TEM Modes, Transmission line modes

$$\mathbf{E},\mathbf{H} = \mathrm{Re}\left\{\left(\hat{\mathbf{E}}(x,y),\hat{\mathbf{H}}(x,y)\right)\exp\left[i\left(k_z z-\omega t\right)\right]\right\}$$

$$\frac{\partial}{\partial t} \Rightarrow -i\omega, \quad \frac{\partial}{\partial z} \Rightarrow ik_z$$

$$\frac{\partial \hat{E}_z}{\partial y} - ik\hat{E}_y = i\omega\mu\hat{H}_x \qquad \frac{\partial \hat{H}_z}{\partial y} - ik\hat{H}_y = -i\omega\varepsilon\hat{E}_x$$

$$ik\hat{E}_x - \frac{\partial \hat{E}_z}{\partial x} = i\omega\mu\hat{H}_y \qquad ik\hat{H}_x - \frac{\partial \hat{H}_z}{\partial x} = -i\omega\varepsilon\hat{E}_y$$

$$\frac{\partial \hat{E}_y}{\partial x} - \frac{\partial \hat{E}_x}{\partial y} = i\omega\mu\hat{H}_z \qquad \frac{\partial \hat{H}_y}{\partial x} - \frac{\partial \hat{H}_x}{\partial y} = -i\omega\varepsilon\hat{E}_z$$

Suppose: $E_z=0$, $H_z=0$

TEM Modes, Transmission line modes

$$\mathbf{E},\mathbf{H} = \mathrm{Re}\left\{\left(\hat{\mathbf{E}}(x,y),\hat{\mathbf{H}}(x,y)\right)\exp\left[i\left(k_z z - \omega t\right)\right]\right\}$$

$$\frac{\partial}{\partial t} \Rightarrow -i\omega, \quad \frac{\partial}{\partial z} \Rightarrow ik_z$$

$$\frac{\partial \hat{E}_z}{\partial y} - ik\hat{E}_y = i\omega\mu\hat{H}_x \qquad \frac{\partial \hat{H}_z}{\partial y} - ik\hat{H}_y = -i\omega\varepsilon\hat{E}_x$$

$$ik\hat{E}_x - \frac{\partial \hat{E}_z}{\partial x} = i\omega\mu\hat{H}_y \qquad ik\hat{H}_x - \frac{\partial \hat{H}_z}{\partial x} = -i\omega\varepsilon\hat{E}_y$$

$$\frac{\partial \hat{E}_y}{\partial x} - \frac{\partial \hat{E}_x}{\partial y} = i\omega\mu\hat{H}_z \qquad \frac{\partial \hat{H}_y}{\partial x} - \frac{\partial \hat{H}_x}{\partial y} = -i\omega\varepsilon\hat{E}_z$$

Suppose: E_z=0, H_z=0

TEM Modes, Transmission line modes

$$\frac{\partial \hat{E}_y}{\partial x} - \frac{\partial \hat{E}_x}{\partial y} = 0$$

$$\hat{E}_x = -\frac{\partial \Phi}{\partial x}, \quad \hat{E}_y = -\frac{\partial \Phi}{\partial y}$$

$$\frac{\partial \hat{E}_x}{\partial x} + \frac{\partial \hat{E}_y}{\partial y} = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Phi = 0$$

Fields like a plane wave in z-direction

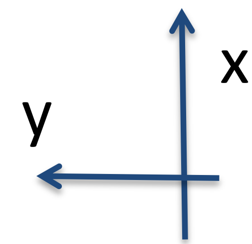
$$\hat{\mathbf{H}} = \sqrt{\frac{\varepsilon}{\mu}}\, \hat{\mathbf{z}} \times \hat{\mathbf{E}}, \quad \omega = k_z v = k_z / \sqrt{\varepsilon\mu}$$

Boundary condition

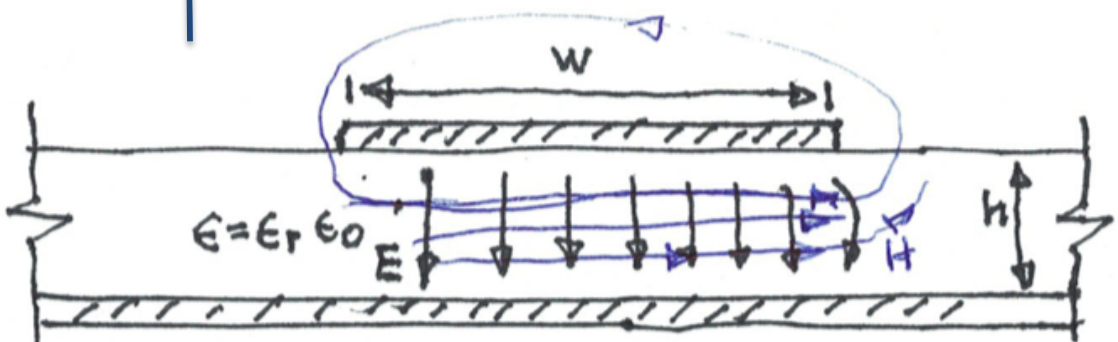
$$\Phi\big|_B = const.$$

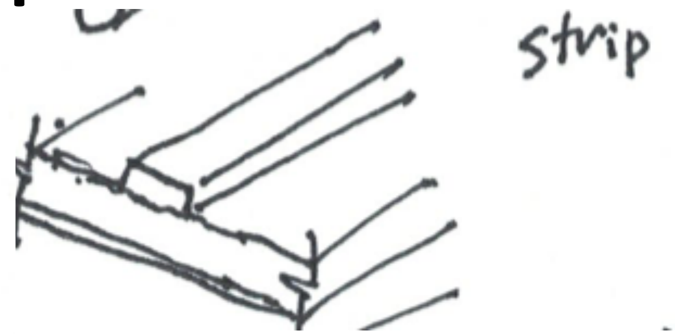
With one boundary potential is constant every where. No solution.

With two or more boundaries. Solutions possible with different potentials on different boundaries.



Example: Strip Line





$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi=0, \quad \Phi=0 \text{ on ground plane,}$$

Very Approximate treatment: w >> h. $\varepsilon_r > 1$

$$\Phi = V \text{ on conducting strip, } \Phi=\frac{x}{h}V, \quad \mathbf{E}=-\hat{\mathbf{x}}V/h$$

Magnetic field $$\mathbf{H}=\sqrt{\frac{\varepsilon_r\varepsilon_0}{\mu_0}}\,\hat{\mathbf{z}}\times\mathbf{E}=-\hat{\mathbf{y}}\sqrt{\frac{\varepsilon_r\varepsilon_0}{\mu_0}}\frac{V}{h}$$

Current in conducting strip (out of plane)

$$I=\oint\mathbf{H}\cdot d\mathbf{l}=w(-H_y)=\sqrt{\frac{\varepsilon_r\varepsilon_0}{\mu_0}}\frac{w}{h}V, \quad Z_0=\sqrt{\frac{\mu_0}{\varepsilon_r\varepsilon_0}}\frac{h}{w}$$

Losses on WG modes

There are two sources of loss in waveguides.

Dielectric losses

Conductor losses

Conductor losses are almost always the largest.

To treat dielectric losses make dielectric complex

$$\omega^2 \varepsilon \mu = k_z^2 + k_{\perp n}^2$$

$$\varepsilon = \varepsilon' + i\varepsilon''$$

$$k_z = k_{zr} + ik_{zi}$$

For $\varepsilon'' << \varepsilon'$

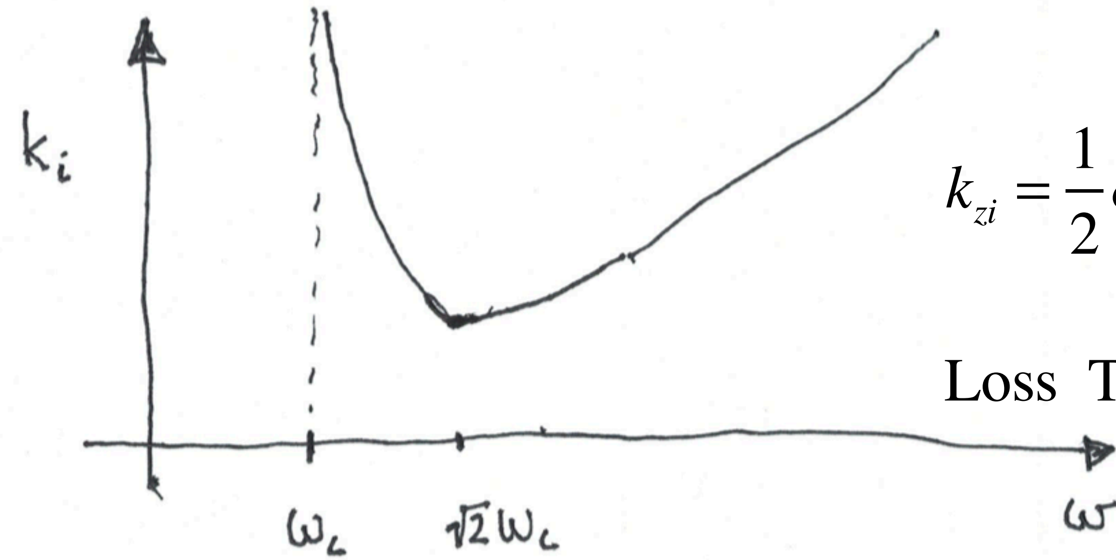
$$\text{Re}: \quad \omega^2 \varepsilon' \mu = k_{zr}^2 + k_{\perp n}^2$$

$$\text{Im}: \quad \omega^2 \varepsilon'' \mu = 2k_{zr}k_{zi}$$

$$k_{zi} = \frac{\varepsilon''}{2\varepsilon'} \frac{k_{zr}^2 + k_{\perp}^2}{k_{zr}} = \frac{1}{2}\delta \frac{\omega / v}{\sqrt{1-\omega_c^2/\omega^2}}$$

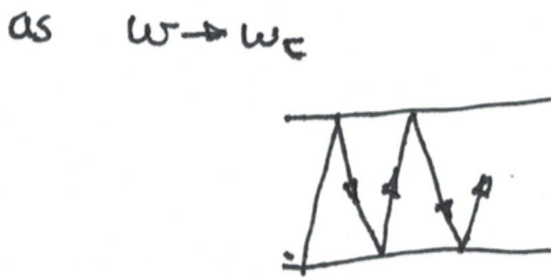
$$\text{Loss Tangent} \quad \delta = \frac{\varepsilon''}{\varepsilon'}$$

Loss rate vs frequency



$$k_{zi} = \frac{1}{2}\delta \frac{\omega / v}{\sqrt{1-\omega_c^2/\omega^2}}$$

$$\text{Loss Tangent} \;\; \delta = \frac{\varepsilon''}{\varepsilon'}$$



Damping is high because near cut-off waves bounce back and forth with little progress in z direction. Also true for conductor losses.

Conductor Losses

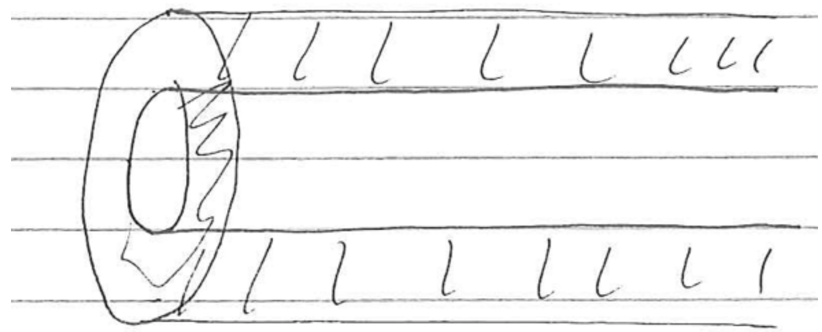
$$\nabla \times \vec{\mathbf{H}} = \varepsilon \frac{\partial \vec{\mathbf{E}}}{\partial t} + \sigma(\mathbf{x})\vec{\mathbf{E}}$$

$$\frac{\partial \hat{H}_z}{\partial y} - ik\hat{H}_y = \left(\sigma(\mathbf{x}) - i\omega\varepsilon\right)\hat{E}_x$$

$$ik\hat{H}_x - \frac{\partial \hat{H}_z}{\partial x} = \left(\sigma(\mathbf{x}) - i\omega\varepsilon\right)\hat{E}_y$$

$$\frac{\partial \hat{H}_y}{\partial x} - \frac{\partial \hat{H}_x}{\partial y} = \left(\sigma(\mathbf{x}) - i\omega\varepsilon\right)\hat{E}_z$$

To calculate losses need to include finite conductivity of WG walls.



Too hard to solve rigorously. Need to make approximation

Skin Depth $\delta = 1/\sqrt{2\omega\sigma\mu}$

$\delta << \lambda = c/f$, WG size

Apply Poyntings Theorem

$$\frac{\partial}{\partial t}\left[u_E + u_M\right] + \nabla \cdot \vec{\mathbf{S}} = -\vec{\mathbf{E}} \cdot \vec{\mathbf{J}}$$

$\mathbf{S} = \mathbf{E} \times \mathbf{H}$:

Poynting vector

$$u_E + u_M = \frac{\varepsilon_0}{2}\mathbf{E}\cdot\mathbf{E} + \frac{1}{2\mu_0}\mathbf{B}\cdot\mathbf{B}$$

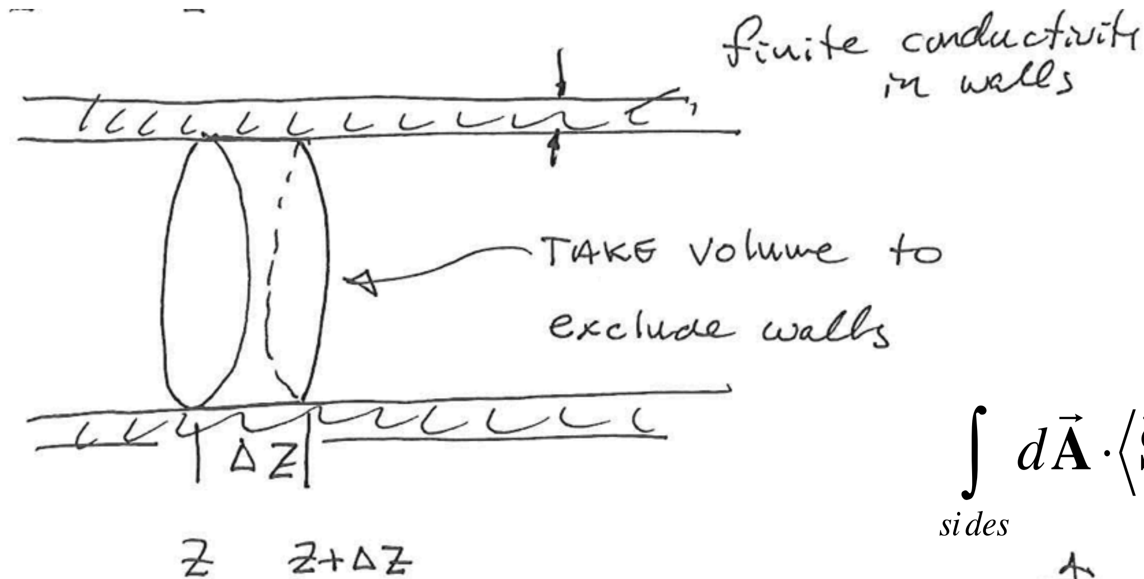
Time average for sinusoidal fields

$$\nabla \cdot \left\langle \vec{\mathbf{S}} \right\rangle = -\left\langle \vec{\mathbf{E}} \cdot \vec{\mathbf{J}} \right\rangle$$

$$\left\langle \mathbf{S} \right\rangle = \left\langle \mathbf{E} \times \mathbf{H} \right\rangle = \frac{1}{2}\mathrm{Re}\left[\hat{\mathbf{E}}^* \times \hat{\mathbf{H}}\right]$$

$$\left\langle \mathbf{J} \cdot \mathbf{E} \right\rangle = \frac{1}{2}\mathrm{Re}\left[\hat{\mathbf{J}}^* \cdot \hat{\mathbf{E}}\right]$$

Apply PT to a disk-like volume in WG



$$\nabla \cdot \langle \vec{\mathbf{S}} \rangle = -\langle \vec{\mathbf{E}} \cdot \vec{\mathbf{J}} \rangle$$

$$\int_{Surface} d\vec{\mathbf{A}} \cdot \langle \vec{\mathbf{S}} \rangle = -\int_{Volume} d^3 r \langle \vec{\mathbf{E}} \cdot \vec{\mathbf{J}} \rangle = 0$$

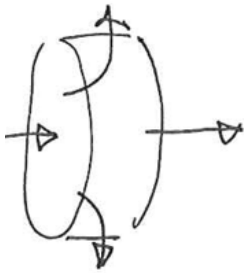
$$\int_{sides} d\vec{\mathbf{A}} \cdot \langle \vec{\mathbf{S}} \rangle$$

$$\langle \vec{S}_z \rangle\Big|_z$$

$$\langle \vec{S}_z \rangle\Big|_{z+\Delta z}$$

$$\int_{\substack{cross \\ section}} dA \cdot \left[\langle \vec{S}_z \rangle\Big|_{z+\Delta z} - \langle \vec{S}_z \rangle\Big|_z \right] + \int_{sides} d\vec{\mathbf{A}} \cdot \langle \vec{\mathbf{S}} \rangle = 0$$

$$\int_{sides} d\vec{\mathbf{A}} \cdot \langle \vec{\mathbf{S}} \rangle$$



$$\int_{\substack{cross\\section}} dA \cdot \left[\langle \vec{S}_z \rangle \big|_{z+\Delta z} - \langle \vec{S}_z \rangle \big|_z \right] + \int_{sides} d\vec{\mathbf{A}} \cdot \langle \vec{\mathbf{S}} \rangle = 0$$

Losses cause power to decay at a rate $2k_i$.

$$\langle \vec{\mathbf{S}}(z) \rangle = \langle \vec{\mathbf{S}}(0) \rangle \exp[-2k_i z]$$

$$\int_{\substack{\text{cross}\\\text{section}}} dA \cdot \left[\langle \vec{S}_z \rangle \big|_{z+\Delta z} - \langle \vec{S}_z \rangle \big|_z \right] = \left[\exp(-2k_i \Delta z) - 1 \right] \int_{\substack{\text{cross}\\\text{section}}} dA \cdot \left[\langle \vec{S}_z \rangle \big|_z \right]$$

$$\simeq -2k_i \Delta z \int_{\substack{\text{cross}\\\text{section}}} dA \cdot \left[\langle \vec{S}_z \rangle \big|_z \right] = -2k_i \Delta z \frac{1}{2} Z_{TE,TM} \int_{\substack{\text{cross}\\\text{section}}} dA \left| \hat{\mathbf{H}}_\perp \right|^2$$

$$\int_{\text{sides}} d\vec{\mathbf{A}} \cdot \langle \vec{\mathbf{S}} \rangle = \Delta z \int_{\text{Perimeter}} dl \frac{1}{2} R_s \left| \hat{\mathbf{H}}_{\tan} \right|^2$$

$$k_i = \frac{1}{2} \frac{R_s \int_{\text{Perimeter}} dl \left| \hat{\mathbf{H}}_{\tan} \right|^2}{Z_{TE,TM} \int_{\substack{\text{cross}\\\text{section}}} dA \left| \hat{\mathbf{H}}_\perp \right|^2}$$

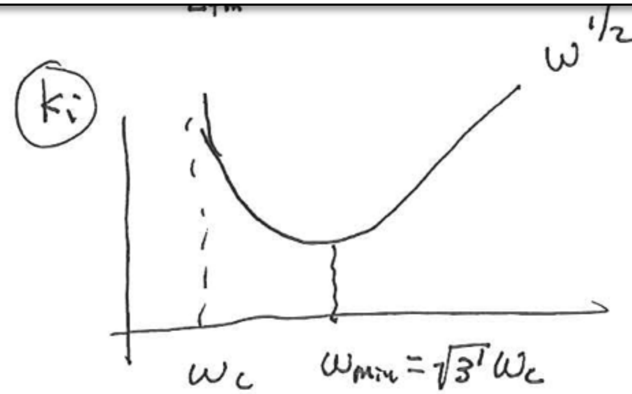
TM conductor losses

$$k_i = \frac{1}{2}\frac{R_s \int_{\text{Perimeter}} dl \left|\hat{\mathbf{H}}_{\tan}\right|^2}{Z_{TM} \int_{\substack{\text{cross}\\\text{section}}} dA \left|\hat{\mathbf{H}}_{\perp}\right|^2} \sim \frac{1}{2}\frac{R_s}{Z_{TM}} \times \frac{\text{perimeter}}{\text{crossection}} \text{f(mode\#, geometry)}$$

$$k_i \sim \frac{1}{2}\frac{R_s}{\sqrt{\mu/\varepsilon}}\frac{1}{\left(1-\omega_c^2/\omega^2\right)^{1/2}} \times \frac{\text{perimeter}}{\text{crossection}} \text{f(mode\#, geometry)}$$

$$\frac{R_s}{\eta} \sim \omega^{1/2}$$

$$R_s = \sqrt{\frac{\omega\mu_2}{2\sigma}}$$



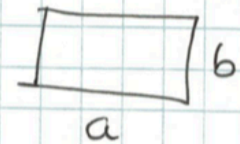
TE Conductor Losses

$$k_i = \frac{1}{2}\frac{R_s \int\limits_{\text{Perimeter}} dl \left|\hat{\mathbf{H}}_{\tan}\right|^2}{Z_{TE} \int\limits_{\substack{\text{cross}\\ \text{section}}} dA \left|\hat{\mathbf{H}}_{\perp}\right|^2} = \frac{1}{2}\frac{R_s}{Z_{TE}}\frac{\int\limits_{\text{Perimeter}} dl \left|\hat{\mathbf{H}}_{\tan,\perp}\right|^2 + \int\limits_{\text{Perimeter}} dl \left|\hat{\mathbf{H}}_{\tan,z}\right|^2}{\int\limits_{\substack{\text{cross}\\ \text{section}}} dA \left|\hat{\mathbf{H}}_{\perp}\right|^2}$$

$$k_i = \frac{R_s}{2\sqrt{\mu/\varepsilon}}\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{1/2} \frac{Per.}{Area}\left\{ f_1 + \frac{f_2}{\left(1 - \frac{\omega_c^2}{\omega^2}\right)} \right\}$$

Examples

TE_{10} of Rec wave guide



$$k_i = \frac{R_s}{b\eta\left(1-\frac{\omega_c^2}{\omega^2}\right)^{1/2}}\left[1+\frac{2b}{a}\frac{\omega_c^2}{\omega^2}\right]$$

$$k_i = \frac{R_s}{a\eta\left(1-\frac{\omega_c^2}{\omega^2}\right)^{1/2}}\left[\frac{\omega_c^2}{\omega^2}+\frac{n^2}{j'^2_{nm}-m^2}\right]$$

j'_{nm} = m^{th} zero of the derivative of the n^{th} order Bessel function