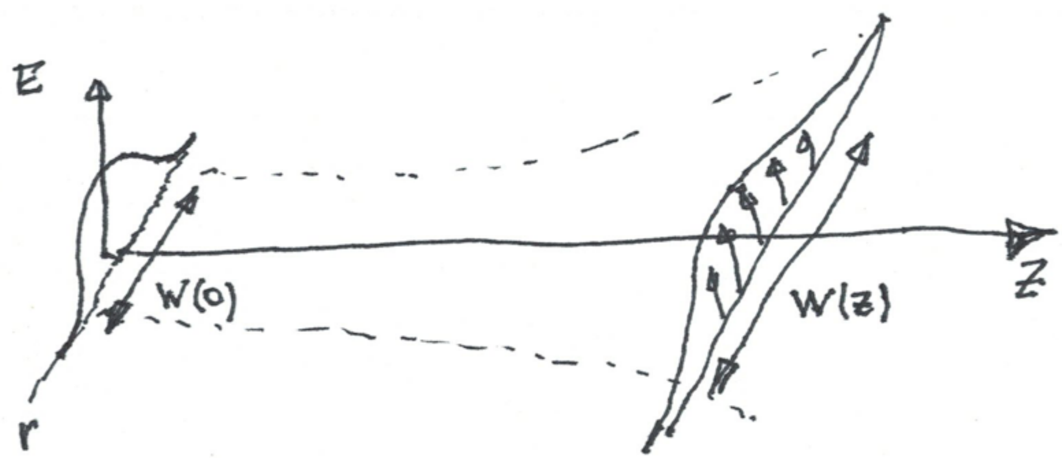
ENEE681

Lecture 08

Diffraction, Interference

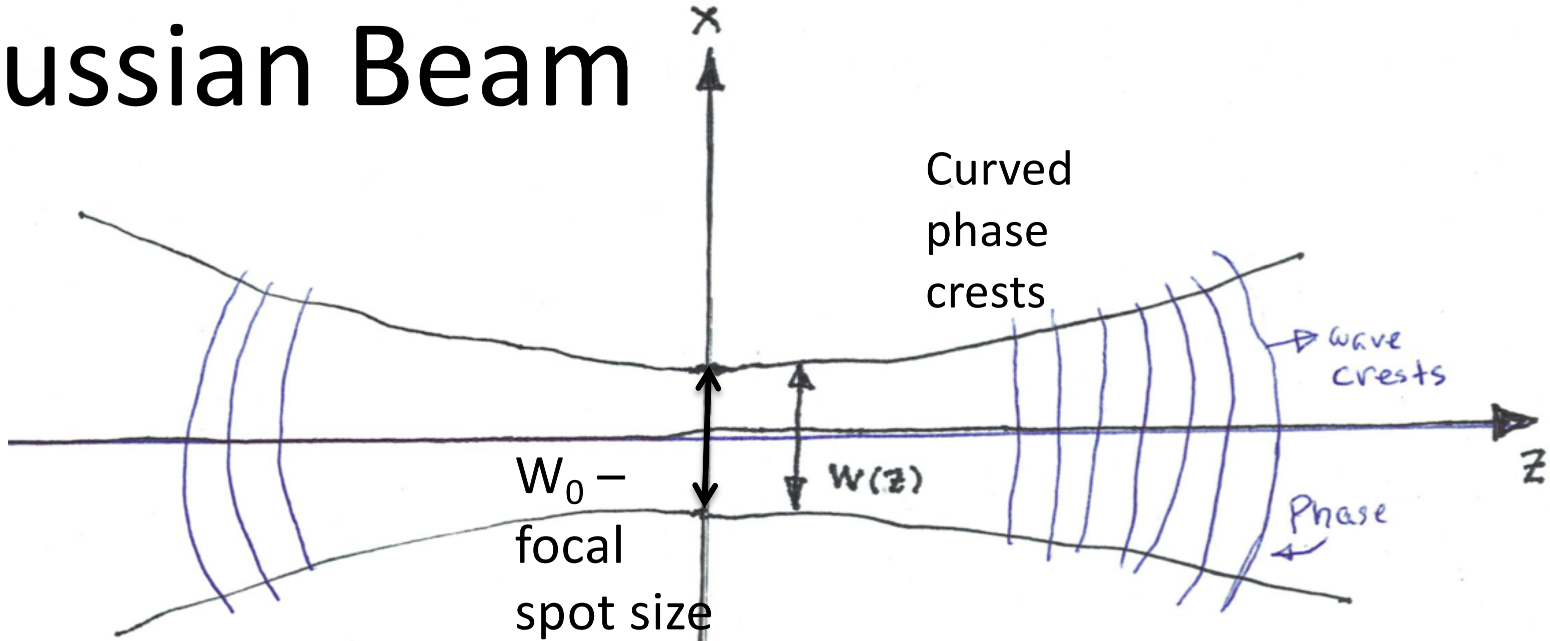
Diffraction

Waves launched from a finite size source spread out as they propagate.



$$W(z) = W(0)\sqrt{1 + z^2 / Z_R^2} \qquad Z_R = \frac{1}{2} k W^2(0)$$ Rayleigh Length

Gaussian Beam



$$E_{x,y}(x,y,z) = \frac{E_0}{1+iz/Z_R}\exp\left[-\frac{\left(x^2+y^2\right)}{W_0^2\left(1+iz/Z_R\right)}+ikz\right]$$

$$W(z) = W_0\sqrt{1+z^2/Z_R^2} \qquad Z_R = \frac{1}{2}kW_0^2$$

Rayleigh Length

$$E_{x,y}(0,0,z) = \frac{E_0}{1+iz/Z_R}\exp\left[+ikz\right] = \frac{E_0}{\sqrt{1+\left(z/Z_R\right)^2}}\exp\left[+ikz+i\phi(z)\right]$$

Guoy Phase $\tan\phi = -z/Z_R$

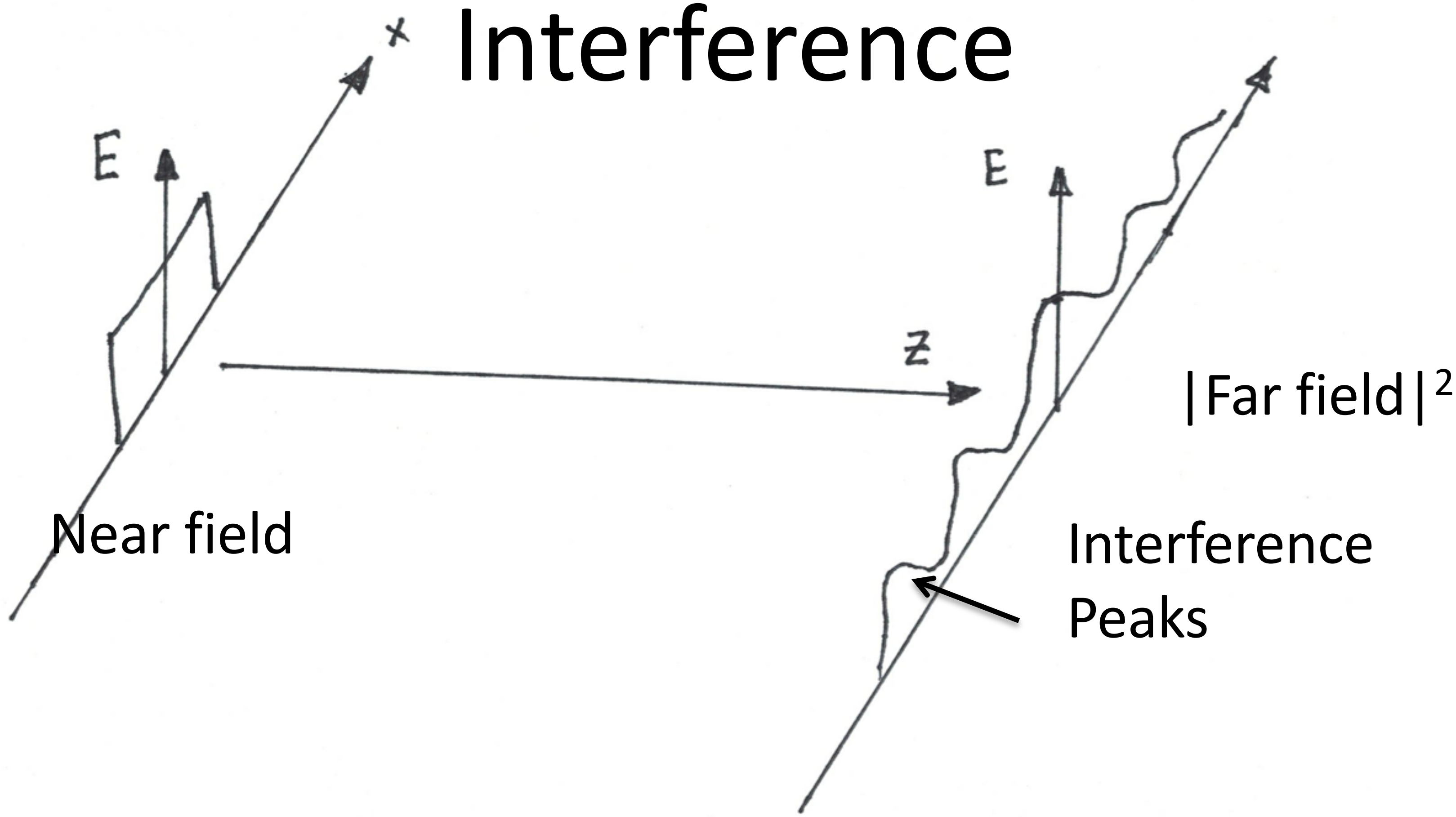
$$E_{x,y}(x,y,z) = \frac{E_0}{1 + iz/Z_R} \exp\left[-\frac{\left(x^2 + y^2\right)}{W_0^2\left(1 + iz/Z_R\right)} + ikz \right]$$

$$= \frac{E_0}{\sqrt{1 + z^2/Z_R^2}} \exp\left[-\frac{\left(x^2 + y^2\right)}{W_0^2\left(1 + z^2/Z_R^2\right)} \right] \exp\left\{ i \left[kz + \frac{z\left(x^2 + y^2\right)}{Z_R W_0^2\left(1 + z^2/Z_R^2\right)} + \phi_G \right] \right\}$$

Amplitude   Phase

As $z \rightarrow \infty$ $ik\left[z + \frac{\left(x^2 + y^2\right)}{2z} \right] + \phi_G = ikr + \phi_G$

Interference



Far field is Fourier transform of near field.

$$E_{x,y}(x,y,z) = \frac{k}{2i\pi z}\bar{E}_{x,y}\left(\frac{kx}{z},\frac{ky}{z},0\right)\exp\left[ik\left(z+\frac{x^2+y^2}{2z}\right)\right]$$

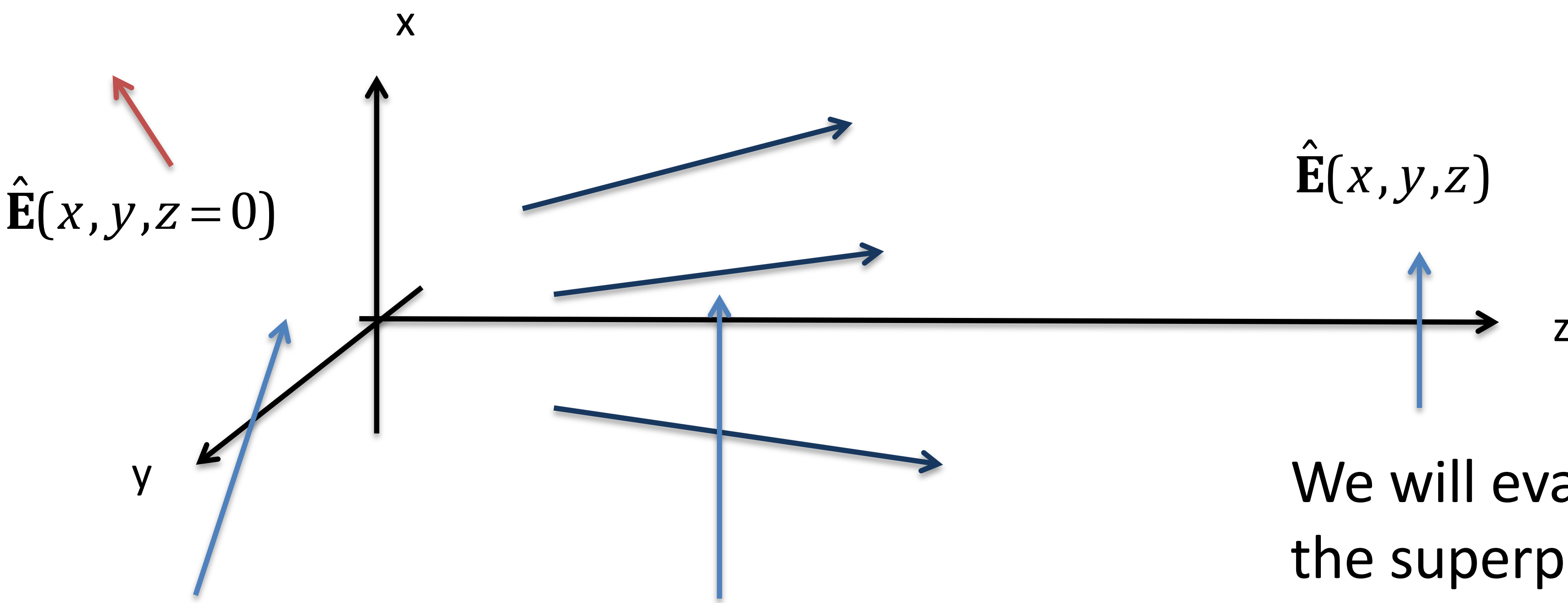
Problem

A certain infrared (wavelength 1 micrometer) laser beam can be focused to a spot size (W_0 = 15 micrometers).

1. What is the Rayleigh distance?

2. Suppose the central intensity at the focal point is 10^{18} W/cm^2, What is the central intensity 3 meters from the focus? What is the RMS electric field?

Approach



We will assume we know E_x and E_y in plane z=0

Fourier Transform E(z=0). Construct a superposition of plane waves giving E_x and E_y in plane z=0

We will evaluate the superposition of plane waves as a function of z. Inverse Fourier transform

Wave Equation

$$\nabla \cdot \vec{\mathbf{E}} = 0 \qquad \nabla \cdot \vec{\mathbf{H}} = 0 \qquad \nabla \times \vec{\mathbf{E}} = -\mu \frac{\partial \vec{\mathbf{H}}}{\partial t} \qquad \nabla \times \vec{\mathbf{H}} = \varepsilon \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

To get equations for phasor amplitudes

$$\vec{\mathbf{E}}, \vec{\mathbf{H}} \Rightarrow \mathrm{Re}\left\{ \left(\hat{\mathbf{E}}(\mathbf{x}), \hat{\mathbf{H}}(\mathbf{x}) \right) e^{-i\omega t} \right\}$$

$$\frac{\partial}{\partial t}, \nabla \Rightarrow -i\omega, \nabla$$

$$\nabla \cdot \hat{\mathbf{E}} = 0 \qquad \nabla \cdot \hat{\mathbf{H}} = 0 \qquad \nabla \times \hat{\mathbf{E}} = i\omega\mu \hat{\mathbf{H}} \qquad \nabla \times \hat{\mathbf{H}} = -i\omega\varepsilon \hat{\mathbf{E}}$$

Combine

$$\nabla \times \left(\nabla \times \hat{\mathbf{E}} \right) = i\omega\mu \nabla \times \hat{\mathbf{H}} = \omega^2 \varepsilon\mu \hat{\mathbf{E}} = k^2 \mathbf{E}$$

$$\nabla \left(\nabla \cdot \hat{\mathbf{E}} \right) - \nabla^2 \hat{\mathbf{E}} = \boxed{-\nabla^2 \hat{\mathbf{E}} = k^2 \hat{\mathbf{E}}}$$

$$k^2 = \frac{\omega^2}{v^2}, \quad \lambda = \frac{2\pi}{k}$$

Consider transverse components, E_x E_y

$$-\nabla^2\hat{\mathbf{E}} = k^2\hat{\mathbf{E}} \quad \rightarrow \quad -\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_{x,y} = k^2 E_{x,y} \quad k^2 = \omega^2\varepsilon\mu$$

Take spatial Fourier transform in x and y

$$-\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \exp\left[-ik_x x - ik_y y\right]\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_{x,y} = k^2\bar{E}_{x,y}(k_x,k_y,z)$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \exp\left[-ik_x x - ik_y y\right]E_{x,y} = \bar{E}_{x,y}(k_x,k_y,z)$$

$$\boxed{\left(k_x^2 + k_y^2 - \frac{\partial^2}{\partial z^2}\right)\bar{E}_{x,y} = k^2\bar{E}_{x,y}(k_x,k_y,z)}$$

Solutions

$$\left(k_x^2 + k_y^2 - \frac{\partial^2}{\partial z^2}\right)\bar{E}_{x,y} = k^2 \bar{E}_{x,y}(k_x, k_y, z)$$

Second order DEQ- 2 solutions

$$\bar{E}_{x,y} = A\exp\left(ik_z z\right) + B\exp\left(-ik_z z\right), \qquad k_z = \sqrt{k^2 - \left(k_x^2 + k_y^2\right)}, \qquad k^2 > \left(k_x^2 + k_y^2\right)$$

$$\bar{E}_{x,y} = A\exp\left(-\kappa_z z\right) + B\exp\left(+\kappa_z z\right), \qquad \kappa_z = \sqrt{\left(k_x^2 + k_y^2\right) - k^2}, \qquad k^2 < \left(k_x^2 + k_y^2\right)$$

Boundary condition as z goes to infinity B=0

$$\bar{E}_{x,y} = \bar{E}_{x,y}(k_x, k_y, z=0)\exp\left(ik_z z\right), \qquad k_z = \sqrt{k^2 - \left(k_x^2 + k_y^2\right)}, \qquad k^2 > \left(k_x^2 + k_y^2\right)$$

$$\bar{E}_{x,y} = \bar{E}_{x,y}(k_x, k_y, z=0)\exp\left(-\kappa_z z\right), \qquad \kappa_z = \sqrt{\left(k_x^2 + k_y^2\right) - k^2}, \qquad k^2 < \left(k_x^2 + k_y^2\right)$$

We know this

Divergence E = 0

$\nabla \cdot \hat{\mathbf{E}} = 0$ Fourier Transform in x and y $\Rightarrow$ $\frac{\partial}{\partial z}\bar{E}_z = -i\left(k_x \bar{E}_x + k_y \bar{E}_y\right)$

$$\bar{E}_z = \frac{-1}{k_z}\left(k_x \bar{E}_x + k_y \bar{E}_y\right)\exp\left(ik_z z\right), \qquad k^2 > \left(k_x^2 + k_y^2\right)$$

$$\bar{E}_{x,y} = \frac{i}{\kappa_z}\left(k_x \bar{E}_x + k_y \bar{E}_y\right)\exp\left(-\kappa_z z\right), \qquad k^2 < \left(k_x^2 + k_y^2\right)$$

We can find E_z after the problem for Ex and Ey is solved

Fourier Inversion

$$E_{x,y}(x,y,z) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_x \bar{E}_{x,y}(k_x,k_y,z)\exp\left[ik_x x + ik_y y\right]$$

$$\bar{E}_{x,y} = \bar{E}_{x,y}(k_x,k_y,z=0)\exp\left(ik_z z\right), \qquad k_z = \sqrt{k^2 - \left(k_x^2 + k_y^2\right)}, \qquad k^2 > \left(k_x^2 + k_y^2\right)$$

$$\bar{E}_{x,y} = \bar{E}_{x,y}(k_x,k_y,z=0)\exp\left(-\kappa_z z\right), \qquad \kappa_z = \sqrt{\left(k_x^2 + k_y^2\right) - k^2}, \qquad k^2 < \left(k_x^2 + k_y^2\right)$$

Let's assume $\bar{E}_{x,y}(k_x,k_y,z=0) \quad \rightarrow 0 \qquad \text{for} \quad k^2 < \left(k_x^2 + k_y^2\right)$

$$E_{x,y}(x,y,z) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_x \bar{E}_{x,y}(k_x,k_y,0)\exp\left[ik_x x + ik_y y + iz\sqrt{k^2 - \left(k_x^2 + k_y^2\right)}\right]$$

Consider Gaussian Dependence on x,y in plane z=0

Gaussian E

$$E_{x,y} = E_0 \exp\left[-\frac{x^2+y^2}{W_0^2}\right]$$

Fourier Transform

$$\bar{E}_{x,y}(k_x,k_y,0) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \exp\left[-ik_x x - ik_y y\right] E_0 \exp\left[-\frac{x^2+y^2}{W_0^2}\right]$$

Complete Square

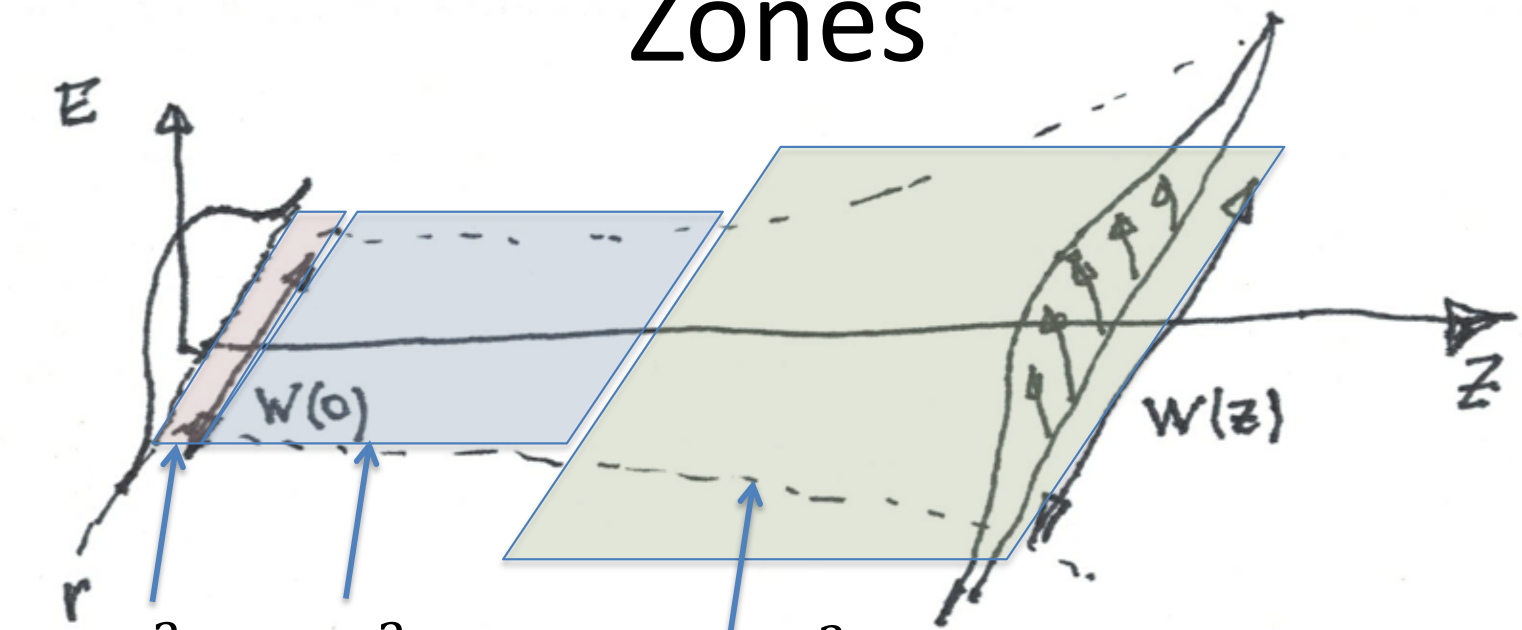
$$\frac{x^2}{W_0^2} + ik_x x = \frac{\left(x + ik_x W_0^2/2\right)^2}{W_0^2} + \frac{k_x^2 W_0^2}{4}$$

K-space E

$$\bar{E}_{x,y}(k_x,k_y,0) = \pi W_0^2 E_0 \exp\left[-\frac{\left(k_x^2+k_y^2\right)W_0^2}{4}\right]$$

W_0 - width in space $2/W_0$ - Width in k_x, k_y

Zones



$z \simeq \lambda = \frac{2\pi}{k}$ — Near field

$\lambda = \frac{2\pi}{k} < z < Z_R$ — Fresnel

$\lambda = \frac{2\pi}{k} < Z_R << z$ — Fraunhoffer

Fresnel: Invented the Fresnel lens. Installed in lighthouses around the world. Saved many lives.

Fraunhoffer: Orphaned at age 11. Worked for glass maker. Buried in collapsed building. Rescued by a Prince. Invented the spectroscope.

1 . Fourier transform $\bar{E}_{x,y}(k_x,k_y,0) = \pi W_0^2 E_0 \exp\left[-\frac{\left(k_x^2+k_y^2\right)W_0^2}{4}\right]$

2 . Inverse transform

$$E_{x,y}(x,y,z) = \int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\int_{-\infty}^{\infty}\frac{dk_y}{2\pi}\bar{E}_{x,y}(k_x,k_y,0)\exp\left[ik_x x + ik_y y + iz\sqrt{k^2-\left(k_x^2+k_y^2\right)}\right]$$

Note: $\bar{E}_{x,y}(k_x,k_y,0) \to 0 \text{ for } k_x^2+k_y^2 > W_0^{-2} << k^2$

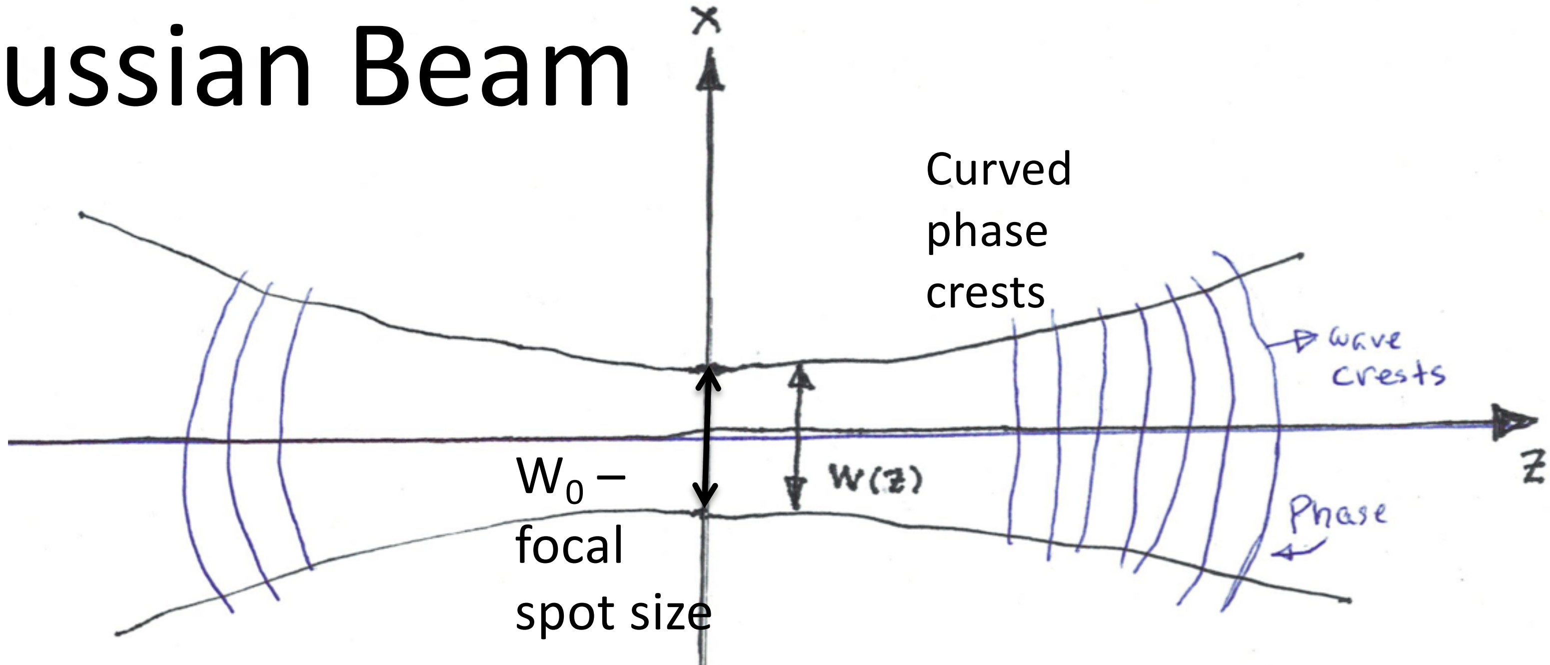
Expand: $\sqrt{k^2-\left(k_x^2+k_y^2\right)} \simeq k - \frac{k_x^2+k_y^2}{2k}$

$$E_{x,y}(x,y,z) = \pi W_0^2 E_0 \int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\int_{-\infty}^{\infty}\frac{dk_y}{2\pi}\exp\left[-\left(k_x^2+k_y^2\right)\left[\frac{W_0^2}{4}+\frac{iz}{2k}\right]+ik_x x+ik_y y+ikz\right]$$

Complete square:

$$\boxed{E_{x,y}(x,y,z) = \frac{E_0}{1+iz/Z_R}\exp\left[-\frac{\left(x^2+y^2\right)}{W_0^2\left(1+iz/Z_R\right)}+ikz\right]}$$

Gaussian Beam



$$E_{x,y}(x,y,z)=\frac{E_0}{1+iz/Z_R}\exp\left[-\frac{\left(x^2+y^2\right)}{W_0^2\left(1+iz/Z_R\right)}+ikz\right]$$

$$W(z)=W_0\sqrt{1+z^2/Z_R^2} \qquad Z_R=\frac{1}{2}kW_0^2$$ Rayleigh Length

$$E_{x,y}(0,0,z)=\frac{E_0}{1+iz/Z_R}\exp[+ikz]=\frac{E_0}{\sqrt{1+\left(z/Z_R\right)^2}}\exp[+ikz+i\phi(z)]$$

Guoy Phase $\tan\phi=-z/Z_R$

Far Field Radiation Pattern

$$E_{x,y}(x,y,z)=\int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\int_{-\infty}^{\infty}\frac{dk_y}{2\pi}\bar{E}_{x,y}(k_x,k_y,0)\exp\left[ik_x x+ik_y y+iz\sqrt{k^2-\left(k_x^2+k_y^2\right)}\right]$$

$$\bar{E}_{x,y}(k_x,k_y,0)\to 0 \text{ for } k_x^2+k_y^2<<k^2 \qquad \sqrt{k^2-\left(k_x^2+k_y^2\right)}\simeq k-\frac{k_x^2+k_y^2}{2k}$$

$$\lim_{z\to\infty}\exp\left[-\frac{iz}{2k}\left(k_x^2+k_y^2\right)+ik_x x+ik_y y\right]=$$

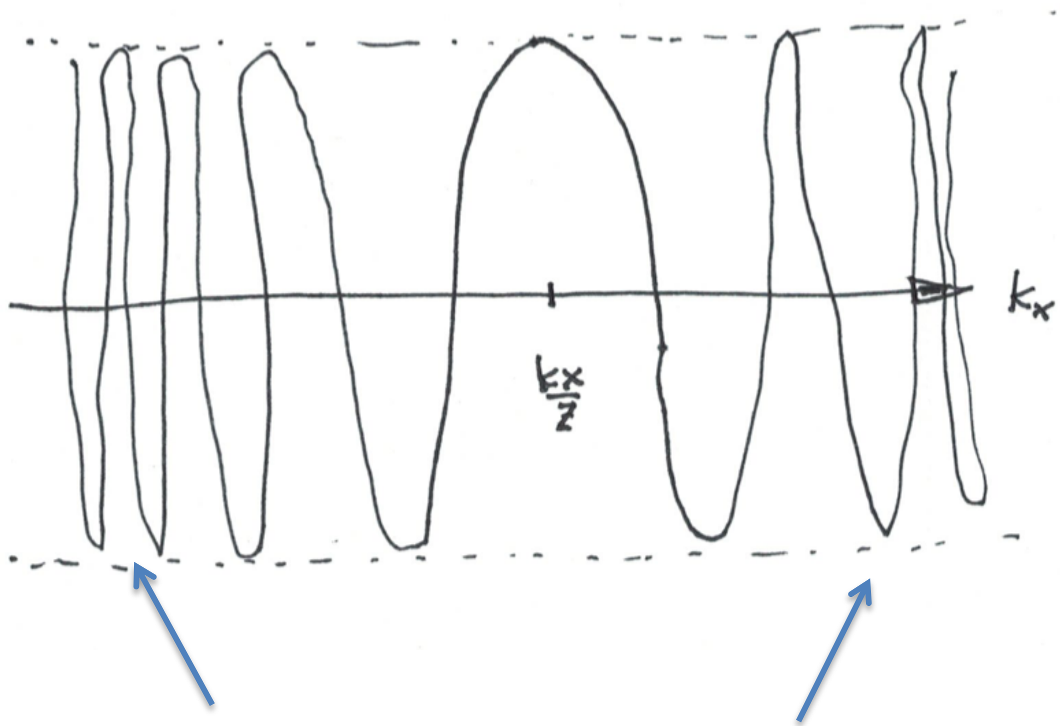
Delta functions

$$\frac{2\pi k}{iz}\exp\left[ik\frac{x^2+y^2}{2z}\right]\delta\left(k_x-\frac{kx}{z}\right)\delta\left(k_y-\frac{ky}{z}\right)$$

$$\boxed{E_{x,y}(x,y,z)=\frac{k}{2i\pi z}\bar{E}_{x,y}(\frac{kx}{z},\frac{ky}{z},0)\exp\left[ik\left(z+\frac{x^2+y^2}{2z}\right)\right]}$$

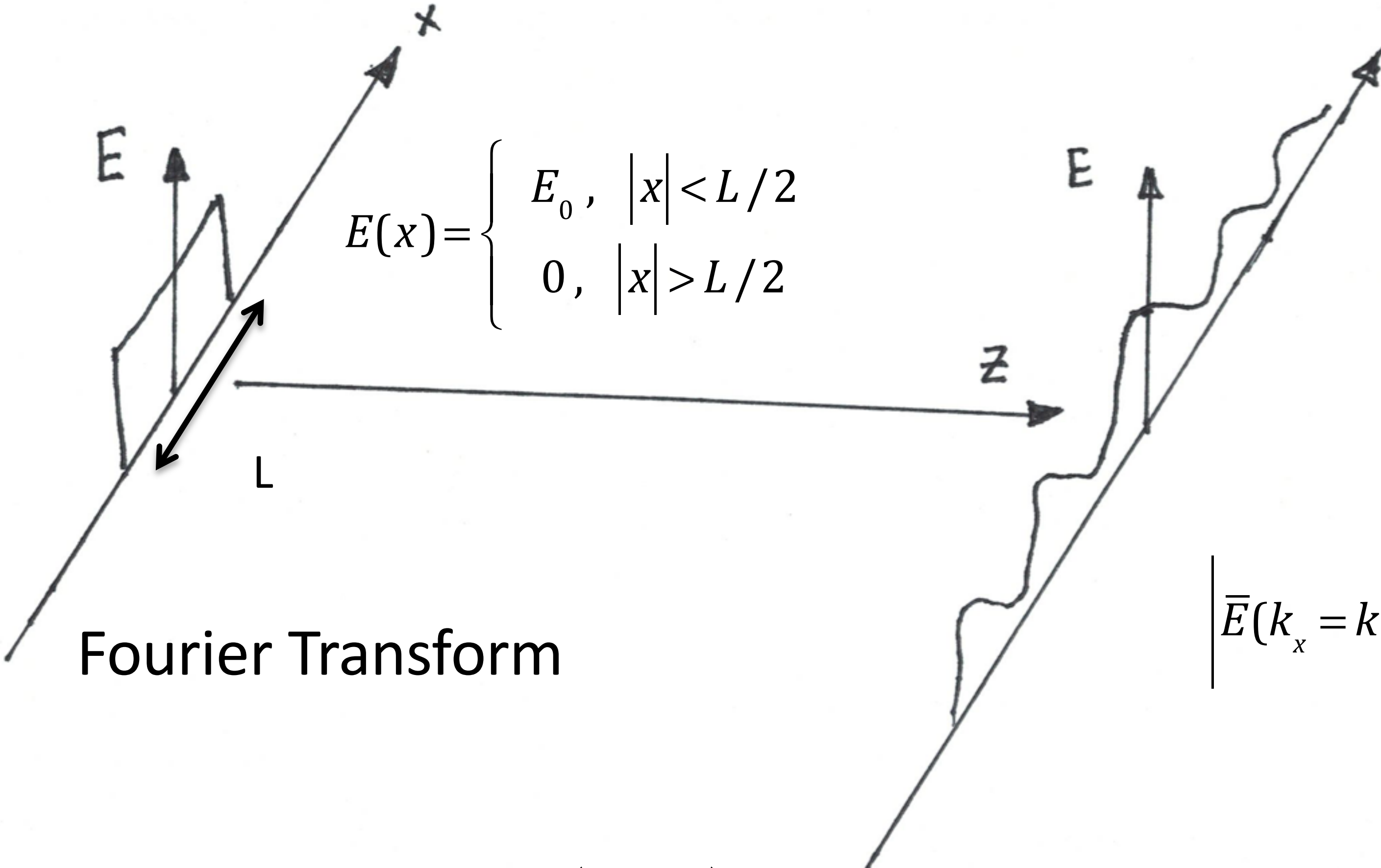
Stationary Phase

$$\lim_{z\to\infty} \exp\left[-\frac{iz}{2k}k_x^2 + ik_x x\right] = \lim_{z\to\infty} \exp\left[-\frac{iz}{2k}\left(k_x - \frac{kx}{z}\right)^2 + i\frac{kx^2}{2z}\right]$$



Rapid oscillations integrate to zero

Example



$$E(x) = \begin{cases} E_0, & |x| < L/2 \\ 0, & |x| > L/2 \end{cases}$$

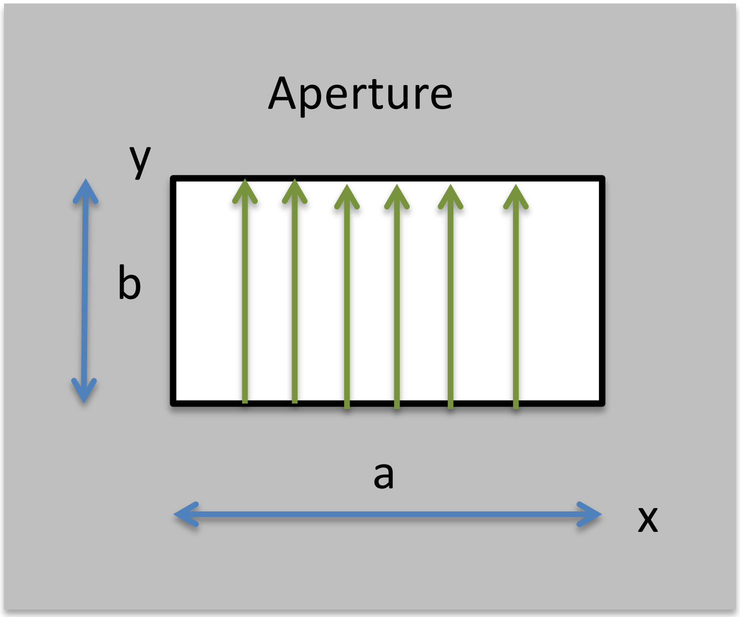
Fourier Transform

$$\left|\bar{E}(k_x = k\frac{x}{z})\right|^2$$

$$\bar{E}(k_x) = \int_{-L/2}^{L/2} dx E_0 \exp\left[-ik_x x\right] = E_0 \frac{\sin\left(k_x L/2\right)}{k_x/2}$$

zero when $\frac{k_x L}{2} = p\pi \rightarrow \tan\theta = \frac{x}{z} = \frac{2\pi p}{L}$

Problem



The electric field in an aperture must satisfy boundary conditions on the edges of the aperture.

What are they?

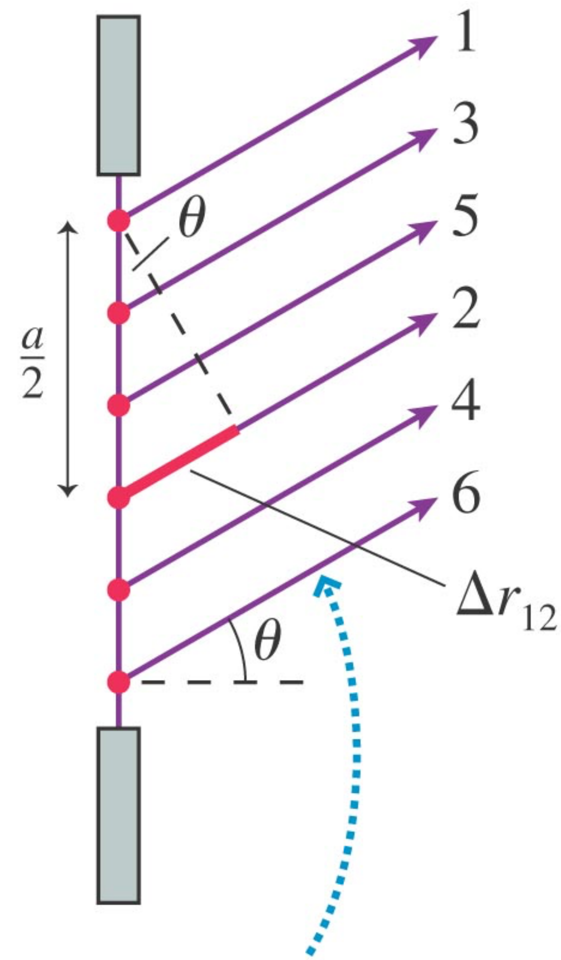
Make up a simple function that satisfies the BC's

Find (for Thursday) the far field diffraction pattern

When is there perfect destructive interference?

(c)

Each point on the wave front is paired with another point distance $a/2$ away.



These wavelets all meet on the screen at angle θ. Wavelet 2 travels distance $\Delta r_{12} = (a/2)\sin\theta$ farther than wavelet 1.

Destructive when

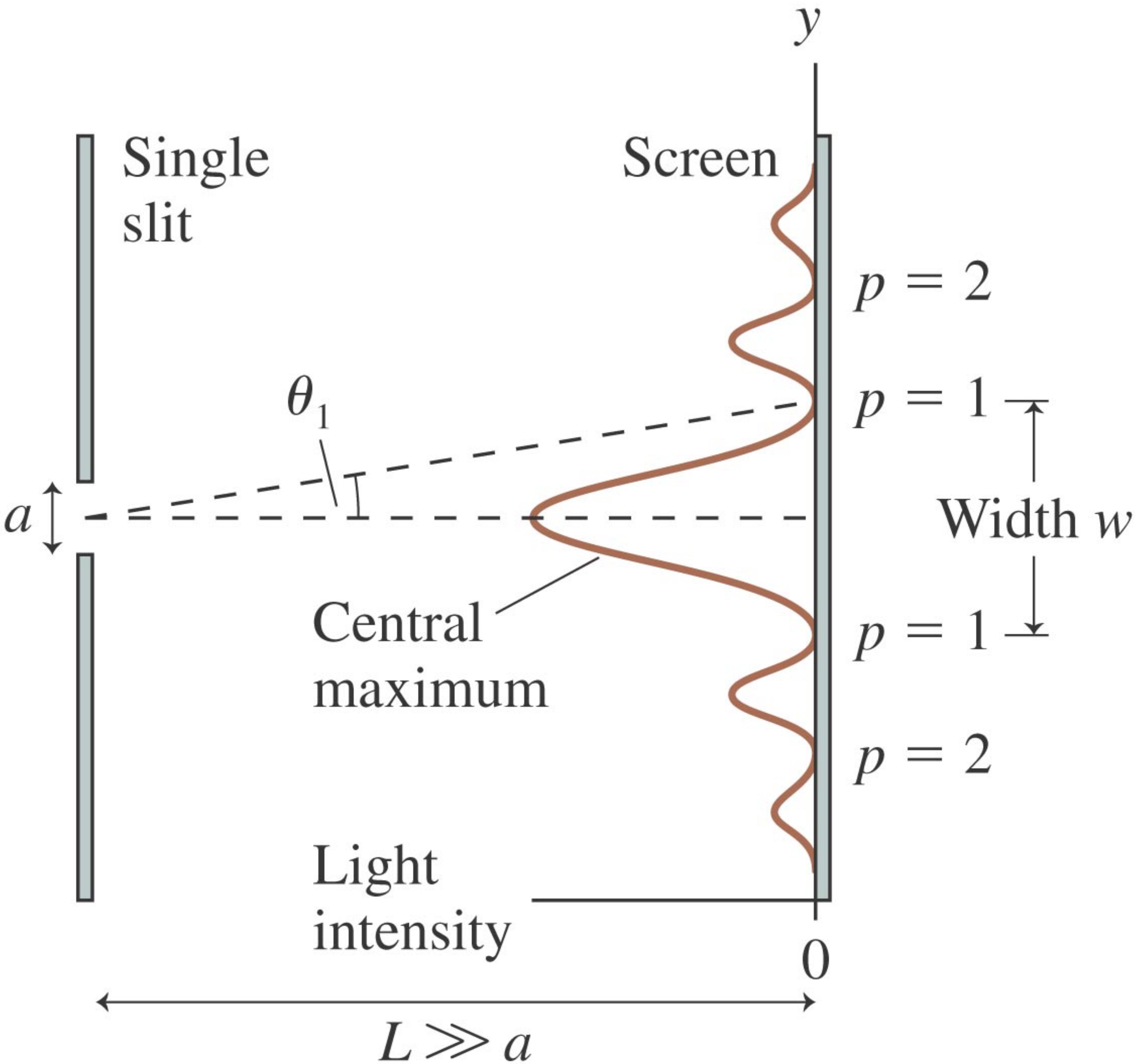
$$\Delta r_{12} = \frac{a}{2}\sin\theta = \frac{\lambda}{2}$$

1 cancels 2
3 cancels 4
5 cancels 6
Etc.

Also:

$$\frac{a}{2p}\sin\theta_p = \frac{\lambda}{2}$$

$$p = 1,2,3.....$$


y
Single
slit
Screen
p = 2
p = 1
θ1
a
Width w
Central
maximum
p = 1
p = 2
Light
intensity
0
L ≫ a