ENEE681

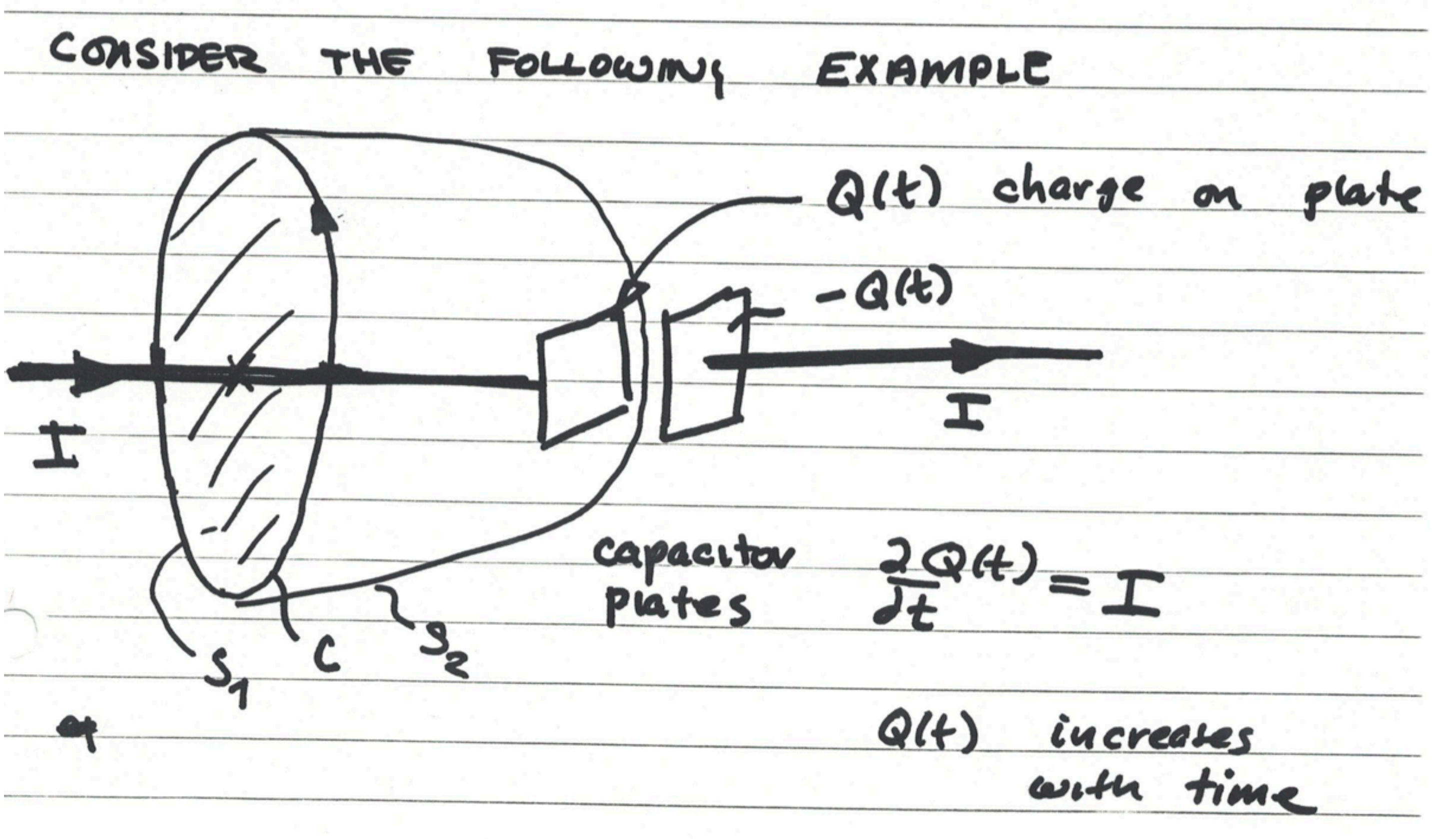
Lecture 2
Displacement Current
Fields in Matter
Boundary Conditions

Maxwell's Displacement Current

It is not really a current. It just acts like one.

Maxwell determined the static Ampere's Law could not be correct.
Inconsistent with charge conservation

$$\oint_{Loop} \vec{\mathbf{B}}(\vec{\mathbf{r}})\cdot d\vec{\mathbf{l}} = \mu_0 \int_S d\vec{\mathbf{A}}\cdot\vec{\mathbf{J}} = \mu_0 I$$



$$\int_{S_1} d\vec{\mathbf{A}}\cdot\vec{\mathbf{J}} = I$$

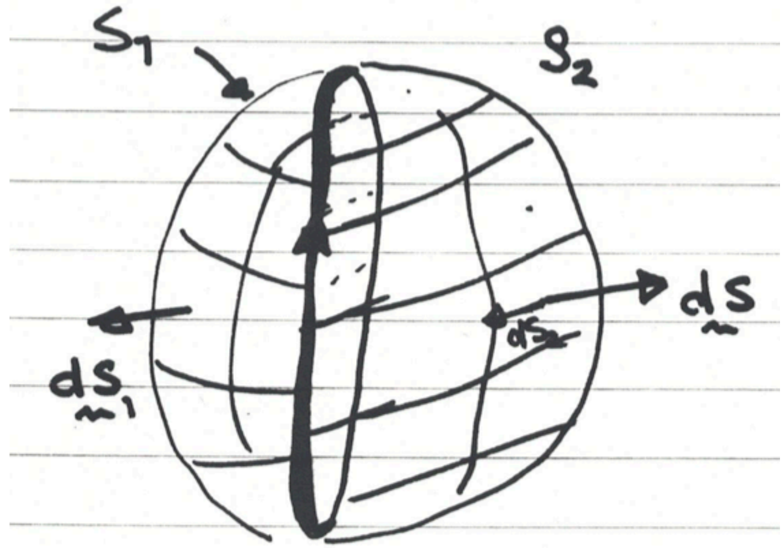
$$\int_{S_2} d\vec{\mathbf{A}}\cdot\vec{\mathbf{J}} = 0$$

Remember for Faraday's Law
Any surface with the same perimeter gave the correct answer.

$$\oint_{loop} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\int_S d\vec{\mathbf{A}} \cdot \frac{\partial \vec{\mathbf{B}}}{\partial t}$$

From Gauss' Law

$$\int_{S_1+S_2} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$

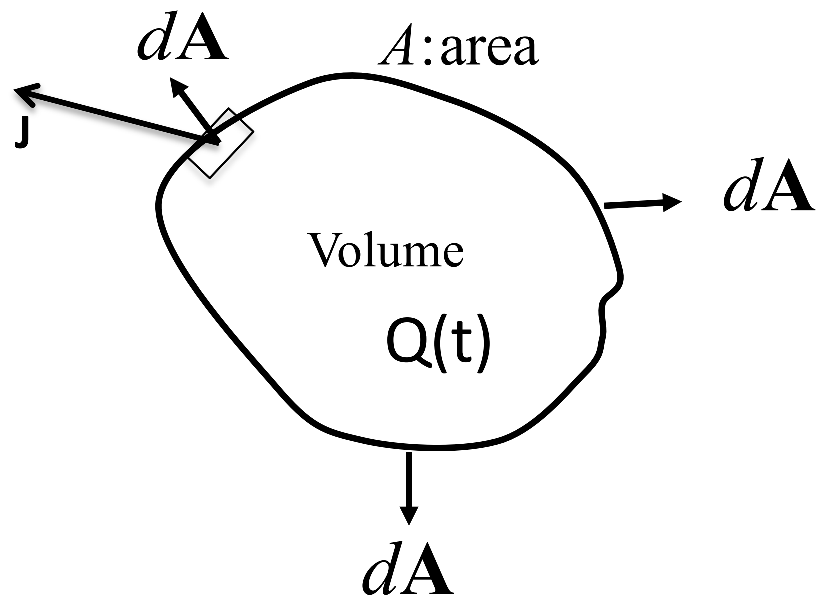


$$d\vec{\mathbf{S}}_1 = d\vec{\mathbf{S}}$$

$$d\vec{\mathbf{S}}_2 = -d\vec{\mathbf{S}}$$

$$\int_{S_1+S_2} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = 0 \Rightarrow \int_{S_1} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}_1 = \int_{S_2} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}_2$$

Conservation of Charge



$$\int_S d\vec{\mathbf{A}} \cdot \vec{\mathbf{J}} + \frac{dQ}{dt} = 0$$

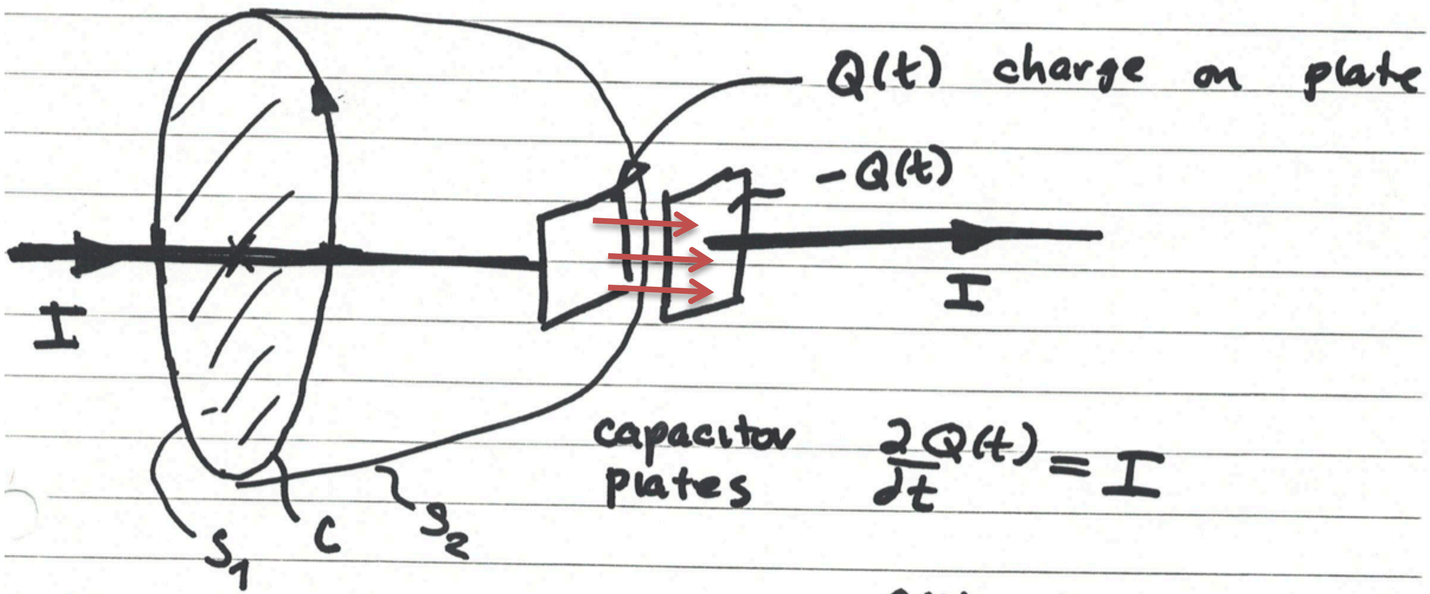
But

$$\oint_S \varepsilon_0 \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = Q$$

So

$$\int_S d\vec{\mathbf{A}} \cdot \left(\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial}{\partial t} \vec{\mathbf{E}} \right) = 0$$

CONSIDER THE FOLLOWING EXAMPLE



$$\frac{\partial Q(t)}{\partial t} = I$$

Q(t) increases with time

$$\int_S d\vec{\mathbf{A}}\cdot\left(\vec{\mathbf{J}}+\varepsilon_0\frac{\partial}{\partial t}\vec{\mathbf{E}}\right)=0$$

Faraday: time varying B makes an E

$$\oint_{loop} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} == -\frac{d}{dt}\int_{\mathbf{S}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}$$

example

$$E_\theta(r) = -\frac{r}{2}\frac{\partial B_z}{\partial t}$$

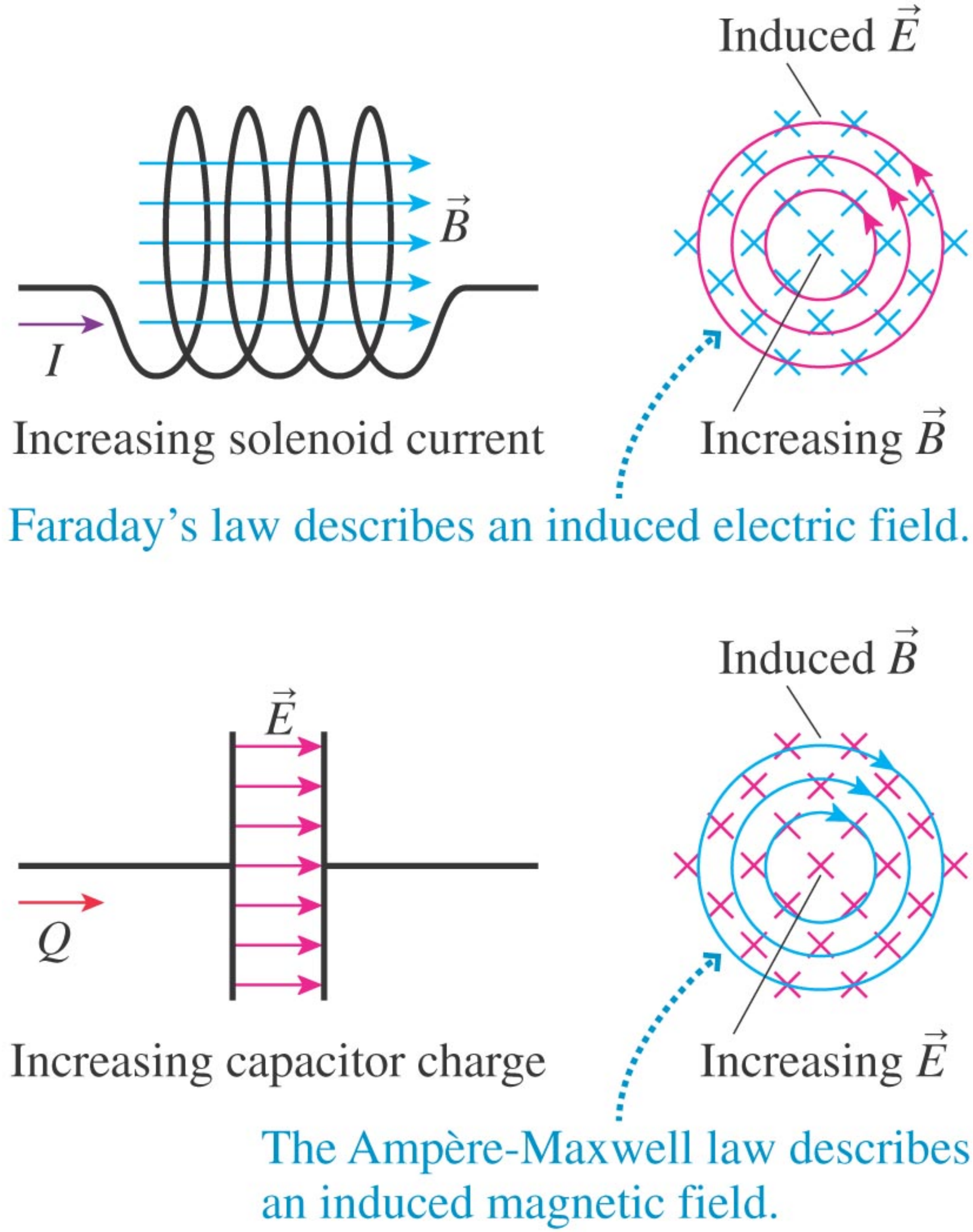
Ampere-Maxwell: time varying E makes a B

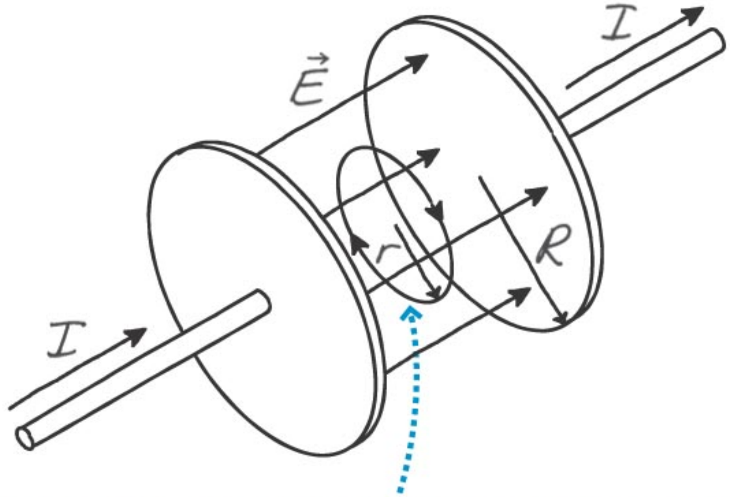
$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 \varepsilon_0 \frac{d}{dt}\int_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

example

$$B_\theta(r) = \frac{\mu_0 \varepsilon_0 r}{2}\frac{\partial E_z}{\partial t}$$

Put together, fields can sustain themselves - Electromagnetic Waves


Induced $\vec{E}$
$\vec{B}$
I
Increasing solenoid current
Increasing $\vec{B}$
Faraday's law describes an induced electric field.
Induced $\vec{B}$
$\vec{E}$
Q
Increasing capacitor charge
Increasing $\vec{E}$
The Ampère-Maxwell law describes an induced magnetic field.



The magnetic field line is a circle concentric with the capacitor. The electric flux through this circle is $\pi r^2 E$.

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = B_\theta(r) 2\pi r$$

$$\Phi_e = \int_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \pi r^2 E_z$$

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 (I + \varepsilon_0 \frac{d\Phi_e}{dt})$$

$$B_\theta(r) = \frac{\mu_0 \varepsilon_0 r}{2} \frac{\partial E_z}{\partial t}$$

Recall from Faraday:

$$E_\theta(r) = -\frac{r}{2} \frac{\partial B_z}{\partial t}$$

Conduction current, Displacement current, Polarization current

In vacuum

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Flow of charge "real" current ($\mathbf{J}$)

Displacement "current" ($\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$)

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_p$$

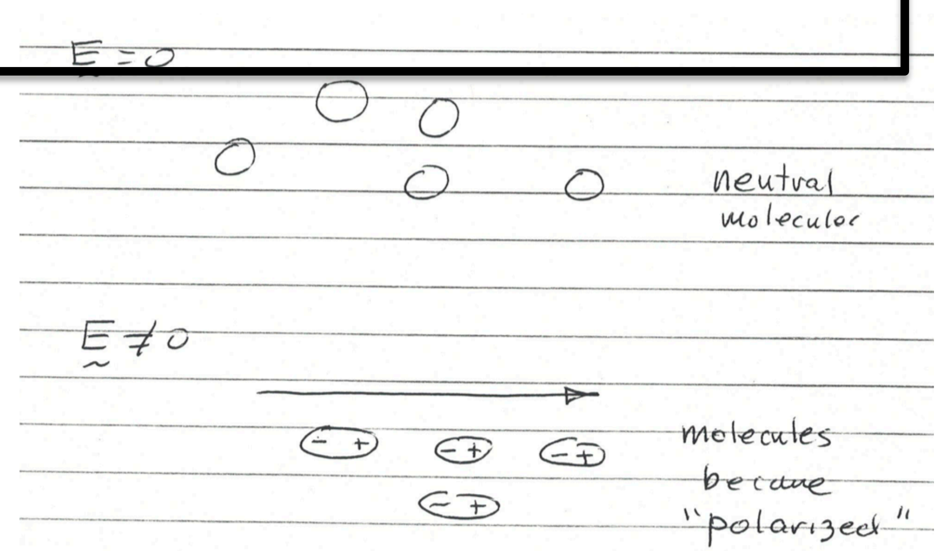
"Free" current ($\mathbf{J}_f$)

polarization current ($\mathbf{J}_p$)

$$\mathbf{J}_p = \varepsilon_0 \chi_E \frac{\partial \mathbf{E}}{\partial t}$$

$$\mathbf{J}_p = \frac{\partial}{\partial t}\left[\varepsilon_0 \chi_E \mathbf{E}\right]$$

Electric Susceptibility χ_E



Dielectric Terminology

$$\nabla \times \frac{\mathbf{B}}{\mu_0} = \mathbf{J}_f + \frac{\partial}{\partial t}\left[\varepsilon_0\left(1+\chi_E\right)\mathbf{E}\right]$$

$$\nabla \times \frac{\mathbf{B}}{\mu_0} = \mathbf{J}_f + \frac{\partial}{\partial t}\mathbf{D}$$

$$\mathbf{D} = \varepsilon_0\left(1+\chi_E\right)\mathbf{E} = \varepsilon\mathbf{E}$$

$\mathbf{D}$ Electric flux density

$\varepsilon = \varepsilon_0\left(1+\chi_E\right)$ Dielectric contant

$\left(1+\chi_E\right)$ Relative Dielectric constant

Maxwell's Equations in Vacuum

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = Q / \varepsilon_0$$

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$

$$\oint_{loop} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\int_S d\vec{\mathbf{A}} \cdot \frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\oint_{Loop} \vec{\mathbf{B}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{l}} = \mu_0 \int_S d\vec{\mathbf{A}} \cdot \left[\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \right]$$

$$\nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \vec{\mathbf{B}} = 0$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\nabla \times \vec{\mathbf{B}} = \mu_0 \left[\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \right]$$

Here ρ and J are the total charge and current densities

Includes charge and current densities induced in dielectric and magnetic materials

Separate charge and current densities into “free” and “induced” components

Somewhat arbitrary but very useful

magnetization current

polarization current

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_m + \mathbf{J}_p$$

“Free” current

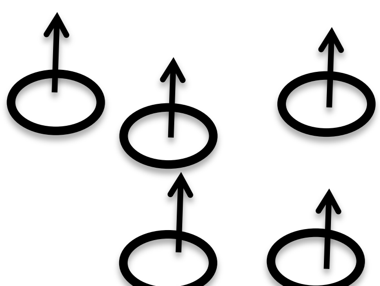
polarization charge density

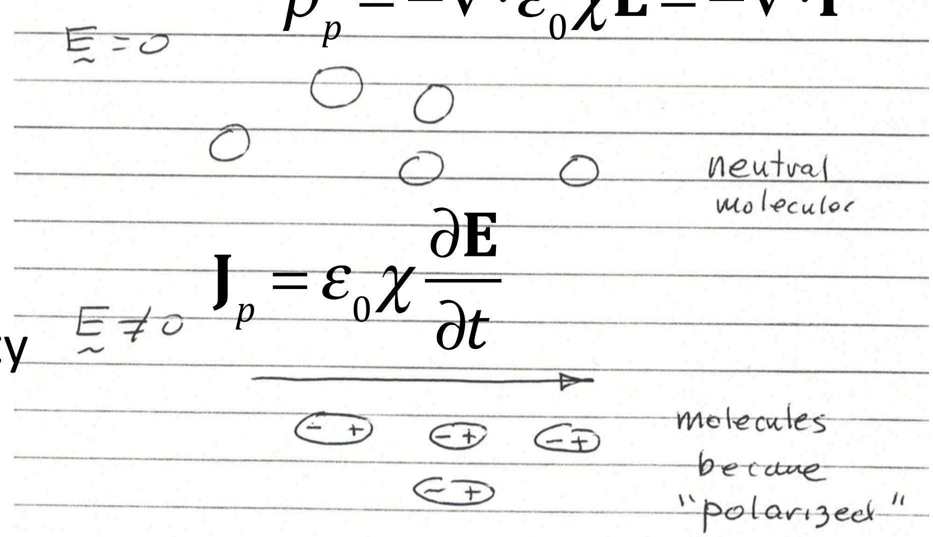
$$\rho = \rho_f + \rho_p$$

“Free” charge density

polarization density

$$\rho_p = -\nabla \cdot \varepsilon_0 \chi \mathbf{E} = -\nabla \cdot \mathbf{P}$$





$$\mathbf{J}_m = \nabla \times \mathbf{M}$$

$$\mathbf{J}_p = \varepsilon_0 \chi \frac{\partial \mathbf{E}}{\partial t}$$

magnetization density

$$\mathbf{M} = \mu_0 \chi_m \mathbf{H}$$

Maxwell's Equations in Matter

Equations in linear media

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon \mathbf{E} \qquad \mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M} = \mu \mathbf{H}$$

$$\mathbf{P} = \varepsilon_0 \chi_E \mathbf{E} \qquad \mathbf{M} = \mu_0 \chi_M \mathbf{H}$$

$$\oint \vec{\mathbf{D}} \cdot d\vec{\mathbf{A}} = Q_{free}$$

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$

$$\oint_{loop} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\int_S d\vec{\mathbf{A}} \cdot \frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\oint_{Loop} \vec{\mathbf{H}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{l}} = \int_S d\vec{\mathbf{A}} \cdot \left[\vec{\mathbf{J}}_{free} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \right]$$

$$\nabla \cdot \vec{\mathbf{D}} = \rho_{free}$$

$$\nabla \cdot \vec{\mathbf{B}} = 0$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_{free} + \frac{\partial \vec{\mathbf{D}}}{\partial t}$$

Maxwell's Equations in Matter

$$\nabla \cdot \mathbf{D} = \rho_f \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

(linear and isotropic matter)

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} = \frac{1}{\mu}\mathbf{B}\ , \quad \mathbf{J}_b = \nabla \times \mathbf{M}$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon \mathbf{E}\ , \quad \rho_b = -\nabla \cdot \mathbf{P}$$

$\mathbf{M} = \chi_m \mathbf{H}$: Magnetization field

$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$: Polarzation field

$\mu = \mu_0(1+\chi_m)$: permeability

$\varepsilon = \varepsilon_0(1+\chi_e)$:permittivity

$\mu_r = 1+\chi_m$: relative permeability

$\varepsilon_r = 1+\chi_e$: relative permittiity

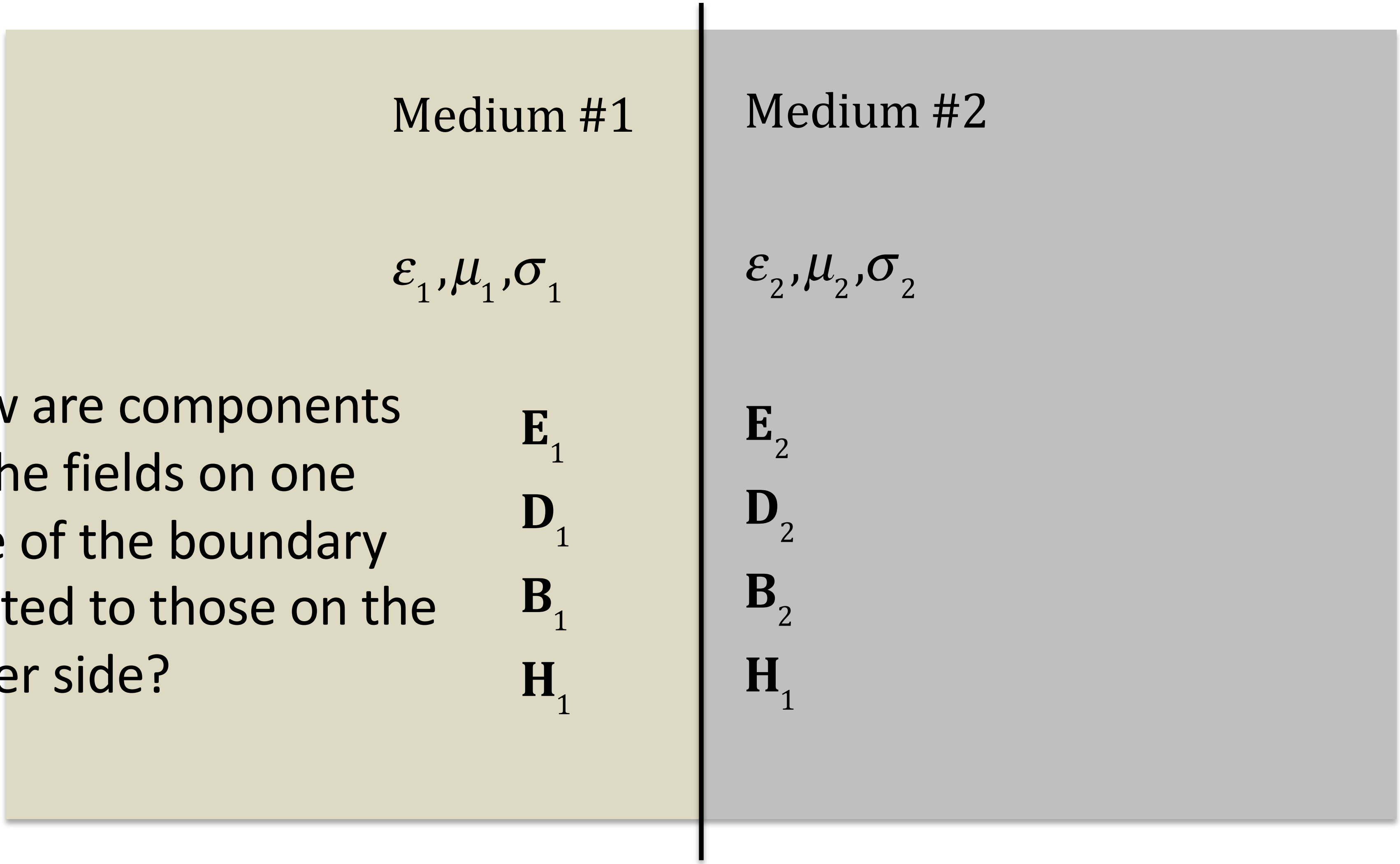
χ_m : magnetic susceptibility

χ_e : electric suscepibility

In a good conductor – Ohms' Law (point version)

$$\mathbf{J}_f = \sigma\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Boundary Conditions



How are components of the fields on one side of the boundary related to those on the other side?

General Comments

Tangential components of **E** are always equal.

Normal components of **B** are always equal.

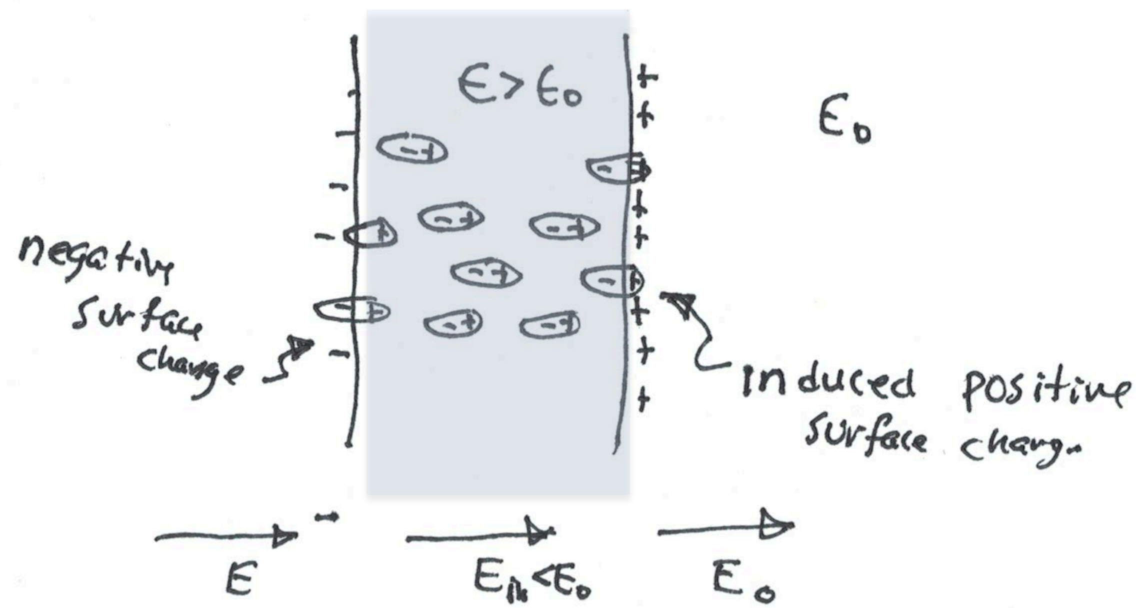
Normal component of **E** discontinuous implies a **surface charge density.**

Normal components of **D** discontinuous implies a **free surface charge density**

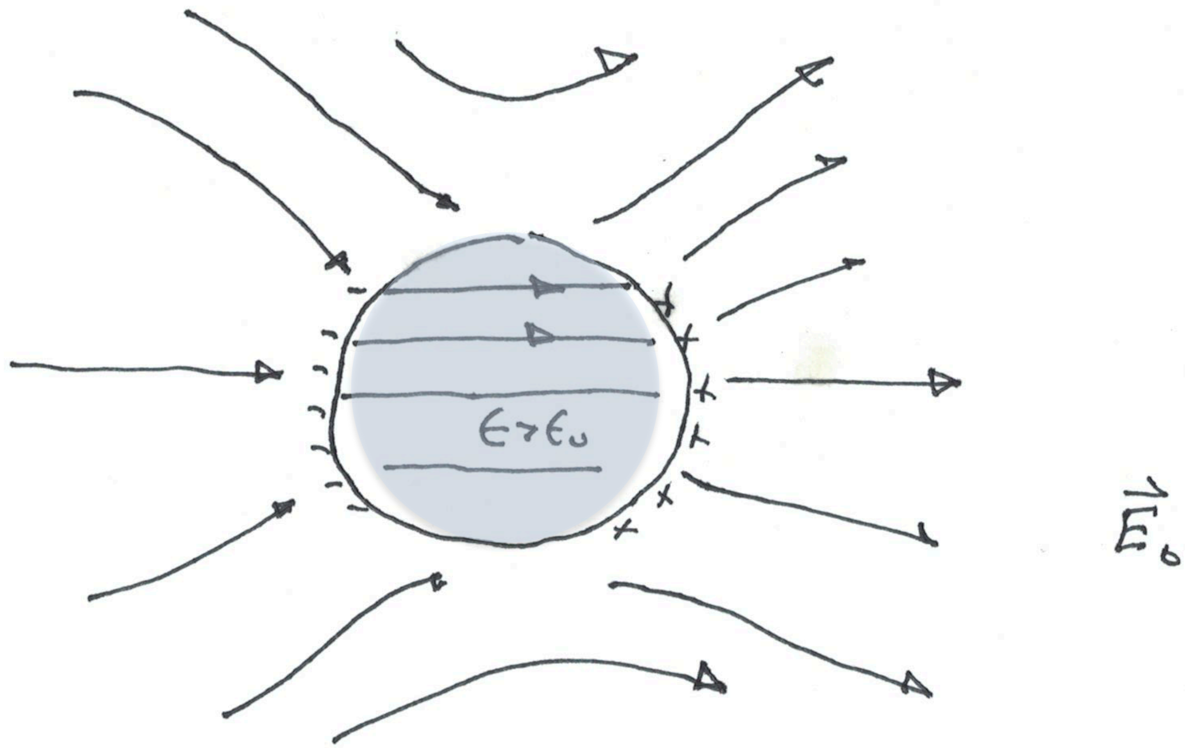
Tangential components of **B** discontinuous implies a **surface current density.**

Tangential components of **H** discontinuous implies a **free surface current density.**

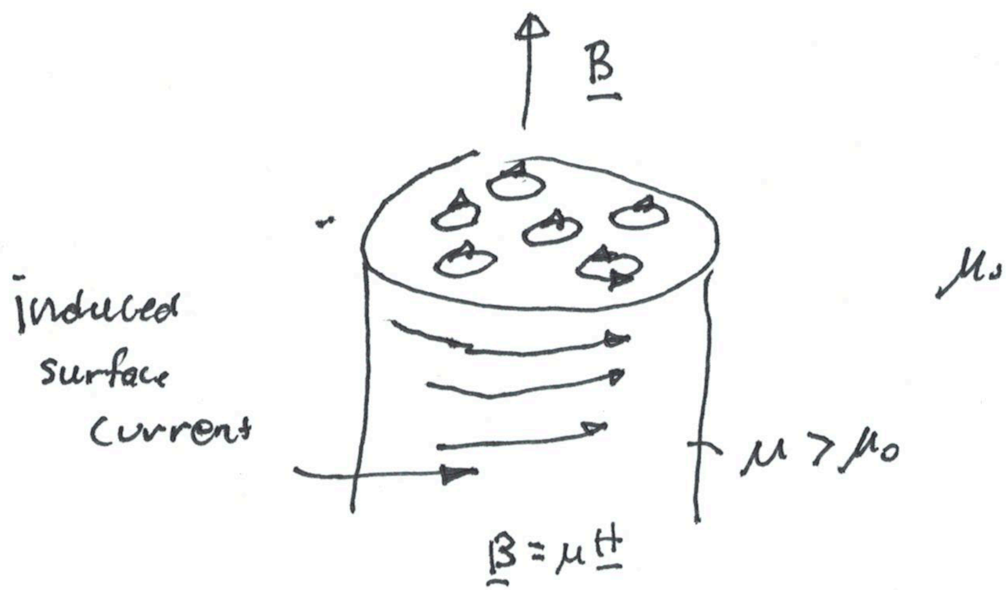
Induced surface charge density



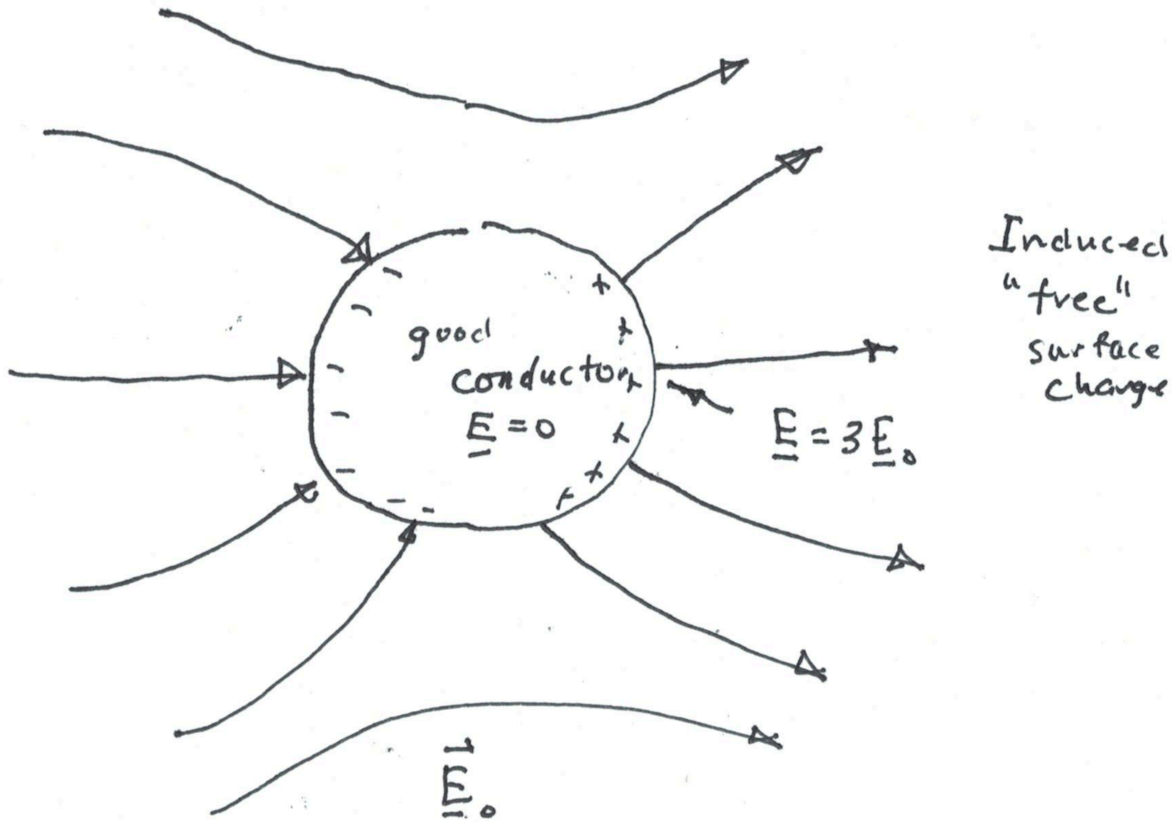
Dielectric Sphere



Magnetized Rod



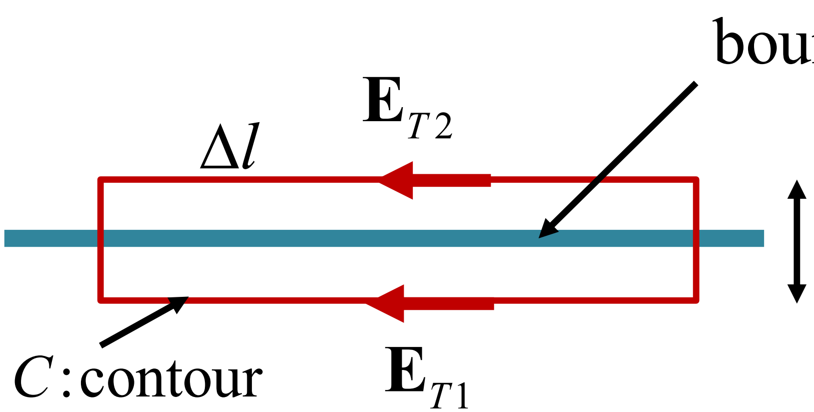
Conducting Sphere



Tangential Component of **E**

Boundary Conditions

Applying Stokes's theorem $\int_A (\nabla \times \mathbf{E}) \cdot \mathbf{da} = -\int_A \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{da} = \oint_C \mathbf{E} \cdot \mathbf{dl}$



$h \rightarrow 0$

$$E_{T2}\Delta l + E_{T1}(-\Delta l) = 0$$

$$\int_A \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{da} \rightarrow 0 \;\text{ since }\; h \rightarrow 0$$

$$\rightarrow \boxed{E_{T2} = E_{T1}}$$

- **The tangential component of electric field E is continuous across a boundary**

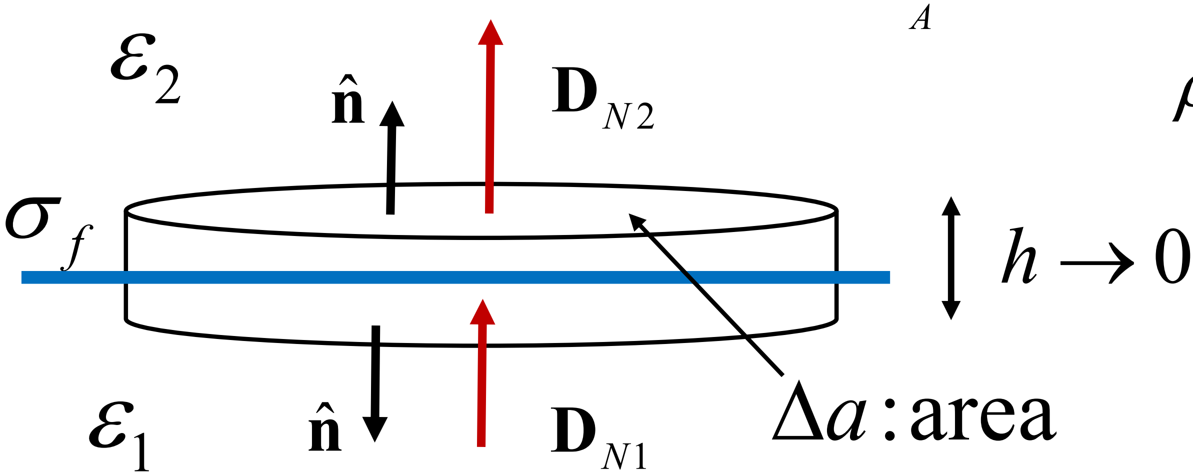
Normal Component of $\mathbf{D}$

Consider the electric flux field: $\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_0(1+\chi_e)\mathbf{E}$ (linear/isotropic matter)

$\nabla \cdot \mathbf{D} = \rho_f \qquad \rho_f =$ free charge density

Applying the Divergence theorem $\int_A \mathbf{D}\cdot\hat{\mathbf{n}}da = \int_V \rho_f(\mathbf{r})d^3r$

$$\rho_f d^3r = \sigma_f da$$



σ_f : free surface charge density

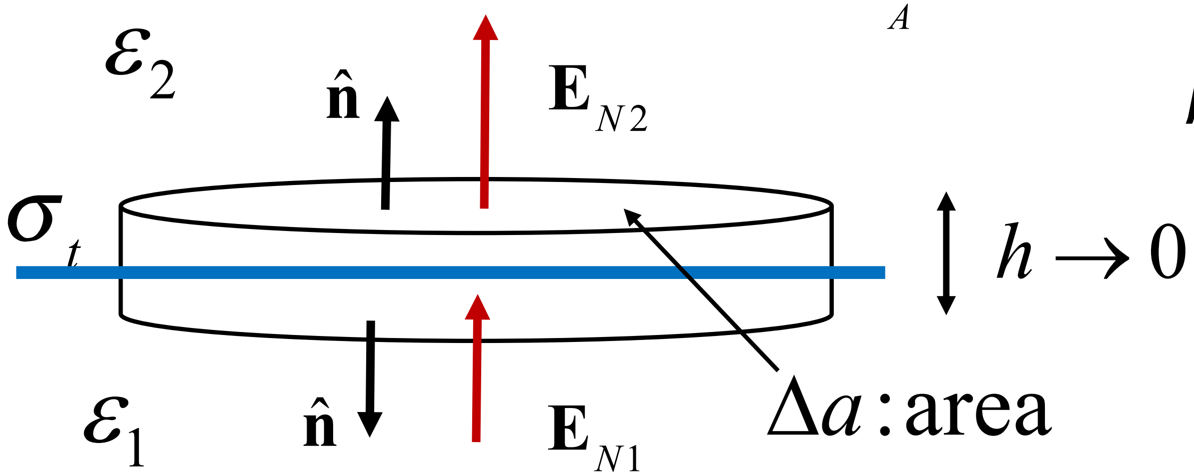
$$(\mathbf{D}_{N2} - \mathbf{D}_{N1})\cdot\hat{\mathbf{n}}\Delta a = Q_f = \sigma_f \Delta a \quad \to \quad \boxed{\varepsilon_2 E_{N2} - \varepsilon_1 E_{N1} = \sigma_f}$$

- The **normal** component of the **electric flux field D** is discontinuous by the **free surface charge density**

Normal E?

$\nabla \cdot \varepsilon_0 \mathbf{E} = \rho_t$ ρ_t = total charge density

Applying the Divergence theorem $\int_A \varepsilon_0 \mathbf{E} \cdot \hat{\mathbf{n}} da = \int_V \rho_t(\mathbf{r}) d^3 r$



$\rho_t d^3 r = \sigma_t da$

σ_t : total surface charge density

$$\varepsilon_0(\mathbf{E}_{N2} - \mathbf{E}_{N1}) \cdot \hat{\mathbf{n}} \Delta a = Q_t = \sigma_t \Delta a \quad \to \quad \boxed{\varepsilon_0 E_{N2} - \varepsilon_0 E_{N1} = \sigma_t}$$

- The **normal** component of the **electric field E** is discontinuous by the **total surface charge density/** ε_0

Tangential Component of $\mathbf{H}$

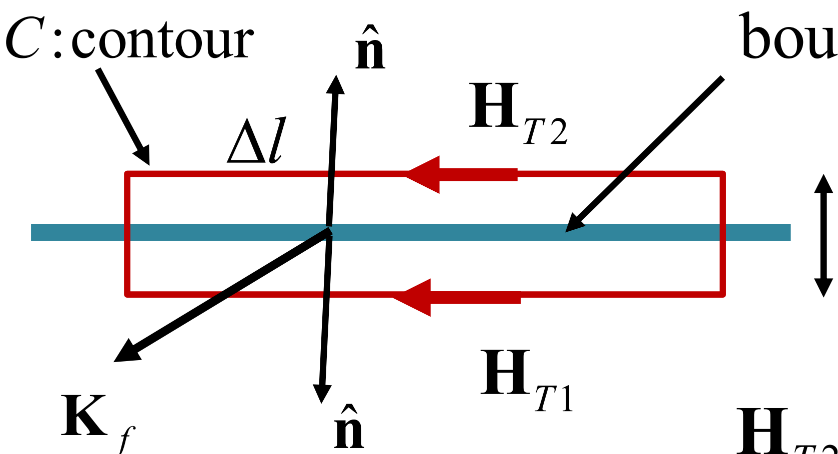
Magnetic field: $\nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t}$

$\mathbf{D} = \varepsilon \mathbf{E}$ (linear/isotropic matter)

$\mathbf{H}=\mathbf{B}/\mu$ (linear/isotropic matter and nonferromagnetic)

Applying Stokes's theorem

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_A \mathbf{J}_f \cdot d\mathbf{a} + \int_A \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{a}$$



free surface current density

$$\int_A \mathbf{J}_f \cdot d\mathbf{a} \;\rightarrow\; \mathbf{J}_f \Delta l h \;\rightarrow\; \mathbf{K}_f \Delta l$$

as $h \rightarrow 0$

$h \rightarrow 0$ $\quad \displaystyle\int_A \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{a} \rightarrow 0$ since $h \rightarrow 0$

$$\mathbf{H}_{T2}\Delta l + \mathbf{H}_{T1}(-\Delta l) = \Delta l \mathbf{K}_f \times \hat{\mathbf{n}}$$

$$\boxed{\mathbf{H}_{T2} - \mathbf{H}_{T1} = \mathbf{K}_f \times \hat{\mathbf{n}}}$$

$\hat{\mathbf{n}}$: outward normal to the surface boundary

- **The tangential** component of **magnetic field H is discontinuous by the free surface current**

Tangential B

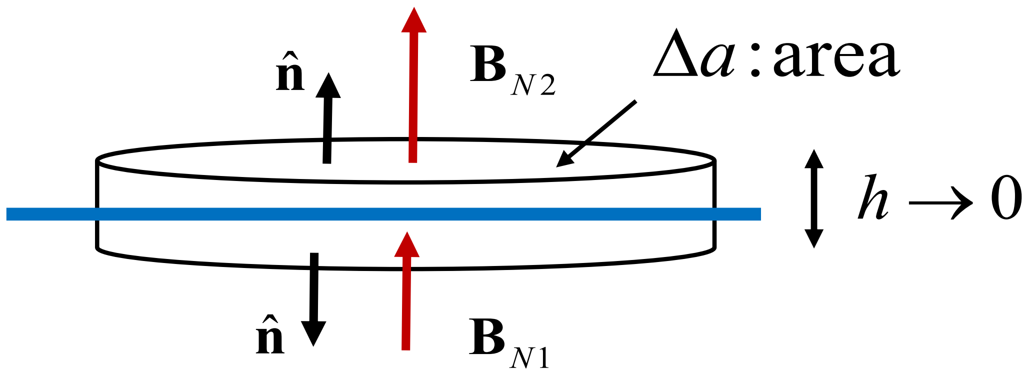
$$\mathbf{H}_{T2} - \mathbf{H}_{T1} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

$$\mathbf{B}_{T2} - \mathbf{B}_{T1} = \mu_0 \mathbf{K}_t \times \hat{\mathbf{n}}$$

- **The tangential** component of **magnetic flux density B is discontinuous by the total surface current** $\times \mu_0$

Normal Component of **B**

$\nabla \cdot \mathbf{B} = 0$ Magnetic flux density : **B**



$$\oint_A \mathbf{B} \cdot \hat{\mathbf{n}} da = 0$$

$$(\mathbf{B}_{N2} - \mathbf{B}_{N1}) \cdot \hat{\mathbf{n}} \Delta a = 0 \quad \rightarrow \quad \boxed{\mathbf{B}_{N2} = \mathbf{B}_{N1}}$$

Normal compontent of **H** $\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}$

$$\rightarrow \quad \boxed{H_{N2} - H_{N1} = -(M_{N2} - M_{N1})}$$

- The **normal** component of the **magnetic flux field B** is **continuous** across boundary

Boundary Condition Summary

Tangential Components

$$\mathbf{E}_{T2} = \mathbf{E}_{T1} \qquad \mathbf{H}_{T2} - \mathbf{H}_{T1} = \mathbf{K}_f \times \hat{\mathbf{n}} \qquad \mathbf{B}_{T2} - \mathbf{B}_{T1} = \mu_0 \mathbf{K}_t \times \hat{\mathbf{n}}$$

Normal Components

$$(\mathbf{D}_{N2} - \mathbf{D}_{N1}) \cdot \hat{\mathbf{n}} \Delta a = Q_f = \sigma_f \Delta a \quad \rightarrow \quad \boxed{\varepsilon_2 E_{N2} - \varepsilon_1 E_{N1} = \sigma_f}$$

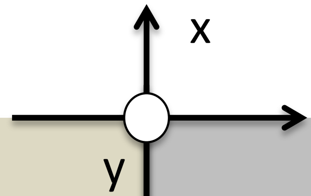
$$\varepsilon_0 (\mathbf{E}_{N2} - \mathbf{E}_{N1}) \cdot \hat{\mathbf{n}} \Delta a = Q_t = \sigma_t \Delta a \quad \rightarrow \quad \boxed{\varepsilon_0 E_{N2} - \varepsilon_0 E_{N1} = \sigma_t}$$

$$(\mathbf{B}_{N2} - \mathbf{B}_{N1}) \cdot \hat{\mathbf{n}} \Delta a = 0 \quad \rightarrow \quad \boxed{\mathbf{B}_{N2} = \mathbf{B}_{N1}}$$

$$\boxed{H_{N2} - H_{N1} = -(M_{N2} - M_{N1})}$$

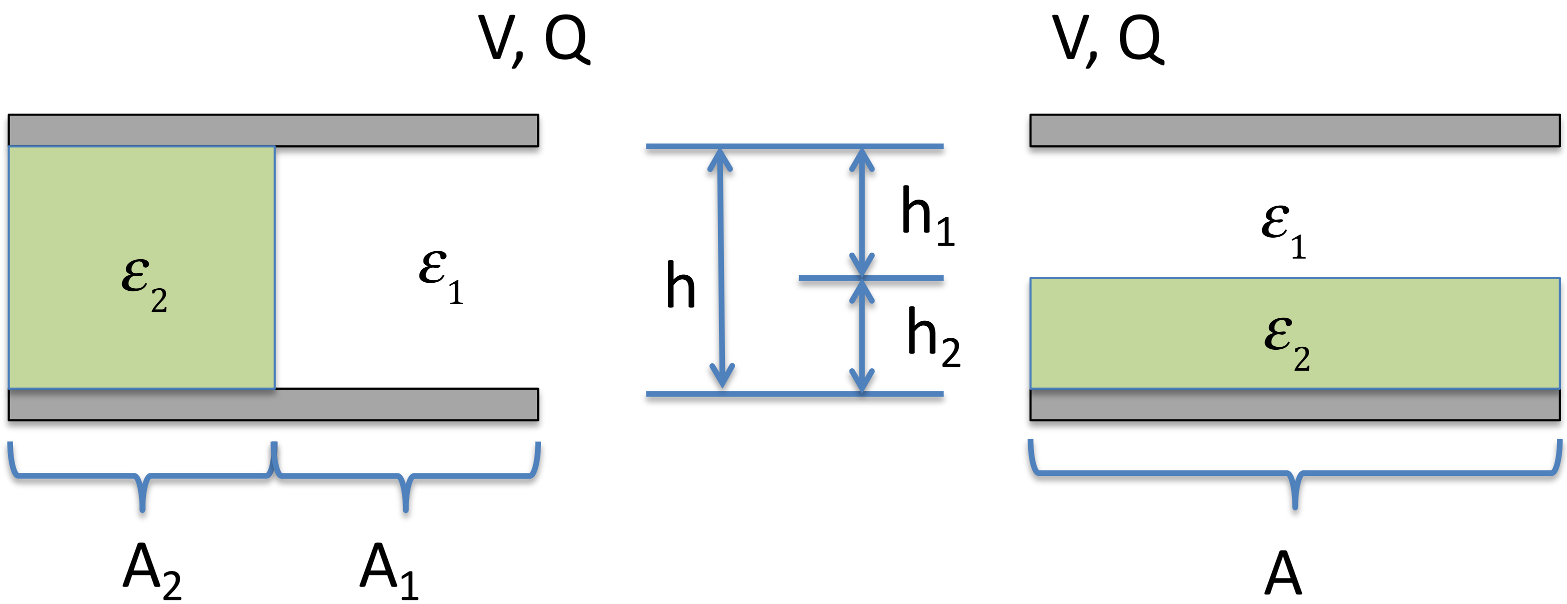
Let's play the boundary condition game!

No free surface charge of current



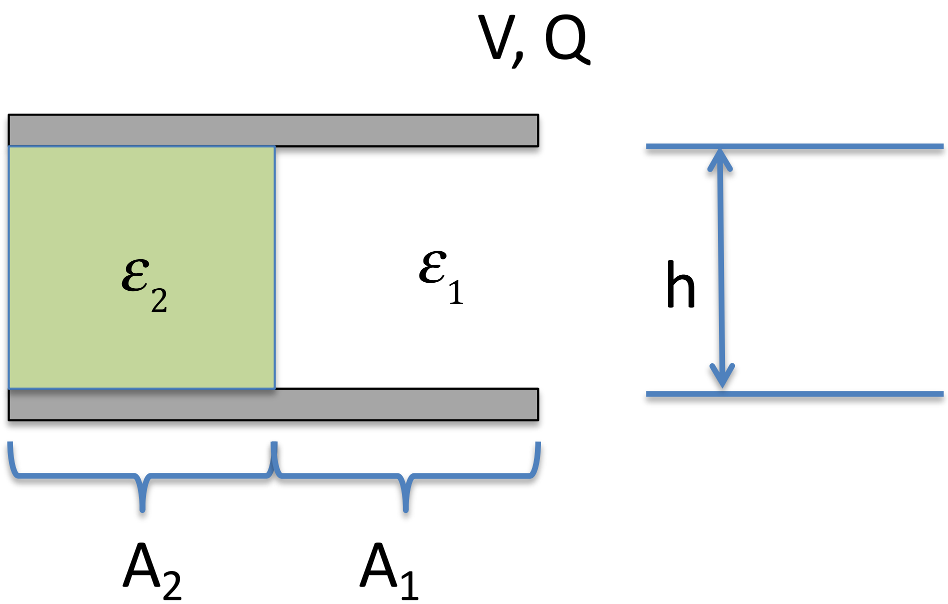
Medium #1	Medium #2
$\varepsilon_1 = 2\varepsilon_0, \mu_1 = \mu_0, \sigma_1 = 0$	$\varepsilon_2 = 4\varepsilon_0, \mu_2 = 2\mu_0, \sigma_2 = 0$
$\mathbf{E}_1 = 2\hat{\mathbf{x}} + 4\hat{\mathbf{y}} + 6\hat{\mathbf{z}}$	$\mathbf{E}_2 = \quad \hat{\mathbf{x}} + \quad \hat{\mathbf{y}} + \quad \hat{\mathbf{z}}$
$\mathbf{D}_1 / \varepsilon_0 = \quad \hat{\mathbf{x}} + \quad \hat{\mathbf{y}} + \quad \hat{\mathbf{z}}$	$\mathbf{D}_2 / \varepsilon_0 = \quad \hat{\mathbf{x}} + \quad \hat{\mathbf{y}} + \quad \hat{\mathbf{z}}$
$\mathbf{B}_1 / \mu_0 = 6\hat{\mathbf{x}} + 7\hat{\mathbf{y}} + 8\hat{\mathbf{z}}$	$\mathbf{B}_2 / \mu_0 = \quad \hat{\mathbf{x}} + \quad \hat{\mathbf{y}} + \quad \hat{\mathbf{z}}$
$\mathbf{H}_1 = \quad \hat{\mathbf{x}} + \quad \hat{\mathbf{y}} + \quad \hat{\mathbf{z}}$	$\mathbf{H}_2 = \quad \hat{\mathbf{x}} + \quad \hat{\mathbf{y}} + \quad \hat{\mathbf{z}}$

Boundary Conditions in a Capacitor



Which boundary Conditions Apply?

Boundary Conditions in a Capacitor



Case #1

$$E_1 = E_2 = V/h \quad \text{tangential E}$$

$$Q_1 = A_1 D_1 = A_1 \varepsilon_1 V/h$$

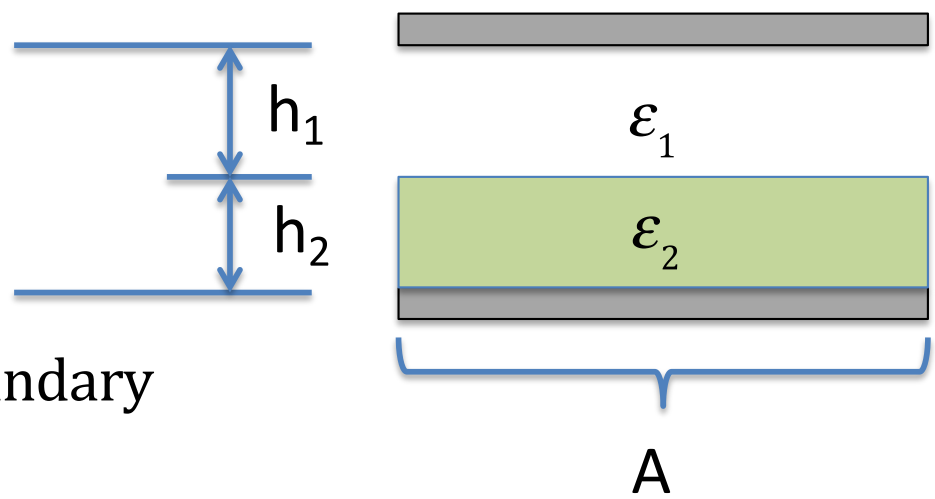
$$Q_2 = A_2 D_2 = A_2 \varepsilon_2 V/h$$

Case #1

$$Q = Q_1 + Q_2 = \left(\frac{A_1 \varepsilon_1 + A_2 \varepsilon_2}{h} \right) V$$

$$C = \left(\frac{A_1 \varepsilon_1 + A_2 \varepsilon_2}{h} \right)$$ Capacitors in parallel

Boundary Conditions in a Capacitor



Case #2

No free surface charge on boundary between ε_1 and ε_2.

$$D_1 = D_2 = Q/A$$

$$E_1 = D_1/\varepsilon_1 = Q/\left(\varepsilon_1 A\right)$$

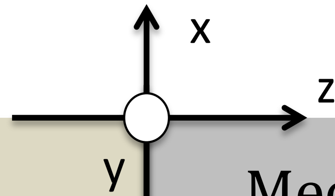
$$E_2 = D_2/\varepsilon_2 = Q/\left(\varepsilon_2 A\right)$$

Case #2

$$V = h_1 E_1 + h_2 E_2 = Q\left(\frac{h_1}{A\varepsilon_1} + \frac{h_2}{A\varepsilon_2}\right)$$

$$C^{-1} = \left(\frac{h_1}{A\varepsilon_1} + \frac{h_2}{A\varepsilon_2}\right)$$ Capacitors in series

Let's play the boundary condition game! With conductivity!!



Medium #1

$$\varepsilon_1 = 2\varepsilon_0, \mu_1 = \mu_0, \sigma_1 = \sigma$$

$$\mathbf{E}_1 = 2\hat{\mathbf{x}} + 4\hat{\mathbf{y}} + 6\hat{\mathbf{z}}$$

$$\mathbf{D}_1 / \varepsilon_0 = \quad \hat{\mathbf{x}} + \quad \hat{\mathbf{y}} + \quad \hat{\mathbf{z}}$$

$$\mathbf{B}_1 / \mu_0 = 6\hat{\mathbf{x}} + 7\hat{\mathbf{y}} + 8\hat{\mathbf{z}}$$

$$\mathbf{H}_1 = \quad \hat{\mathbf{x}} + \quad \hat{\mathbf{y}} + \quad \hat{\mathbf{z}}$$

$$\mathbf{J}_1 / \sigma = \quad \hat{\mathbf{x}} + \quad \hat{\mathbf{y}} + \quad \hat{\mathbf{z}}$$

Medium #2

$$\varepsilon_2 = 4\varepsilon_0, \mu_2 = \mu_0, \sigma_2 = 2\sigma$$

$$\mathbf{E}_2 = \quad \hat{\mathbf{x}} + \quad \hat{\mathbf{y}} + \quad \hat{\mathbf{z}}$$

$$\mathbf{D}_2 / \varepsilon_0 = \quad \hat{\mathbf{x}} + \quad \hat{\mathbf{y}} + \quad \hat{\mathbf{z}}$$

$$\mathbf{B}_2 / \mu_0 = \quad \hat{\mathbf{x}} + \quad \hat{\mathbf{y}} + \quad \hat{\mathbf{z}}$$

$$\mathbf{H}_2 = \quad \hat{\mathbf{x}} + \quad \hat{\mathbf{y}} + \quad \hat{\mathbf{z}}$$

$$\mathbf{J}_2 / \sigma = \quad \hat{\mathbf{x}} + \quad \hat{\mathbf{y}} + \quad \hat{\mathbf{z}}$$