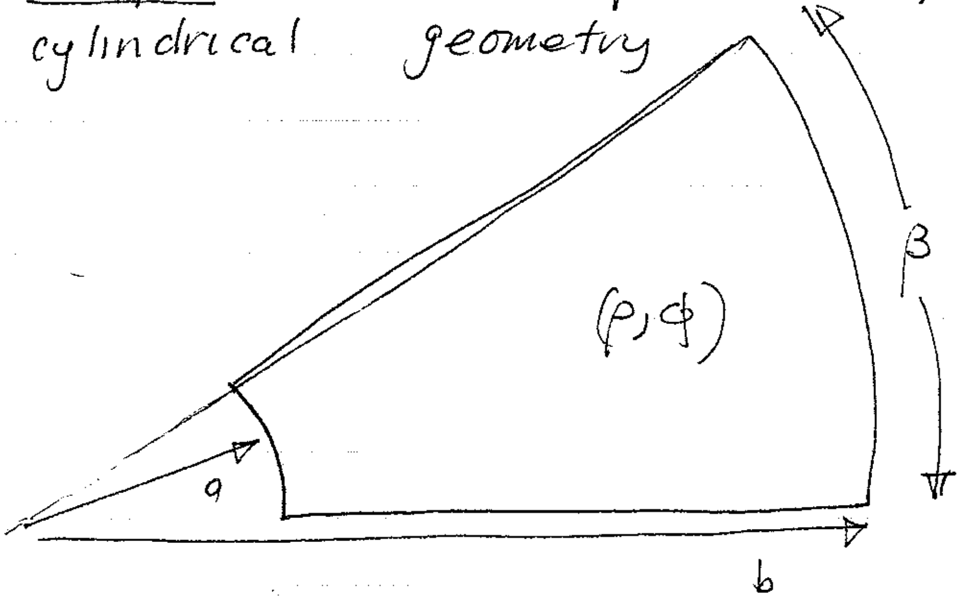
Separation in Cylindrical Coord

Examples of Laplace's Equation in cylindrical geometry



Separation $\qquad \Phi(\rho,\varphi) = R(\rho)\psi(\varphi)$

$$\nabla^2\Phi = 0 \qquad \text{Laplaces Equation}$$

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\Phi + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2}$$

$$= \frac{R\psi}{\rho^2}\left\{ \underbrace{\frac{\rho}{R}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}R}_{\text{separation constant } \nu^2} + \underbrace{\frac{1}{\psi}\frac{\partial^2}{\partial\varphi^2}\psi}_{-\nu^2} \right\}$$

ν^2 is arbitrary

for a particular $\nu^2 > 0$

~~possi~~ solutions can be written

$$\psi = C\exp(i\nu\varphi) \text{ , or } A\sin\nu\phi + B\cos\nu\varphi$$

$$R = \cancel{A r^{\pm\nu}} \quad a\rho^{\nu} + b\rho^{-\nu}$$

arbitrary constants

~~general so~~

IF $\nu^2 = 0$

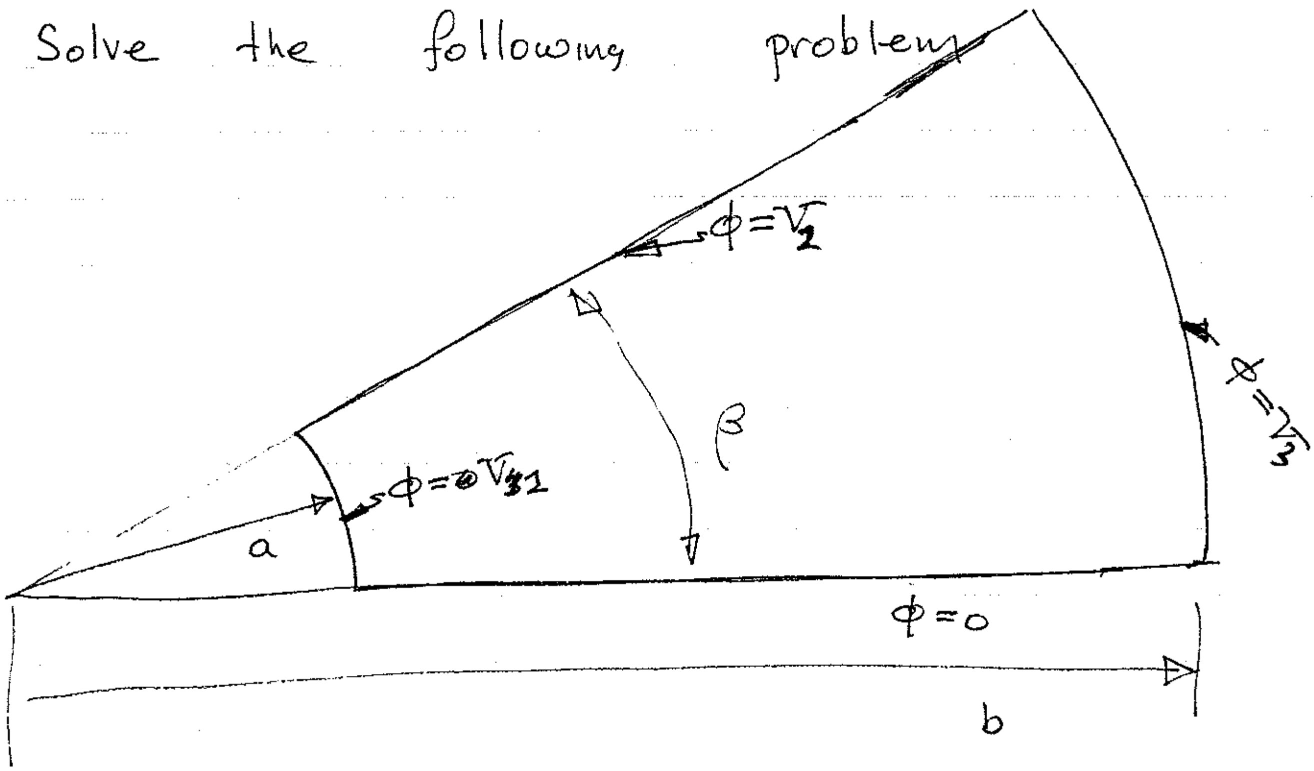
$$\frac{\partial^2\psi}{\partial\varphi^2} = 0 \qquad \psi = A_0 + B_0\varphi$$

$$\rho\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}R = 0 \qquad R = a_0 + b_0\ln\rho$$

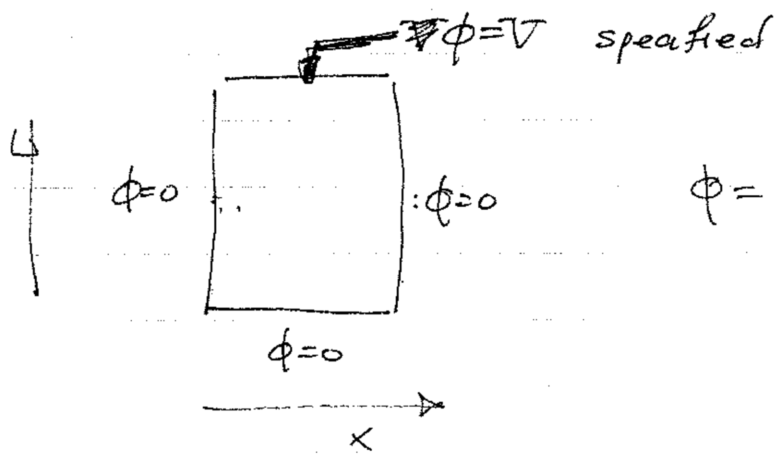
if Region of solution corresponds to full range of φ such that

$\psi(\phi + 2\pi) = \psi(\phi)$ (periodic) ν = integer

Solve the following problem



Similar to our problem



$$\phi = \sum_n a_n \sin\left(\frac{n\pi x}{a}\right) \sinh \frac{n\pi y}{a}$$

$\cancel{\phi = B_0 r}$

Start with a ~~separa~~ solution corresponding to zero separation constant,

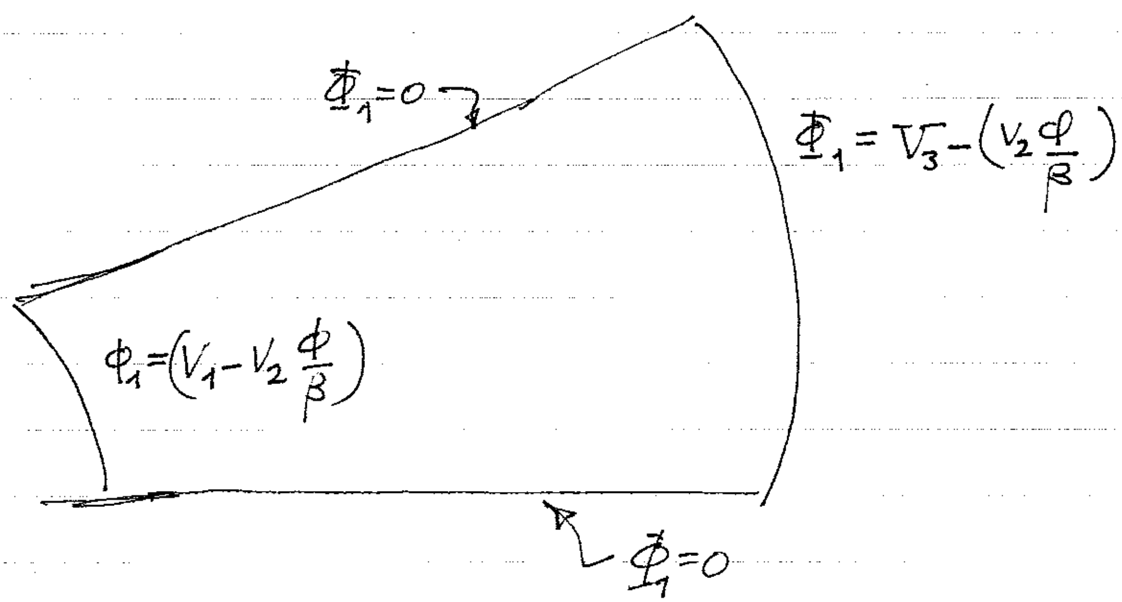
VI3 $\psi = B_0 \varphi \qquad R = a_0$

Why? this gives Φ = constant on upper boundary

$$\Phi = V_2 \frac{\varphi}{\beta} + \Phi_1$$

~~whe~~ $$\nabla^2 \Phi = \cancelto{0}{\nabla^2 V_2 \frac{\varphi}{\beta}} + \nabla^2 \Phi_1 = 0$$

$$\nabla^2 \Phi_1 = 0$$



$$\Phi_1 = \sum_{n=1}^{\infty} \left[a_n \rho^{\nu_n} + b_n \rho^{-\nu_n} \right] \sin \nu_n \varphi$$

$$\nu_n = \frac{n\pi}{\beta} \qquad n^{th} \text{ separation constant}$$

at $\rho = a$

$$\Phi_1 = \left(V_1 - V_2 \frac{\varphi}{\beta}\right) = \sum_{n=1} \left[a_n a^{\nu_n} + b_n a^{-\nu_n} \right] \sin \nu_n \phi$$

multipy by $\sin \nu_m \varphi$ ~~and~~ integrate 0 to β

$$\left[a_m a^{\nu_m} + b_m a^{-\nu_m}\right] \beta/2 = \int_0^\beta d\varphi \, \sin \nu_m \varphi \left[V_1 - V_2 \frac{\varphi}{\beta}\right]$$

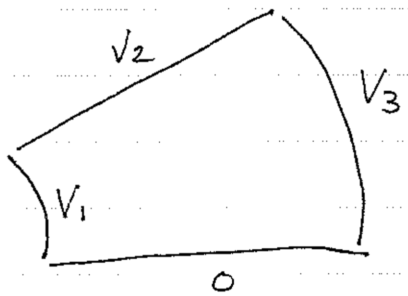
$$\equiv \beta/2 \; d_m$$

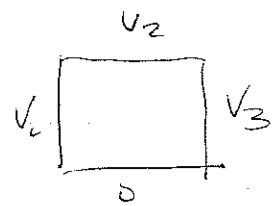
at $\rho = b$

$$\equiv \frac{\beta}{2} e_m$$

$$\left[a_m b^{\nu_m} + b_m b^{-\nu_m}\right] \beta/2 = \int_0^\beta d\phi \, \sin \nu_m \varphi \left[V_3 - V_2 \frac{\varphi}{\beta}\right]$$

We are going to do the following problems:

1) 



2) $$\nabla^2 G = -4\pi\,\delta(\underline{x}-\underline{x}') \quad \text{in } 2D$$

WRITE

$$\phi_1 = \sum_{n=1}^{\infty}\Big[c_n \underbrace{\left[\left(\frac{\rho}{a}\right)^{\nu_n} - \left(\frac{a}{\rho}\right)^{\nu_n}\right]}_{=0 \text{ on } a} + d_n \underbrace{\left[\left(\frac{b}{\rho}\right)^{\nu_n} - \left(\frac{\rho}{b}\right)^{\nu_n}\right]}_{=0 \text{ on } b}\Big]$$

$$\sin \nu_n \varphi$$

at $\rho = a$ $\quad \Phi_1 = V_1 - V_2 \frac{\varphi}{\beta}$

$$= \sum_{n=1}^{\infty} d_n \left[\left(\frac{b}{a}\right)^{\nu_n} - \left(\frac{a}{b}\right)^{\nu_n}\right] \sin \nu_n \varphi$$

multiply by $\sin \nu_m \varphi$ and integrate 0 to β

$$d_m \left[\left(\frac{b}{a}\right)^{\nu_m} - \left(\frac{a}{b}\right)^{\nu_m}\right] \frac{\beta}{2} = \int_0^{\beta} d\beta \sin \nu_m \varphi \left[V_1 - V_2 \frac{\varphi}{\beta}\right]$$

$$\equiv \frac{\beta}{2} \tilde{d}_m \qquad \text{can be done}$$

Like wize

$$C_m \left[\left(\frac{b}{a}\right)^{\nu_m} - \left(\frac{a}{b}\right)^{\nu_m} \right] \frac{\beta}{2} = \int_0^{\beta} d\beta \, \sin \nu_m \varphi \left[V_3 - V_2 \frac{\varphi}{\beta} \right]$$

$$\equiv \frac{\beta}{2} e_m$$

lets suppose $b \ggg a$

$$\phi_1 = \sum_{n=1}^{\infty} e_m$$

Suppose $b >> a, \rho$

and ~~$b \rho \simeq a$~~

$$C_m \to 0 \quad \text{as} \quad \left(\frac{a}{b}\right)^{\nu_m} \to 0$$

$$\phi_1 \simeq \sum_{n=1}^{\infty} \left(\frac{a}{\rho}\right)^{\nu_m} d_n \sin \nu_n \varphi$$

when $a = \rho$ gives $V_1 - V_2 \frac{\varphi}{\beta}$ as it should

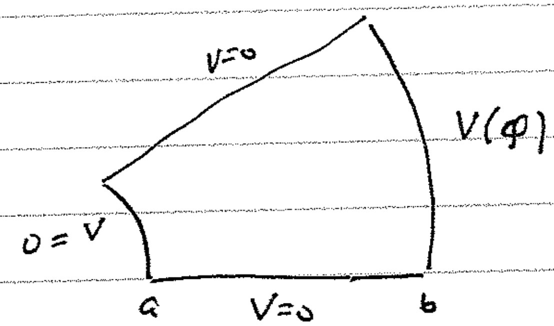
as $\rho / a \to \infty$ (but $\rho/b << 1$)

$\phi_1 \to 0$

as $\quad b/a \to \infty$

$$d_m \sim \left(\frac{a}{b}\right)^{\nu_m}$$

$$c_m \sim \left(\frac{a}{b}\right)^{\nu_m}$$



skip

$$\Phi = \sum_{n=1}^{\infty} \left[c_n \left(\frac{\rho}{a}\right)^{\nu_n} + d_n \left(\frac{a}{\rho}\right)^{\nu_n} \right] \sin \nu_n \varphi$$

when @ $\rho = a$ $\quad \Phi_1 = 0 \qquad d_n = -c_n$

as $\quad \rho = b$

$$V(\varphi) = \sum_n c_n \left[\left(\frac{b}{a}\right)^{\nu_n} - \left(\frac{a}{b}\right)^{\nu_n} \right] \sin \nu_n \varphi$$

$$= \sum_n V_n \sin \nu_n \varphi$$

$$C_n = \frac{V_n}{\left[\left(\frac{b}{a}\right)^{\nu_n} - \left(\frac{a}{b}\right)^{\nu_n} \right]}$$

$$\phi = \sum_{n=1}^{\infty} \frac{\left[(\rho/a)^{\nu_n} - (a/\rho)^{\nu_n} \right]}{\left[\left(\frac{b}{a}\right)^{\nu_n} - \left(\frac{a}{b}\right)^{\nu_n} \right]} V_n \sin n\varphi$$

$\nu_n = \frac{n\pi}{\beta}$ as $(b/a) \to \infty$ $\quad \frac{\left[(\rho/a)^{\nu_n} - (a/\rho)^{\nu_n} \right] V_n}{(b/a)^{\nu_n}}$

leading term is

one with smallest ν_n $(n=1)$

$$\boxed{\phi \simeq \left(\frac{(\rho/a) - (a/\rho)}{(b/a)} \right) V_1 \sin(\pi \phi/\beta)}$$

Green's Function in cylindrical coordinates (2Dimensional case)

$$\nabla^2 G = -4\pi\,\delta(\underset{\sim}{r}-\underset{\sim}{r}')$$

in two demensions ρ, ϕ

two dimension delta fun

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}G + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}G = -4\pi \boxed{\frac{\delta(\varphi-\varphi')\,\delta(\rho-\rho')}{\rho}}$$

$$\int d\phi'\rho' d\rho'\,\frac{\delta(\varphi-\phi')\,\delta(\rho-\rho')}{\rho'} = 1$$

assume periodic in φ

$$\text{write}\quad G = \sum_{\nu=-\infty}^{\infty} \hat{G}_{\nu}(\rho,\rho',\varphi')\,e^{i\nu\phi}$$

$e^{-i\nu\phi} \int \frac{d\phi}{2\pi}$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\hat{G}_\nu - \frac{\nu^2}{\rho^2}\hat{G}_\nu = -4\pi\int_0^{2\pi}\frac{d\phi}{2\pi}\,\frac{\delta(\phi-\varphi')\,\delta(\rho-\rho')}{\rho}\,e^{-i\nu\phi}$$

multiply by $e^{i\nu\varphi}$

can do ϕ integral

$$= -4\pi \frac{1}{2\pi} e^{-i\nu\phi'} \frac{\delta(\rho-\rho')}{\rho}$$

let $\hat{G}_\nu = e^{-i\nu\phi'} G_\nu(\rho,\rho')$

$$G = \sum_{\bar{\nu}=-\infty}^{\infty} G_{\bar{\nu}}\, e^{i\bar{\nu}(\phi-\phi')}$$

$$\boxed{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}G_\nu - \frac{\nu^2}{\rho^2}G_\nu = -2\,\frac{\delta(\rho-\rho')}{\rho}}$$

How to solve such an equation

Option #1 Represent $G_\nu(\rho)$ in a complete set of functions on the interval $\rho=0$ $\rho=\infty$

Range is infinite separation constant will be continuous

$$G_\nu(\rho) = \int dk\, A_\nu(k)\, G_\nu(\rho, k)$$

Bessel transform (Like Fourier transform)

where

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} G_\nu(\rho, k) - \frac{\nu^2}{\rho^2} G_\nu(\rho, k) + k^2 G_\nu(\rho, k) = 0$$

Bessel's equation

$$G_\nu(\rho, k) = J_\nu(k\rho)$$

Option #2 since we have an ordinary equation solve it on either side of δ function & match for $\rho < \rho'$

$$G_\nu = a_\nu \rho^{|\nu|} + b_\nu \rho^{-|\nu|}$$

as $\rho \to 0$ G_ν finite requires

we set $b_\nu = 0$

$$G_\nu = a_\nu \rho^{|\nu|} \qquad \rho < \rho'$$

for $\rho > \rho'$

$$G_\nu = c_\nu \rho^{|\nu|} + d_\nu \rho^{-|\nu|}$$

as $\rho \to \infty$ G finite gives $c_\nu = 0$.

$$G_\nu = d_\nu \rho^{-|\nu|}$$

conditions at $\rho \simeq \rho'$

G is continuous

but $\dfrac{\partial G}{\partial \rho}$ is discontinuous

why

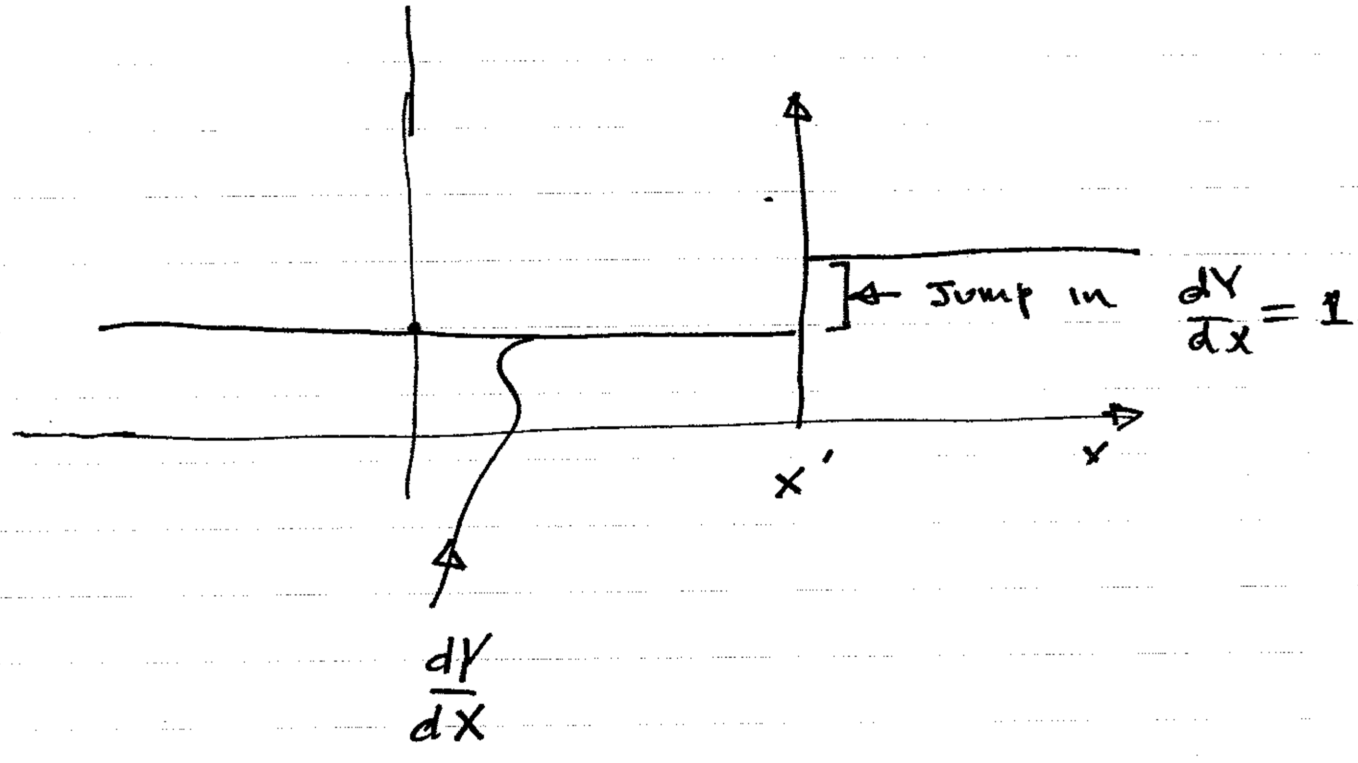
consider the equation

$$\frac{d^2}{dx^2} Y = \delta(x-x')$$

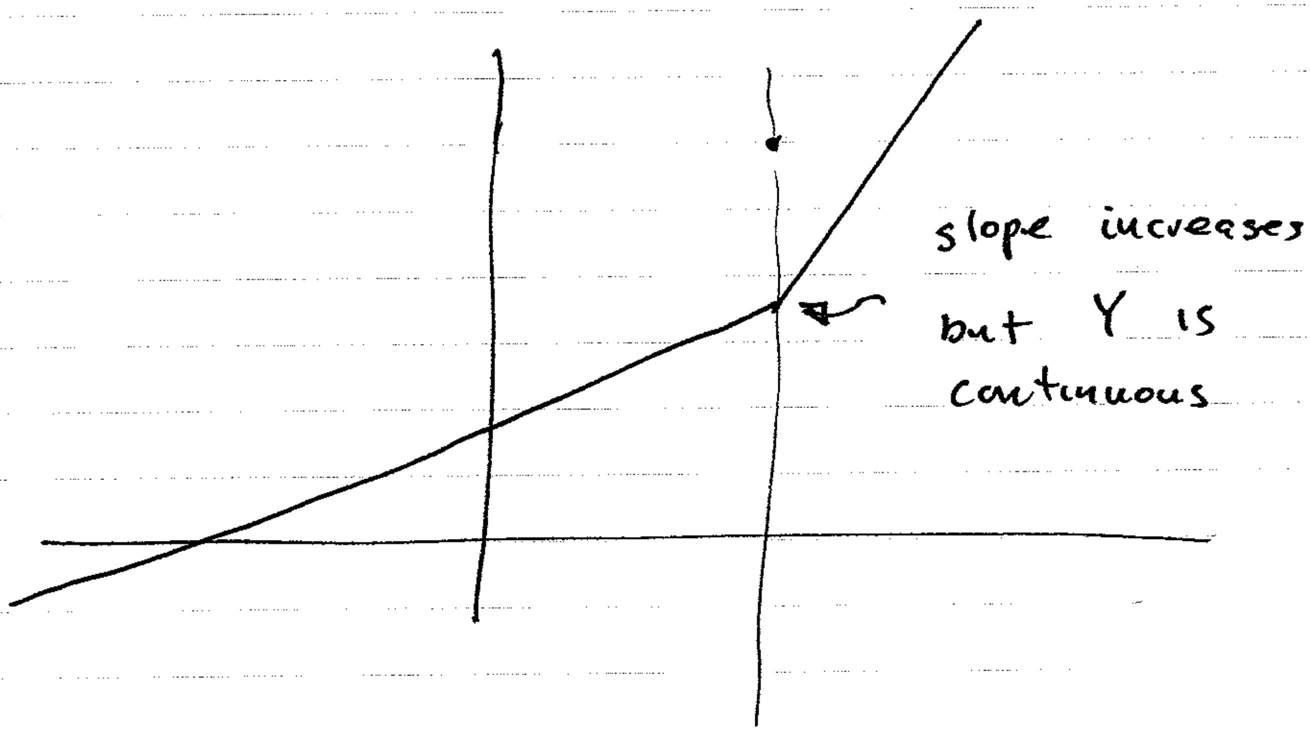
$$\frac{dY}{dx} = \int_{-\infty}^{x} dx' \, \delta(x-x')$$

$$= \left.\frac{dY}{dx}\right|_{-\infty} \qquad x < x'$$

$$= \left.\frac{dY}{dx}\right|_{-\infty} + 1 \qquad x > x'$$



integrate once more to get Y



continuity of G_ν

$$a_\nu \rho'^{|\nu|} = d_\nu \rho'^{-|\nu|}$$

discontinuity of $\partial G / \partial \rho$

for ρ near ρ'

$$\int_{\rho'-\epsilon}^{\rho'+\epsilon} d\rho \left[\frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} G_\nu - \frac{\nu^2}{\rho} G_\nu \right] = -2 \int_{\rho'-\epsilon}^{\rho'+\epsilon} d\rho \, \delta(\rho - \rho')$$

$\epsilon \to 0$

$$\underbrace{\rho \frac{\partial}{\partial \rho} G_\nu \Big|_{\rho'-\epsilon}^{\rho'+\epsilon}}_{\text{Jump}} \quad \sim \epsilon \qquad = \quad -2$$

$$-|\nu| d_\nu \rho'^{-|\nu|} - |\nu| \underbrace{a_\nu \rho'^{|\nu|}}_{= d_\nu \rho'^{-|\nu|}} = -2$$

from continuity of G

$$d_\nu = \frac{1}{|\nu|} \rho'^{|\nu|}$$

$$a_\nu = \frac{1}{|\nu|} \rho'^{-|\nu|}$$

$$G_\nu(\rho,\rho') = \frac{1}{|\nu|} \begin{cases} \left(\frac{\rho}{\rho'}\right)^{|\nu|} & \rho < \rho' \\ \left(\frac{\rho'}{\rho}\right)^{+|\nu|} & \rho > \rho' \end{cases}$$

$$G(\rho,\rho',\phi,\phi') = \sum_{\nu=-\infty}^{\infty} \frac{1}{|\nu|} e^{i\nu(\phi-\phi')} \begin{cases} \left(\frac{\rho}{\rho'}\right)^{|\nu|} & \rho<\rho' \\ \left(\frac{\rho'}{\rho}\right)^{|\nu|} & \rho>\rho' \end{cases}$$

~~symmetric $\rho \leftrightarrow \rho'$~~

~~what if $\nu = 0$ take limit~~

must do $\nu = 0$ dem

$$G = a \qquad \rho < \rho'$$

$$G = b + c \ln \rho \qquad \rho > \rho' \qquad [\text{diverges as } \rho \to \infty]$$

Match G jump $\frac{\partial G}{\partial \rho}$

potential due to line charge diverges

$$G = a \qquad \rho < \rho'$$

$$G = a + c \ln(\rho/\rho') \qquad \rho > \rho'$$

constant undetermined

$$\rho \frac{C}{\rho} = -2$$

$$C = -2$$

$$G = 0 \qquad \rho = b$$

$$G = 0 \qquad \rho = a$$

$\rho < \rho'$ $\quad G = A \ln \frac{\rho}{a}$

G: $\rho > \rho'$ $\quad G = B \ln\left(\frac{\rho}{b}\right)$

$\nu = 0$ case

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial G}{\partial \rho} = -2\,\frac{\delta(\rho-\rho')}{\rho}$$

integrate $\displaystyle\int_0^{\rho}\rho' d\rho'\left\{\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\rho'\frac{\partial G}{\partial\rho'}\right\} = -\begin{cases}-2 & \rho>\rho' \\ 0 & \end{cases}$

$$\rho\frac{\partial G}{\partial\rho} - \cancel{0\left.\frac{\partial G}{\partial\rho}\right|_0}^{\,0} = \begin{cases}-2 & \rho>\rho' \\ 0 & \rho<\rho'\end{cases}$$

$$\frac{\partial G}{\partial\rho} = \begin{matrix}-2/\rho & \rho>\rho' \\ 0 & \rho<\rho'\end{matrix}$$

$$G(\rho) = G(0) \qquad \rho<\rho'$$

$$G(\rho) = G(0) - 2\ln\frac{\rho}{\rho'} \qquad \rho>\rho'$$

note $G \not\to 0$ as $\rho\to\infty$