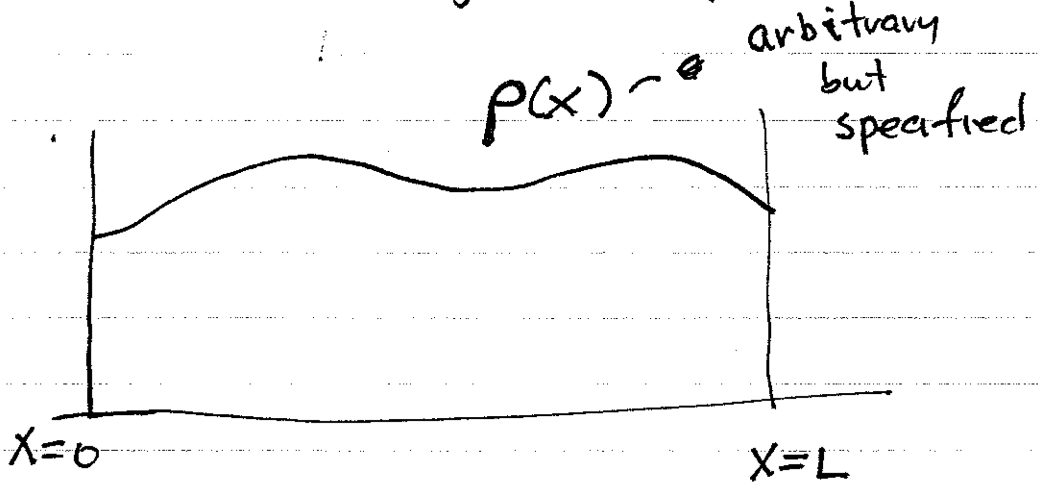
ORTHOGONAL FUNCTIONS

Solution of Poisson's equation based on expansion in terms of orthogonal functions

$$\nabla^2 \phi = -\cancel{4\pi}\rho(\underset{\sim}{x})/\epsilon_0$$

Consider the following 1D example



$$\frac{d^2\phi}{dx^2} = -\cancel{4\pi}\rho(x)/\epsilon_0 \qquad \phi(x=0)=0$$

$$\phi(x=L)=0$$

<u>How to solve</u>

what you would really ~~do~~ is integrate (ordinary equation)

$$\frac{d\phi}{dx} = \left.\frac{d\phi}{dx}\right|_0 - \cancel{4\pi}\int_0^x \frac{\rho(x')}{\epsilon_0}dx'$$

$$\phi(x) = \cancel{\phi(0)}^{\,0} + \int_0^x dx'\frac{d\phi}{dx'}$$

$$= \int_0^x dx'\left[\left.\frac{d\phi}{dx}\right|_0 - \cancel{4\pi}\int_0^{x'} dx''\frac{\rho(x'')}{\epsilon_0}\right]$$

$$= x\left.\frac{d\phi}{dx}\right|_0 - \cancel{4\pi}\underbrace{\int_0^x dx'\int_0^{x'} dx''\rho(x'')}_{\text{can be simplified}}/\epsilon_0$$

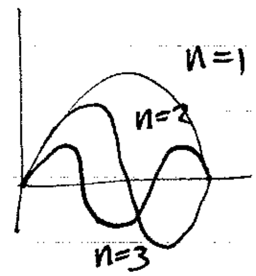
$$= x\left.\frac{d\phi}{dx}\right|_0 - \cancel{4\pi}\int_0^x dx''\rho(x'')(x-x'')/\epsilon_0$$

pick $\left.\frac{d\phi}{dx}\right|_0$ such that $\phi(L)=0$

Alternate method based on Fourier Series

write a Fourier sine series for $\phi(x)$

$$\phi(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}$$



choice of basis functions motivated by B.C's

all terms satisfy b.c.

substitute for $\phi(x)$ in P.E.
multiply by $\sin \frac{m\pi x}{L}$ and integrate from 0 to L

$$\int_0^L dx \, \sin \frac{m\pi x}{L} \left[+ \frac{d^2}{dx^2} \sum_n a_n \sin \frac{n\pi x}{L} \right]$$

assume can be differentiated term by term

$$\sum_n \left(\frac{n\pi}{L}\right)^2 a_n \underbrace{\int_0^L dx \; \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}}_{\begin{cases} \frac{L}{2} & \text{if } m=n \\ 0 & \text{if } m\neq n \end{cases}}$$

$$= -\frac{\cancel{4\pi} L}{\epsilon_0} \int_0^L \frac{dx}{L} \sin\frac{m\pi x}{L}\, \rho(x)$$

algebraic equation $\qquad \frac{d^2}{dx^2} \sin\frac{m\pi x}{L} = -\left(\frac{m\pi}{L}\right)^2 \sin\frac{m\pi x}{L}$

also ~~an~~ orthogonal fn is eigen of Laplacian

$$\frac{L}{2}\left(\frac{m\pi}{L}\right)^2 a_m = -\cancel{4\pi} \; \cancel{\int_0^L} \; \frac{L}{2}\frac{\rho_m}{\epsilon_0}$$

where

$$\rho_m = \frac{2}{L}\int_0^L dx \; \sin\frac{m\pi x}{L}\,\rho(x)$$

Solution

$$\left.\begin{aligned} \rho(x) &= \sum_m \rho_m \sin\frac{m\pi x}{L} \\ \phi(x) &= \sum_m a_m \sin\frac{m\pi x}{L} \end{aligned}\right\}$$ both ϕ and ρ expanded in orthogonal functions

$$a_m = \frac{-\cancel{4\pi}\,\rho_m/\epsilon_0}{(m\pi/L)^2}$$

if solving Poisson Equation

$$\nabla^2\phi = -\cancel{4\pi}\rho/\epsilon_0$$

choose $\cancel{\nabla^2\phi}$ $\nabla^2\cancel{\phi}u_n + k_n^2 u_n = 0$

* ~~manipulate pro~~

eigen value

* choose a set which satisfies appropriate Boundary conditions
  ~~manipulate pro~~

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* exploit symmetry to simplify problem

* ~~practical~~ manipulate problem so that solution converges with minimum number of terms

orthogonal functions

$$f(x) = \sum_{n=0}^{\infty} a_n u_n(x)$$

$$a_n = \int_a^b dx' \, u_n(x') f(x')$$

$$f(x) = \sum_{n=0}^{\infty} \int_a^b dx' \, u_n(x') u_n(x) f(x')$$

complete set $\Rightarrow$ $\displaystyle\sum_{n=0}^{\infty} u_n(x') u_n(x) = \delta(x-x')$

Choice of basis functions

Choose a set which simplifies the differential equation

if solving Laplace Eqn $\nabla^2 \phi = 0$

choose $\nabla^2 u_n = 0$ [u_n not complete]

Return to our example

$$\frac{d^2}{dx^2}\phi = -\cancel{4\pi}\rho(x)\cancel{/\epsilon_0} \qquad 0 < x < L$$

$\phi(0) = \phi_0 \qquad \phi(L) = \phi_1$ note

No brainer approach

expand $\phi(x)$ in basis $u_n(x)$
satisfy

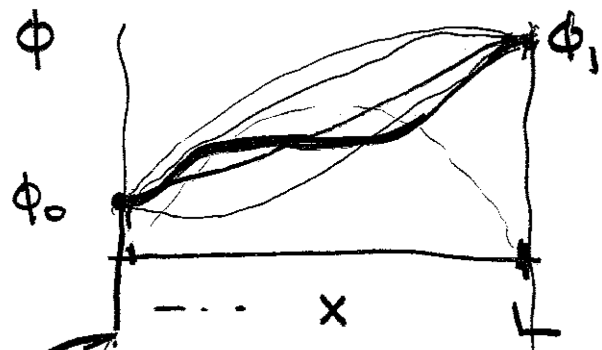
$$\frac{d^2 u_n}{dx^2} = -k_n^2 u_n \qquad \text{and} \qquad \boxed{\begin{aligned} u_n(0) &= 0 \\ u_n(L) &= 0 \end{aligned}}$$

normalisation con

$$u_n(x) = C_n \sin \frac{n\pi x}{L} \qquad n = 1, 2, 3, \ldots$$

$$k_n = \frac{n\pi}{L} \qquad C_n = \sqrt{\frac{2}{L}} \qquad \int_0^L dx\, u_n^2(x) = 1$$

picture of $\phi(x)$ we are creating



~~substit~~ not absolutely convergent !

multiply equation by $u_n(x)$ & integrate over x,

$$\int_0^L dx\, u_n(x)\,\frac{d^2\phi}{dx^2} = -\cancel{4\pi}\,\underbrace{\int_0^L dx\, u_n(x)\,\rho(x)}_{\rho_n}\,/\epsilon_0$$

$$\| $$

$$\underbrace{u_n(x)\,\frac{d\phi}{dx}\Big|_{0+\epsilon}^{L-\epsilon}}_{0} - \phi\,\frac{du_n}{dx}\Big|_{0+\epsilon}^{L-\epsilon} + \int_{0+\epsilon}^{L-\epsilon} dx\,\phi(x)\,\frac{d^2 u_n}{dx^2}$$

$$= -\cancel{4\pi}\rho_n\cancel{/\epsilon_0}$$

$u_n(x) \to 0$

$$\frac{du_n}{dx} = k_n C_n \cos k_n x$$

$$-k_n C_n\left[\phi_1 \cos n\pi - \phi_0\right] + \int_{0+\epsilon}^{L-\epsilon} dx\,\phi(x)\left(-k_n^2 u_n\right)$$

$$= -\cancel{4\pi}\rho_n\cancel{/\epsilon_0}$$

$$\phi_n = \int dx\, u_n \phi(x)$$

$$+k_n^2\phi_n = +\cancel{4\pi}\rho_n/\epsilon_0 + k_n c_n\left[\phi_1 \cos n\pi - \phi_0\right]$$

$$\phi_n = \underbrace{\frac{\cancel{4\pi}\rho_n}{k_n^2\epsilon_0}}_{\sim \frac{1}{n^2}} + \frac{\overset{\sqrt{\frac{2}{L}}}{c_n}}{k_n}\left[\phi_1 \cos n\pi - \phi_0\right]$$

$\sim 1/n$ note poor convergence

$$k_n = \frac{n\pi}{L}$$

<u>Better approach</u>

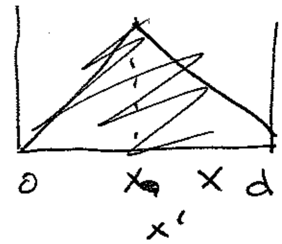
$$\phi(x) = \phi_p(x) + \phi_h(x)$$

$$\frac{d^2\phi_h}{dx^2} = 0 \qquad \phi_h(0) = \phi_0$$

$$\phi_h(L) = \phi_1$$

$$\frac{d^2\phi_p}{dx^2} = -4\pi\rho(x)/\epsilon_0 \qquad \phi_p(0) = \phi_p(L) = 0$$

$$\frac{d^2 G(x,x_0)}{dx^2} = -4\pi\delta(x-x_0)$$



$$G_D(x,x') = \begin{cases} -4\pi\left(1-\frac{x}{d}\right)x' & 0<x'<x \\ -4\pi\left(1-\frac{x'}{d}\right)x & x<x'<d \end{cases}$$

Alternate representation in orthogonal function

$$\phi(x) = \int_0^d dx'\, G_0(x,x')\frac{\rho(x')}{4\pi}$$

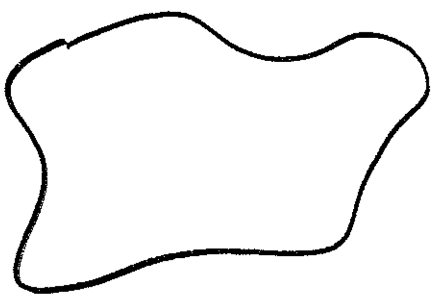
$$\phi_n(x) = \phi_0 + \frac{x}{L}(\phi_1 - \phi_0)$$

$$\boxed{\phi_p(x) = \sum_n \frac{4\pi\rho_n}{k_n^2} u_n(x)}$$

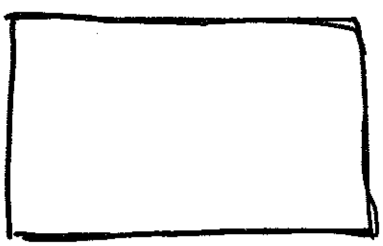
Solutions in higher dimensions

$$\nabla^2 \cancel{\phi} u_n = -k_n^2 u_n$$

how to find



for general boundary
~~not~~ no analytic solutions



for ~~symmetr~~ symmetric ~~separable~~ systems
separation of variables

...mments on separation of variables

Basis Functions For Laplace's Eq.: Rectangular Coordinates

$$\nabla^2 h = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) h = 0$$

separation of variables: assume

$$h(x,y,z) = X(x)\, Y(x)\, Z(z)$$

go through avg

$$\underbrace{\frac{1}{X}\frac{\partial^2}{\partial x^2} X}_{\text{depends only on } x} + \underbrace{\frac{1}{Y}\frac{\partial^2}{\partial y^2} Y}_{\text{only } y} + \underbrace{\frac{1}{Z}\frac{\partial^2}{\partial z^2} Z}_{\text{only } z} = 0$$

$$\frac{1}{X}\frac{\partial^2}{\partial x^2} X = -\alpha^2$$

$$\frac{\partial^2}{\partial x^2} X + \alpha^2 X = 0 \qquad X = e^{\pm i\alpha x} \qquad X = \cos\alpha x, \sin\alpha x$$

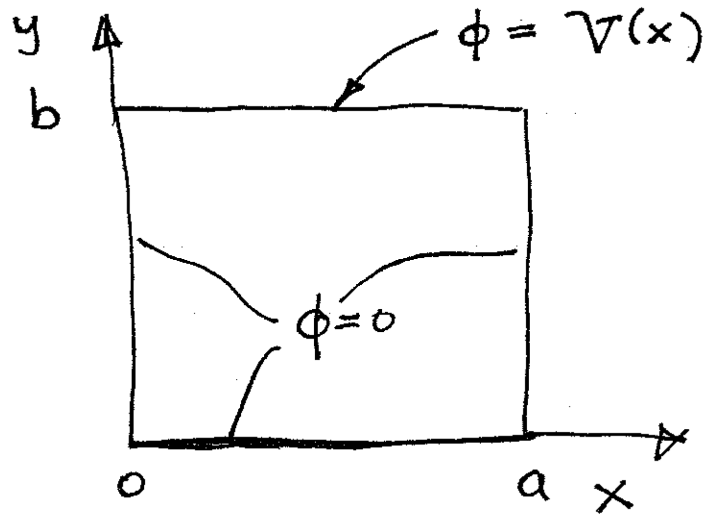
$$\frac{1}{Y}\frac{\partial^2}{\partial y^2} Y = -\beta^2 \qquad Y = e^{\pm i\beta y}$$

$$\frac{\partial^2}{\partial z^2} Z - (\alpha^2+\beta^2) Z = 0 \qquad Z = e^{\pm\sqrt{\alpha^2+\beta^2}\, z}$$

$$\boxed{h(x,y,z) = e^{\pm i\alpha x}\, e^{\pm i\beta y}\, e^{\pm\sqrt{\alpha^2+\beta^2}\, z}}$$

- α and β are arbitrary functions (could be complex)
- note that can permute the variables x, y, z

<u>Example</u> Consider a two dimensional Domain



$$\nabla^2 \phi = 0$$

$$\phi(x,y) = \sum_n A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

why sin? satisfies bc at $x=0$ & $x=a$

why sinh? satisfies bc at $y=0$

(otherwise) $\sum_n \sin\left(\frac{n\pi x}{a}\right)\left[A_n \sinh\left(\frac{n\pi y}{a}\right) + B_n \cosh\left(\frac{n\pi y}{a}\right)\right]$

or $$\sum_n \sin\left(\frac{n\pi x}{a}\right)\left[C_n e^{\frac{n\pi y}{a}} + D_n e^{-\frac{n\pi y}{a}}\right]$$

at $y = b$

$$\phi(x,b) = V(x) = \sum_n \sin\left(\frac{n\pi x}{a}\right) A_n \sinh\left(\frac{n\pi b}{a}\right)$$

$$V(x) = \sum_n V_n \sin\left(\frac{n\pi x}{a}\right)$$ FOURIER

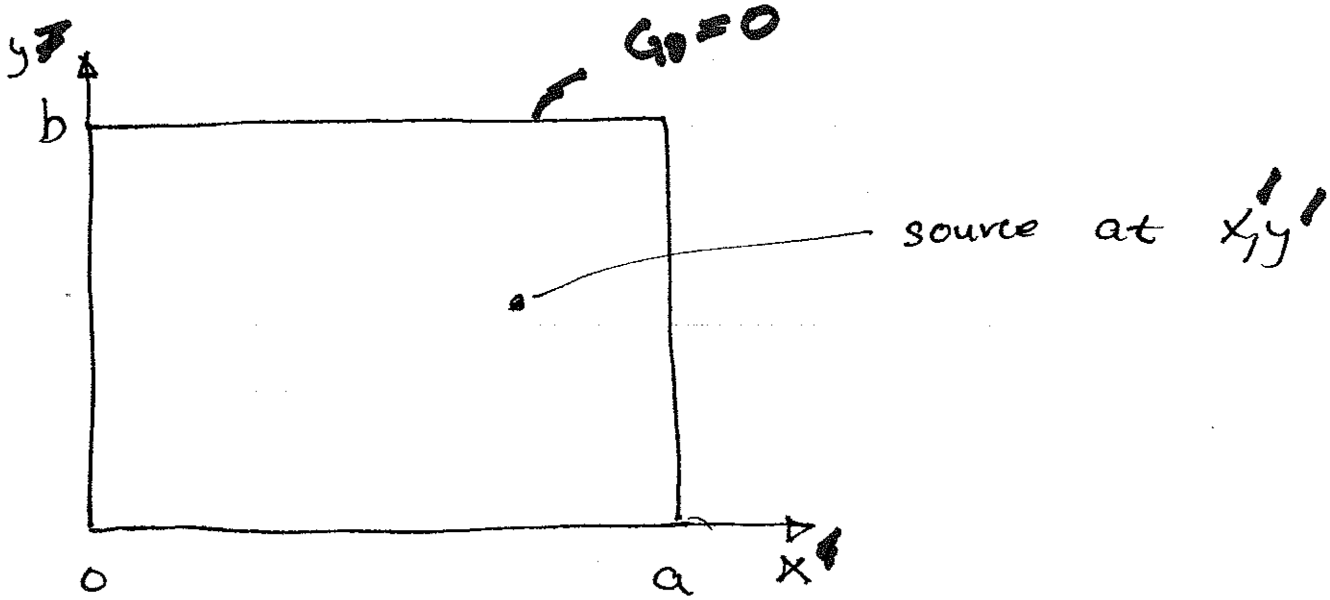
$$V_n = \frac{2}{a}\int_0^a \sin\left(\frac{n\pi x}{a}\right) V(x)\, dx$$

THUS $A_n \sinh\left(\frac{n\pi b}{a}\right) = V_n$

determines A_n

Dirichlet

Green's Function for a Rectangular box



$$\nabla^2 G(\underset{\sim}{x}, \underset{\sim}{x}') = -4\pi\,\delta(x-x')\,\delta(y-y')$$

Here we are not solving Laplacian so separation constants not equal

~~pick~~ ~~basis~~ $\nabla^2 u + k_t^2 u = 0$

$$G(\underset{\sim}{x}, \underset{\sim}{x}') = \sum_{m,n} G_{m,n}(x', y') \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}$$

$$\nabla^2 G(x,x') = \cancel{-\sum_{m,n}\left(k_n^2+k_m^2\right)}$$

$$= -\sum_{m,n}\left(\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right) G_{m,n} \sin\frac{n\pi x'}{a}\sin\frac{m\pi y'}{b}$$

$$= -4\pi\,\delta(x-x')\,\delta(y-y')$$

multiply by $\sin\frac{p\pi x'}{a}\sin\frac{l\pi y'}{b}$

and integrate over x' & y'

only $n=p$, $m=l$ term survives

$$\int_0^a dx'\int_0^b dy'\,\sin^2\frac{p\pi x'}{a}\,\sin^2\frac{l\pi}{b}y' = \frac{ab}{4}$$

$$\int_0^a dx'\int_0^b dy'\,\sin\frac{p\pi x'}{a}\,\sin\frac{l\pi}{b}y'\,\delta(x-x')\,\delta(y-y')$$

$$= \sin\frac{p\pi x}{a}\,\sin\frac{l\pi y}{b}$$

THUS

$$-\frac{ab}{4}\left[\left(\frac{p\pi}{a}\right)^2+\left(\frac{l\pi}{b}\right)^2\right]G_{p,l}(x',y')$$

$$= -4\pi \sin\frac{p\pi x'}{a}\sin\frac{l\pi y'}{b}$$

determines $G_{p,l}$

$$G(x,y,x',y') = \sum_{nm}\frac{16\pi}{ab}\frac{1}{\left[\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right]}$$

$$\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\sin\frac{m\pi y}{b}\sin\frac{m\pi y'}{b}$$