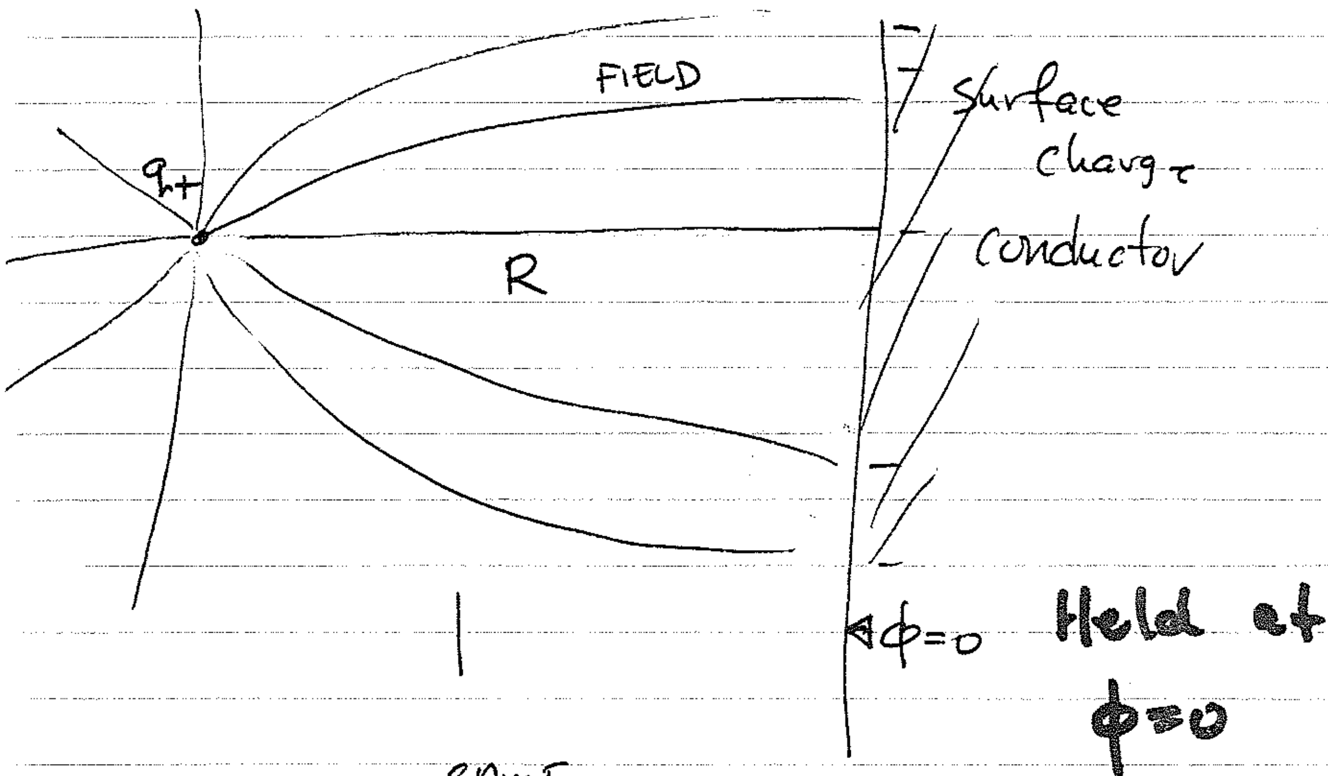
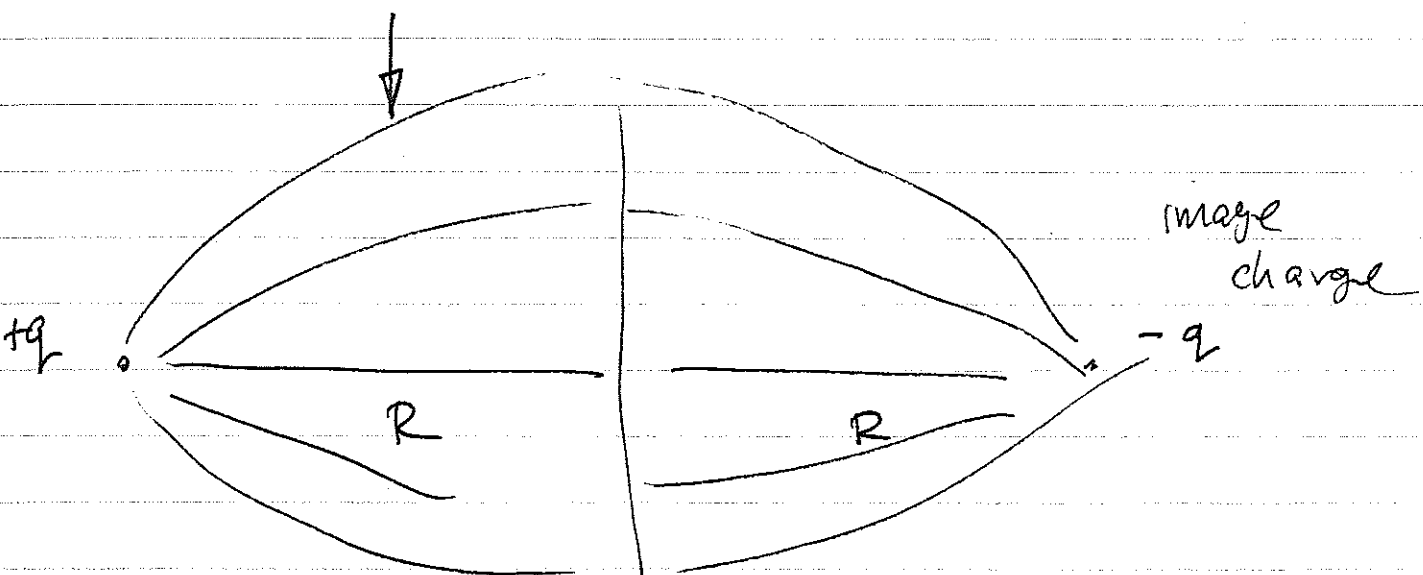
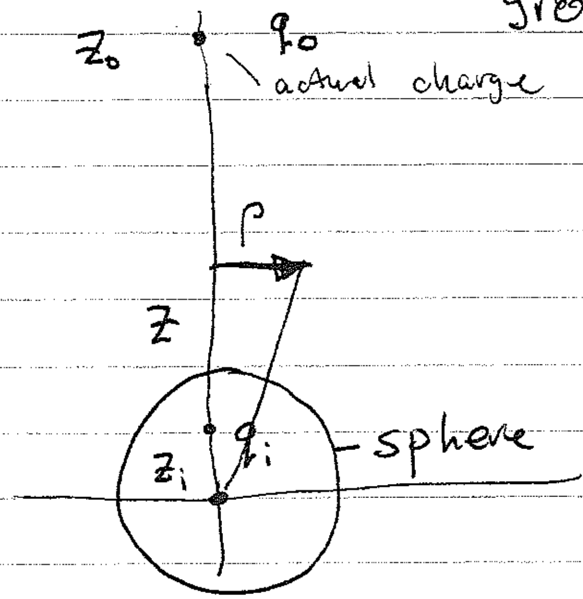
II Method of images works with sufficient symmetry



SAME



suppose we wish to find image charge for a conducting sphere
grounded ($\phi = 0$)



can I find a q_i & Z_i such that

$\phi = 0$ on surface of sphere

of radius a

Use polar coordinates

potential due to act   im

$$\phi(\rho, z) = \frac{q_0}{\sqrt{\rho^2 + (z - z_0)^2}} + \frac{q_i}{\sqrt{\rho^2 + (z - z_i)^2}}$$

$= 0$ on sphere of radius a $q_0 q_i < 0$ opposite charges

FIND LOCUS OF POINTS FOR $\phi = 0$

$$q_0^2\left(\rho^2+z^2+z_i^2-2zz_i\right) - q_i^2\left(\rho^2+z^2+z_0^2-2zz_0\right)=0$$

We want this to describe a sphere of radius a

i.e. $\boxed{\rho^2+z^2-a^2=0}$

THIS REQUIRES

$$q_0^2 z_i - q_i^2 z_0 = 0$$

TERMS LINEAR IN z VANISH

$$q_0^2 z_i^2 - q_i^2 z_0^2 = -(q_0^2-q_i^2)a^2$$

$$z_i = \frac{q_i^2 z_0}{q_0^2}$$

$$\cancel{q_0^2}\left(\frac{q_i^2}{q_0^2}\right)^2 z_0^2 - \frac{q_i^2}{q_0^2} z_0^2 = \left(1-\frac{q_i^2}{q_0^2}\right)a^2$$

Solution $z_i z_0 = a^2$ geometric mean

$$q_i = -q_0\, a/z_0$$

note as $z_0 \to a$

$z_i \to a \qquad q_i \to -q_0$

let $z_0 = a + \epsilon$

$\epsilon \ll a$

then $z_i \simeq a - \epsilon$

$q_i \simeq -q_0$ images close to plane

calculate surface charge densith

$$\phi = \frac{q_0}{|\underset{\sim}{r} - \underset{\sim}{e}_z z_0|} + \frac{q_i}{|\underset{\sim}{r} - \underset{\sim}{e}_z z_i|}$$

normal electric field

$$E_r = - \underset{\sim}{e}_r \cdot \nabla \phi \Big|_{|\underset{\sim}{r}|=a} = \sigma/\epsilon_0$$

$$= + \underset{\sim}{e}_r \cdot \left[\frac{q_0(\underset{\sim}{r} - \underset{\sim}{e}_z z_0)}{|\underset{\sim}{r} - \underset{\sim}{e}_z z_0|^3} + \frac{q_i(\underset{\sim}{r} - \underset{\sim}{e}_z z_i)}{|\underset{\sim}{r} - \underset{\sim}{e}_z z_i|^3} \right]$$

note, on sphere $\underset{\sim}{r} = a\underset{\sim}{e}_r$

$$\frac{q_0}{|\underset{\sim}{r} - e_z z_0|} + \frac{q_i}{|\underset{\sim}{r} - e_z z_i|} = 0$$

$$\frac{1}{|\underset{\sim}{r} - \underset{\sim}{e}_z z_i|^3} = -\frac{q_0^3}{q_i^3 |\underset{\sim}{r} - \ldots}$$

$$|\underset{\sim}{r} - e_z z_i| = - |\underset{\sim}{r} - e_z z_0| \frac{q_0}{q_i}$$

$q_0 - q_i$

$$E_r = \frac{+\,e_r \cdot}{4\pi\epsilon_0 |r - e_z z_0|^3}\left[q_0 (r - e_z z_0) - q_i (r - e_z z_i)\frac{q_0^3}{q_i^3} \right]$$

$|r| = a$

$$= + \cancel{\frac{q_0}{4\pi\epsilon_0 |r - e_z z_0|^3}\, a \,\cos\theta\, z_0 - \left(1 - \frac{q_0^2}{q_i^2}\right)}$$

substitute $q_i^2 = q_0^2 a^2 / z_0^2$

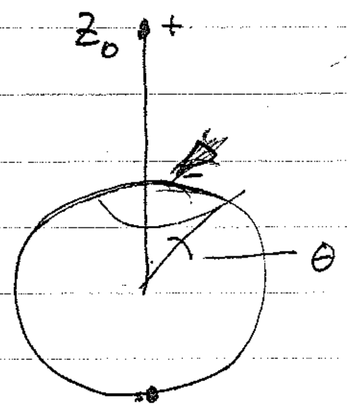
$$= + \frac{q_0}{4\pi\epsilon_0 |r - e_z z_0|^3}\left\{ a\left(1 - \frac{q_0^2}{q_i^2}\right) - e_r \cdot e_z \underbrace{\left(q_0 z_0 - \frac{q_0^3}{q_i^2} z_i\right)}_{0} \right\}$$

$r = a\, e_r$

$$|r - e_z z_0| = \sqrt{a^2 + z_0^2 - 2 z_0 a \cos\theta}$$

$$\cancel{4\pi}\,\sigma = \frac{E_r}{4\pi}$$

σ largest when



$q_0 z_0 - \frac{q_0^3}{q_i^2}$

$z_i = \frac{a^2}{z_0}$

$\cancel{E_x}$

smallest

$$\cancel{\sigma = -\frac{q_0 a (1}{4\pi}}$$

$$E_r = \frac{-\,q_0}{|r}$$

$$E_r = \frac{q_0}{4\pi\epsilon_0 |\underset{\sim}{r} - \underset{\sim}{e}_z z_0|^3} \left\{ a\left(1 - \frac{q_0^2}{q_i^2}\right) - \underset{\sim}{e}_r \cdot \underset{\sim}{e}_z \left(q_0 z_0 - \frac{q_0^3}{q_i^3} z_i \right) \right\}$$

SUBSTITUTE $\quad q_i = -q_0 \left(\frac{a}{z_0}\right)$

$$z_i = \frac{a^2}{z_0}$$

$$1 - \frac{q_0^2}{q_i^2} = 1 - \frac{z_0^2}{a^2}$$

$$q_0 z_0 - \frac{q_0^3}{q_i^2} z_i = q_0 z_0 - \frac{q_0^3 a^2}{z_0} \frac{z_0^2}{q_0^2 a^2} = 0$$

$$E_r = \frac{q_0}{4\pi\epsilon_0 |\underset{\sim}{r} - \underset{\sim}{e}_z z_0|^3} \, a\left(1 - \underbrace{\frac{z_0^2}{a^2}}_{\text{negative}}\right)$$

$$|\underset{\sim}{r} - e_z z_0| = \sqrt{r^2 + z_0^2 - 2 r z_0 \cos\theta}$$

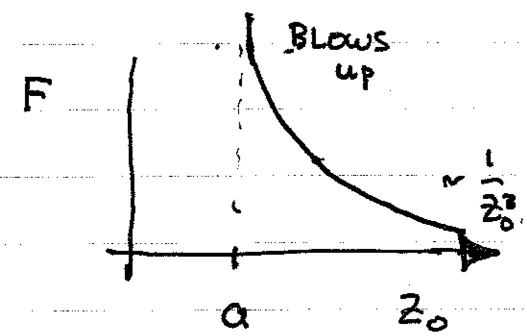
Calculate force ~~on~~ of attraction between charge ~~due~~ and ~~to~~ sphere. FORCE on q_o same as produced by image charge

force on q_o

$$\underset{\sim}{F} = \frac{q_o q_i \hat{e}_z (z_o - z_i)}{|z_o - z_i|^3}$$

where $z_i = a^2/z_o$

$$q_i = -q_o\, a/z_o$$

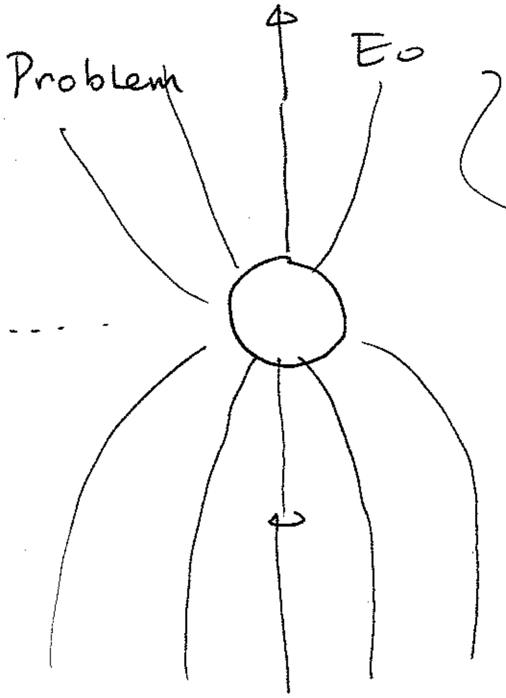


$$\underset{\sim}{F} = -e_z \left(\frac{q_o^2}{z_o^2}\right) \frac{a}{z_o(1-a^2/z_o^2)^2}$$

can also show that it is the same as

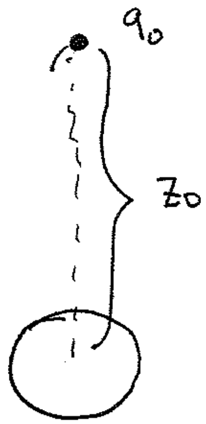
$$F_{sphere} = \frac{1}{2}\int da' \sigma \frac{q_o(\underset{\sim}{r} - z_o e_z)}{|\underset{\sim}{r} - z_o \underset{\sim}{e}_z|^3} = -$$

Problem



uniform electric field $\underset{\sim}{E}_0$

<u>metal</u> sphere of radius a



without sphere

$$\underset{\sim}{E}(0) = -\frac{q_0}{4\pi\epsilon_0 z_0^2}\hat{z}$$

let $q_0 = -4\pi\epsilon_0 E_0 z_0^2$

and $z_0 \to \infty$

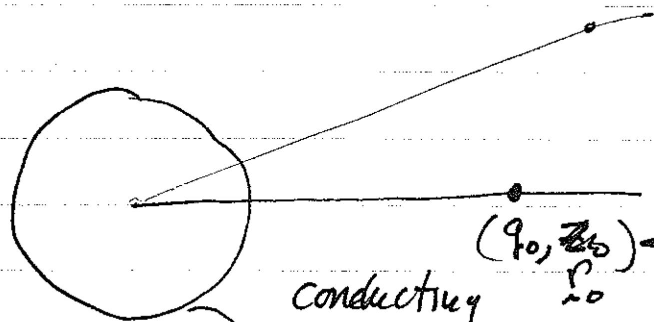
dipole moment of sphere

$$\underset{\sim}{p} = \hat{z}\, q_i z_i = \hat{z}\left(-\frac{q_0 a}{z_0}\right)\frac{a^2}{z_0}$$

$$\boxed{\underset{\sim}{p} = -a^3\hat{z}\,\frac{q_0}{z_0^2} = 4\pi\epsilon_0 E_0 a^3\hat{z}}$$

induced dipole moment

Method of images used to find field surrounding an arbitrarily charged ~~spere~~ sphere



Let $\phi_0(\underset{\sim}{r})$ be the solution we have obtained previously corresponding to a grounded sphere
note generalization

$$\phi_0(\underset{\sim}{r}) = \frac{q_0}{4\pi\epsilon_0|\underset{\sim}{r} - \underset{\sim}{r}_0|} + \frac{q_i}{4\pi\epsilon_0|\underset{\sim}{r} - \underset{\sim}{r}_i|}$$

$$\boxed{q_i = -q_0 \frac{a}{|\underset{\sim}{r}_0|} \qquad \underset{\sim}{r}_i = \underset{\sim}{r}_0 \frac{a^2}{|\underset{\sim}{r}_0|^2}}$$

This satisfies

$$\phi_0(|\underset{\sim}{r}|=a) = 0$$

the total charge on the sphere is q_i.

To obtain a solution for an arbitrary amount of charge add to this the solution corresponding to a uniform charge distribution on surface (same as a point charge

$$\phi(\underset{\sim}{r}) = \phi_0(\underset{\sim}{r}) + \frac{(Q-q_i)}{4\pi\epsilon_0|\underset{\sim}{r}|}$$

The sum corresponds to a total charge Q

satisfies Poisson equation outside sphere

$$\nabla^2\phi = -4\pi\,\delta(\underset{\sim}{r}-\underset{\sim}{r}_0)\,q_0$$

$$\phi = \text{const on surface of sphere.}$$

$\therefore$ Unique solution

$$\nabla^2 G_D = -4\pi\,\delta(\underset{\sim}{x} - \underset{\sim}{x}')$$

$\phi = 0$
on sphere

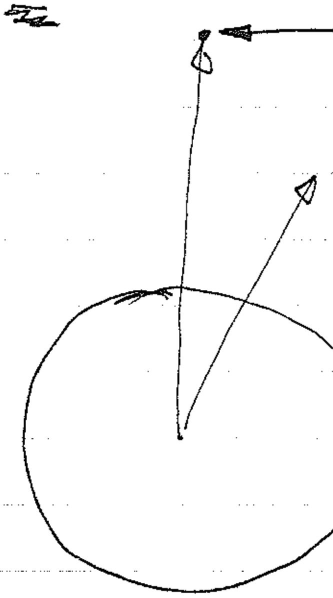
DIRICHILET ~~DIRICHILET~~ Green's FUNCTION ~~or~~ POTENTIAL
due to ~~point cha~~ unit charge

$q_0 = 1$

$$\nabla^2 \phi = -\frac{q_0}{\epsilon_0}\,\delta(\underset{\sim}{x} - \underset{\sim}{x}_0)$$

$$\phi(|\underset{\sim}{x}| = a) = 0$$

$G_D(\underset{\sim}{x}, \underset{\sim}{x}')$ ~~=~~



sorce at $\underset{\sim}{x}'$

observation point at $\underset{\sim}{x}$

$q_0 \overset{1/\epsilon_0}{=} \cancel{\ } 4\pi \qquad q_i / \epsilon_0 = -\dfrac{a}{|\underset{\sim}{x}'|}\, 4\pi$

$\underset{\sim}{r}_0 = \underset{\sim}{x}' \qquad \underset{\sim}{r}_i = \underset{\sim}{x}' \dfrac{a^2}{|\underset{\sim}{x}'|^2}$

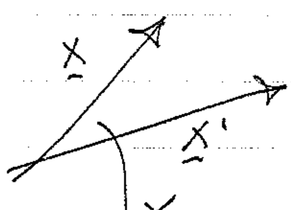
$\underset{\sim}{r} = \underset{\sim}{x}$

$$G_D(\underset{\sim}{x}, \underset{\sim}{x}') = \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|} - \frac{a}{|\underset{\sim}{x}'|}\,\frac{1}{\left|\underset{\sim}{x} - \underset{\sim}{x}' \dfrac{a^2}{|\underset{\sim}{x}'|^2}\right|}$$

G_D

yes
Symmetric?

$$\left|\underset{\sim}{x} - \underset{\sim}{x}' \frac{a^2}{|x'|^2}\right| = \sqrt{x^2 + x'^2\left(\frac{a^2}{x'^2}\right)^2 - 2xx'\frac{a^2}{x'^2}\cos\gamma}$$



$$= \frac{a}{x'}\sqrt{\frac{x^2 x'^2}{a^2} + a^2 - 2xx'\cos\gamma}$$

$$G_D = \frac{1}{|x - x'|} - \frac{1}{\sqrt{\ \ }}$$

Remember the role of $G_D(\underset{\sim}{x}, \underset{\sim}{x}')$

specified charge density outside sphere

$$\phi(\underset{\sim}{x}) = \int \frac{d^3x' \, G_D(\underset{\sim}{x}, \underset{\sim}{x}') \rho(\underset{\sim}{x}')}{4\pi\epsilon_0}$$

$\underset{\sim}{n}$ is normal pointing out of V (into sphere)

$$- \frac{1}{4\pi} \int_S da' \, \underset{\sim}{n} \cdot \nabla' G_D(\underset{\sim}{x}, \underset{\sim}{x}') \phi(\underset{\sim}{x}')$$

potential on sphere

We are now in a position to solve a host of problems

Eample: $\rho(x) = 0$ $\quad \phi$ specified on boundary of sphere

$$\phi(\underset{\sim}{x}) = -\frac{1}{4\pi} \int_S da' \, \phi(\underset{\sim}{x}') \, \underset{\sim}{n} \cdot \nabla' G_D(\underset{\sim}{x}, \underset{\sim}{x}')$$

$\boxed{\text{See book}}$

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