Electrostatic potential Energy

consider $i = 1, j-1$ charges q_i at points $\underset{\sim}{x}_i$

now bring in the j^{th} charge to point $\underset{\sim}{x}_j$ this takes an amount of work

$$W_j = q_j\, \phi(\underset{\sim}{x}_j)$$

$$W_j = q \sum_{i=1}^{j-1} \frac{q_j q_i}{|\underset{\sim}{x}_j - \underset{\sim}{x}_i|} \frac{1}{(4\pi\epsilon_0)}$$

THIS was the work required to bring the j^{th} charge

The work required ~~to~~ to assemble all ~~charges is~~ n is

$$W = \sum_{j=1}^{n} W_j = \sum_{j=1}^{n} \sum_{i=1}^{j-1} \frac{q_i q_j}{|\underset{\sim}{x}_j - \underset{\sim}{x}_i|\, 4\pi\epsilon_0}$$

$$= \frac{1}{2} \sum_{j=1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{q_i q_j}{|\underset{\sim}{x}_j - \underset{\sim}{x}_i|\, 4\pi\epsilon_0}$$

double counting

for continuous charge density

$$W = \frac{1}{2}\int d^3x \int d^3x' \frac{\rho(x)\rho(x')}{4\pi\epsilon_0 |\underset{\sim}{x} - \underset{\sim}{x}'|}$$

(note no longer worry about $\underline{x} = \underline{x}'$

also can be written

$$W = \frac{1}{\cancel{4\pi\epsilon_0}}\int d^3x \frac{\rho(\underset{\sim}{x})\phi(\underset{\sim}{x})}{2}$$

BUT $\qquad \rho(x) = -\frac{\epsilon_0}{\cancel{4\pi}}\nabla^2\phi(x)$

so

$$W = -\frac{1}{\cancel{8\pi}}\int d^3x \; \phi(x)\nabla^2\phi(x)\frac{\epsilon_0}{2}$$

$$= \int d^3x \; \frac{|\nabla\phi|^2}{\cancel{8\pi}}\frac{\epsilon_0}{2}$$

assuming $\phi \to 0$ as $x \to \infty$

$$W = \int d^3x \, \frac{\epsilon_0 |\underline{E}|^2}{\cancel{8\pi}\, 2}$$

$$w = \frac{\epsilon_0 |\underline{E}|^2}{\cancel{8\pi}\, 2}$$

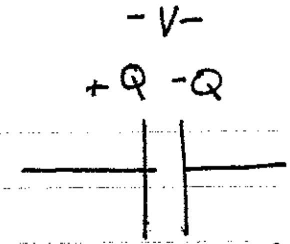
we can identify this as potential energy stored in the form of ~~electric~~ the electric field. One immediately runs into trouble here if one reverts to point charges. Why?

$$\int d^3x \, \frac{|\underline{E}|^2}{8\pi} \to \infty$$

The electric field energy associated with a point charge is ∞

<u>self energy is included</u>

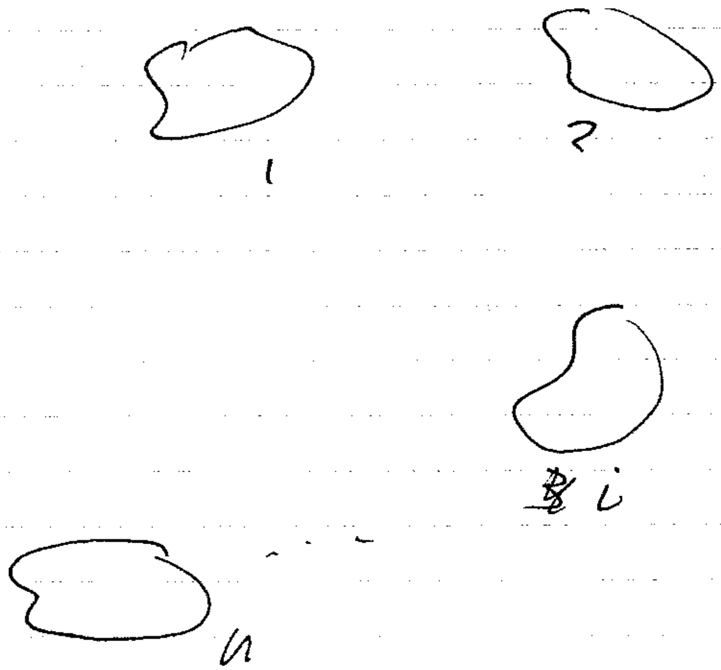
can compare difference of two states



~~Force on a~~

$Q = VC$

Capacitance: Consider a system of n conductors



on each conductor there is a surface charge density $\sigma_i(x)$ $i=1,n$ and a total charge $Q_i = \int da\, \sigma_i(x)$

Assuming the conductors are

$\phi(\infty) = 0$

perfect conductors we have

$\phi(\underset{\sim}{x}) = V_i$ ~~at~~ a constant

on each conductor

THUS ENERGY STORED

$$W = \frac{1}{2}\int d^3x\, \rho(x)\phi(x) = \frac{1}{2}\sum_{i=1}^{n}\int da\, \sigma_i(x)\phi(x)$$

$$= \frac{1}{2}\sum_{i=1}^{n} V_i Q_i$$

$$\cancel{V_i = \phi(\underset{\sim}{x}) = \sum_{j=1}^{n}\int da\, \sigma_j(x)\,/\,|x|}$$

~~$\underset{\sim}{x}$ at ith conductor~~

in general

inverse is C_{ij}

V_i is proportional to each of the Q_j & vice verse

~~capacitance matrix~~

$$V_i = \sum_j M_{ij} Q_j$$

Reciprocity $M_{ij} = M_{ji}$

This implies

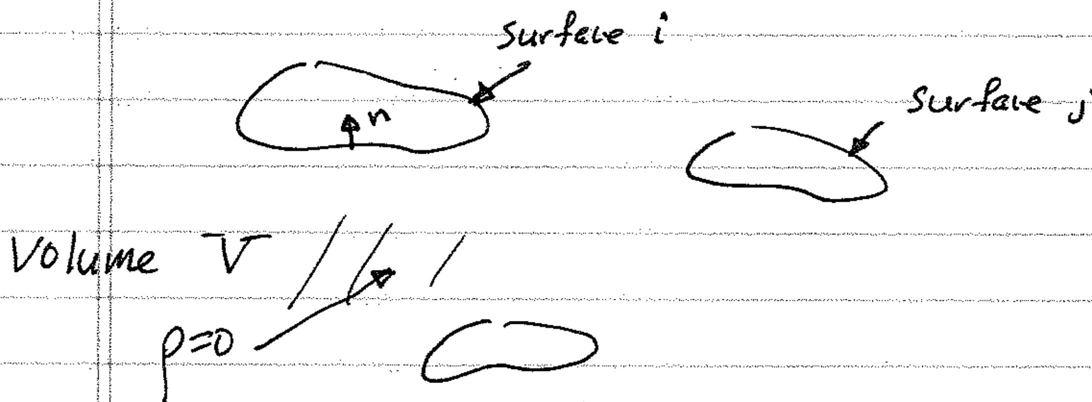
$$Q_i = \sum_j C_{ij} V_j$$

C_{ij} capacitance matrix

Follows from Green's theorem

$$\phi(\underline{x}') = \overbrace{\int \frac{d^3x}{4\pi\epsilon_0}\,\rho(\underline{x})\, G_0(\underline{x},\underline{x}')}^{0} - \frac{1}{4\pi}\int_S dA\, \phi(\underline{x})\, \underline{n}\cdot\nabla G_0(\underline{x},\underline{x}')$$

outward normal



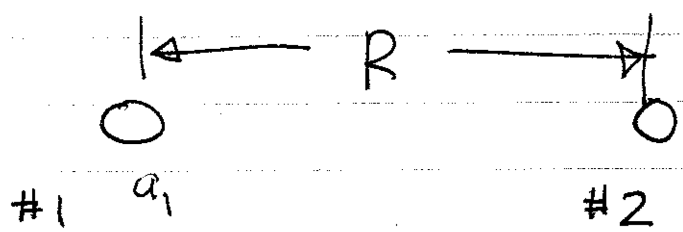
$$Q_i = \int_{S_i} \epsilon_0\, \underline{E}(\underline{x}')\cdot(-\underline{n})\, dA' = \int_{S'} \epsilon_0\, \underline{n}\cdot\nabla'\phi(\underline{x}')\, dA'$$

$$\underline{n}\cdot\nabla'\phi(\underline{x}') = -\frac{1}{4\pi}\sum_j V_j \int_{S_j} dA\, \underline{n}\cdot\nabla\left(\underline{n}\cdot\nabla' G_0(\underline{x},\underline{x}')\right)$$

$$Q_i = \sum_j V_j \underbrace{\left(-\frac{1}{4\pi} \int_{S_i} dA' \int_{S_j} dA \; \underline{n} \cdot \nabla \left(\underline{n} \cdot \nabla G_D(\underline{x}, \underline{x}') \right) \right)}_{C_{ij}}$$

$C_{ij} = C_{ji}$ because $G_D(\underline{x}, x') = G_D(\underline{x}', x)$

Example, calculate the <u>capacitance matrix</u> ~~due to~~ for two small spheres of radius a_1, a_2 ~~and~~ separated by a distance R with $R >> a_1, a_2$



$R >> a$ allows us to neglect the rearrangement of charge on spheres due to presence of other spheres.

Calculate potential due to point charge

$$V_1 = \frac{Q_1}{4\pi\epsilon_0 a_1} + \frac{Q_2}{4\pi\epsilon_0 R} \qquad V_2 = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1}{R} + \frac{Q_2}{a_2}\right)$$

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \frac{1}{4\pi\epsilon_0}\begin{bmatrix} 1/a_1 & 1/R \\ 1/R & 1/a_2 \end{bmatrix}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

Invert

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \frac{4\pi\epsilon_0}{\frac{1}{a_1^{\cancel{2}} a_2} - \frac{1}{R^2}}\begin{bmatrix} \frac{1}{a_2} & -\frac{1}{R} \\ -\frac{1}{R} & \frac{1}{a_1} \end{bmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

neglect

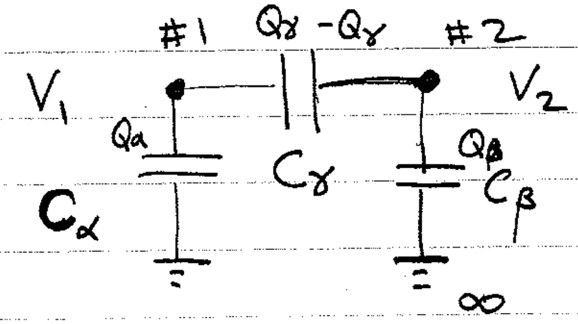
$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = 4\pi\epsilon_0 \begin{array}{c} \begin{array}{cc} C_{11} & C_{12} \end{array} \\ \begin{bmatrix} a_1 & -a_1 a_2/R \\ -a_1 a_2/R & a_2 \end{bmatrix} \\ \begin{array}{cc} & C_{22} \end{array} \end{array} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

[illegible]

Symmetric

SKIP

~~what is the capacitance~~



Equivalent circuit

$$Q_\gamma = C_\gamma (V_1 - V_2)$$

$$Q_\alpha = C_\alpha V_1$$

$$Q_\beta = C_\beta V_2$$

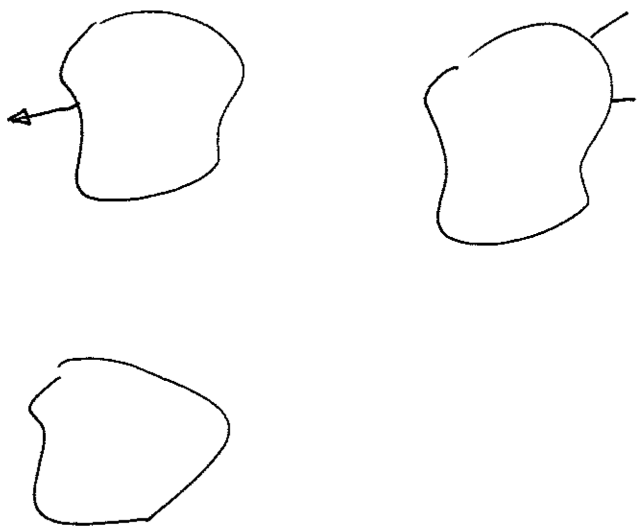
$$Q_1 = (Q_\alpha + Q_\gamma) = C_\alpha V_1 + C_\gamma (V_1 - V_2)$$

$$= \overset{C_{11}}{(C_\alpha + C_\gamma)} V_1 - \overset{C_{12}}{C_\gamma} V_2$$

$$Q_2 = (Q_\beta + -Q_\gamma) = (\cancel{C_\beta +} \; \overset{C_{21}}{-C_\gamma} V_1 + \overset{C_{22}}{(C_\beta + C_\gamma)} V_2$$

(1) <u>1.13</u> Thomson's

surface i



on each surface place Q_i Couloumbs of charge distributed according to surface charge density σ_i

Form of σ_i is unknown

$$\int_{S_i} dS\, \sigma_i = Q_i$$

Minimize

$$W = \int d^3x \frac{\epsilon_0 |\nabla\phi|^2}{2}$$

where $\nabla^2\phi = 0$ except at surfaces where

$$\sigma_i = \epsilon_0 \left[n\cdot\nabla(\phi_{out}) - n\cdot\nabla\phi_{in} \right]$$

let $\sigma_i = \sigma_{io} + \epsilon\, \sigma_{it}$ ← test

actual minimizing charge

2

$$W = \int_{V \text{ outside all}} d^3x\, \epsilon_0 \frac{|\nabla\phi|^2}{2} + \sum_i \int_{V_i} d^3x\, \epsilon_0 \frac{|\nabla\phi|^2}{2}$$

$$\cancel{W} = \int_V d^3x \frac{\epsilon_0}{2} \nabla\phi \cdot \nabla\phi$$

outward normal

$$= \int_S ds \frac{\epsilon_0}{2} \phi\, n \cdot \nabla\phi - \int_V d^3x \frac{\epsilon_0}{2} \phi \nabla^2\phi$$

For our problem $\nabla^2\phi = 0$

$$W = \sum_i \int_{S_i} ds \frac{\epsilon_0}{2} \phi \left(n \cdot \nabla\phi_{out} - n \cdot \nabla\phi_{in} \right)$$

$$\phi = \phi_0 + \epsilon \phi_t$$

$$W(\epsilon) = \int_{\cancel{V_{out}}\ ALL-V} d^3x \frac{\epsilon_0}{2} \left[|\nabla\phi_0|^2 + 2\epsilon \nabla\phi_0 \cdot \nabla\phi_t + \epsilon^2 |\nabla\phi_t|^2 \right]$$

$$\left.\frac{\partial W}{\partial \epsilon}\right|_{\epsilon=0} = \int_{ALL \cdot V} d^3x\, \epsilon_0 \nabla\phi_0 \cdot \nabla\phi_t$$

$$= \sum_i \int_{S_i} ds \, \epsilon_0 \, \phi_0 \underbrace{\left(n \cdot \nabla \phi_{t,out} - n \cdot \nabla \phi_{t,in} \right)}$$

$$= \sum_i \int_{S_i} ds \; \phi_{0i} \, \sigma_{it} \qquad = 0$$

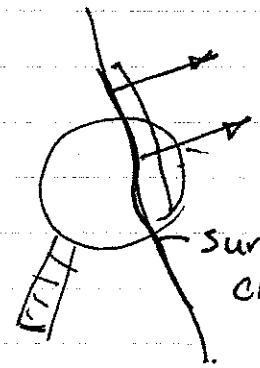
$$\int \sigma_{i0} \, ds = Q_i \qquad\qquad \int \sigma_{it} \, ds = 0$$

only way integral can be zero is if

$\underline{\underline{\phi_{0i} = \text{const}}}$

Force on the surface of a conductor

conductor



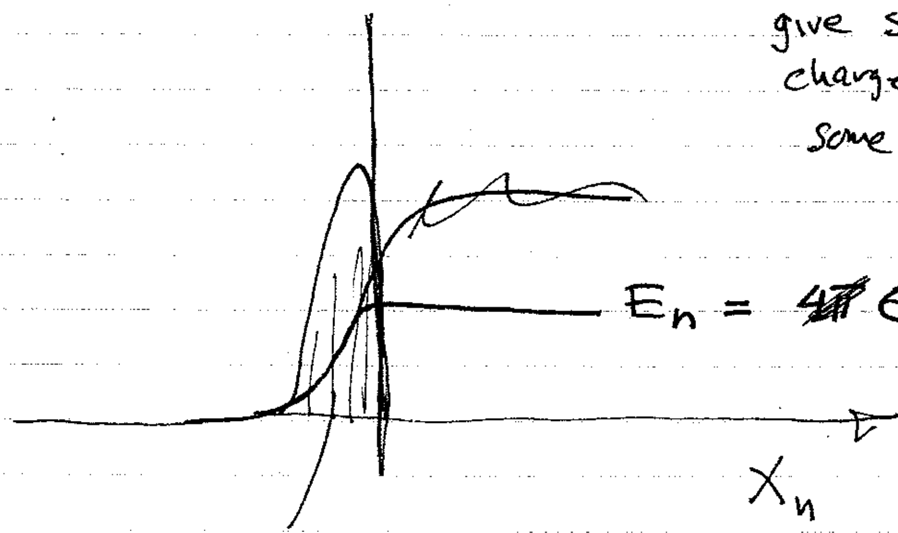
$\underset{\sim}{E_n}$

$\underset{\sim}{n}$ out ward normal

surface charge density $\sigma(\underset{\sim}{x})$

Blow up of surface

give surface charge densith some ~~fin~~ thickner



$$E_n = \cancel{4\pi}\sigma = \cancel{4\pi}\int dx_n \frac{\rho(\underset{\sim}{x})}{\epsilon_0}$$

$\rho(x)$

$$\frac{\partial E_n}{\partial x_n} = \frac{\cancel{4\pi}\rho(x_n)}{\epsilon_0} \qquad \sigma = \int dx_n \rho(\underset{\sim}{x})$$

small force act

$$\cancel{d}F_x =$$

$$\underset{\sim}{F} = \int d^3x \, \rho(\underset{\sim}{x}) \, \underset{\sim}{E}$$

normal direction

$$= \int da \, \underset{\sim}{n} \int dx_n \, \rho \, E_n$$

$$\frac{\partial E_n}{\partial x_n} = \cancel{4\pi} \rho / \epsilon_0$$

$$= \int da \, \underset{\sim}{n} \int \frac{\epsilon_0}{\cancel{4\pi}} E_n \frac{\partial E_n}{\partial x_n}(x_n)$$

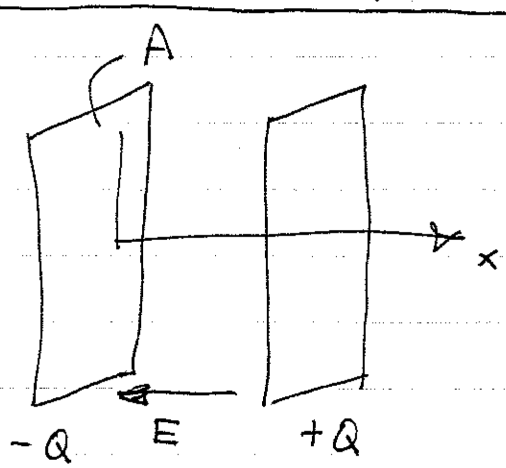
$$= \int da \, \underset{\sim}{n} \, \frac{\epsilon_0}{\cancel{8\pi}} \frac{|E_n|_v^2}{2}$$

out.

electric pressure $= -\frac{\epsilon_0}{\cancel{8\pi}} \frac{|\underset{\sim}{E}|_v^2}{2}$

Skip

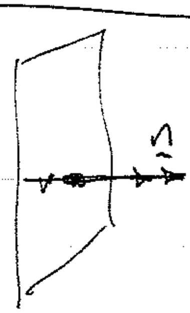
FORCE on a parallel plate



surface charge densit

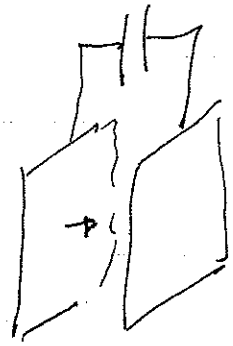
$$\mathbf{E}_n = -4\pi \frac{Q}{A}\, \mathbf{a}_x$$

Force on plate $\quad \mathbf{F} = \int da\, \mathbf{n}\, \frac{|E_n|^2}{8\pi}$



attraction to other plate

Check out Problem 1.9 use virtual work



constant C

$$\delta W = \mathbf{F} \cdot \delta \mathbf{x} = \delta \left[\tfrac{1}{2} C V^2 \text{ or } \frac{Q^2}{2C} \right]$$

constant V

must include $V \delta Q$