END OF LECTURE #1 508

14'

~~Green's Functions~~ Delta-functions

CONSIDER POISSON'S EQUATION

$$\epsilon_0 \nabla^2 \phi = - \rho(\underset{\sim}{x})$$

which has the solution

$$\phi(\underset{\sim}{x}) = \int d^3x' \frac{\rho(\underset{\sim}{x}')}{4\pi\epsilon_0 |\underset{\sim}{x} - \underset{\sim}{x}'|}$$

THIS MEANS

$$\nabla^2 \phi(x) = \int d^3x' \frac{\rho(x')}{4\pi\epsilon_0} \nabla^2 \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

$$= - \frac{4\pi \rho(\underset{\sim}{x})}{\epsilon_0}$$

or $$\rho(\underset{\sim}{x}) = \int d^3x' \rho(\underset{\sim}{x}') \underbrace{\left[-\frac{1}{4\pi} \nabla^2 \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} \right]}$$

True for any $\rho(\underset{\sim}{x})$

A VERY SPECIAL FUNCTION

$$\delta(\underset{\sim}{x}-\underset{\sim}{x}') = -\frac{1}{4\pi} \nabla^2 \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}$$

ok!

$$\cancel{\delta(\underset{\sim}{x}-\underset{\sim}{x}') = \begin{cases} 0 & \text{if } \underset{\sim}{x} \neq \underset{\sim}{x}' \\ \infty & \text{if } \underset{\sim}{x} = \underset{\sim}{x}' \end{cases}}$$

DIRAC DELTA FUNCTION

$\cancel{\int d^3x}$

$$\delta(\underset{\sim}{x}) = \begin{cases} 0 & \text{if } \underset{\sim}{x} \neq 0 \\ \infty & \text{if } \underset{\sim}{x} = 0 \end{cases}$$

$\cancel{\delta(\underset{\sim}{z})}$ −

but $$\int d^3x\, \delta(\underset{\sim}{x}) = \Big\{ \underline{1}$$

TWO USEFUL RELATIONS

$$\nabla_x \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} = \frac{-(\underset{\sim}{x}-\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3}$$

$$\nabla \cdot \nabla \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} = \nabla \cdot \left(- \frac{(\underset{\sim}{x}-\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3} \right) = -4\pi \overset{\text{vector}}{\delta}(\underset{\sim}{x}-\underset{\sim}{x}')$$

Lecture #2 507

Boundary Conditions on & surfaces

~~consider~~

$$\phi(x) = \int \frac{d^3x' \, \rho(\underset{\sim}{x}')}{4\pi\epsilon_0 |\underset{\sim}{x} - \underset{\sim}{x}'|}$$

1.6
1.7
1.8
1.9
1.10
1.11

$$\underset{\sim}{E} = -\nabla\phi(\underset{\sim}{x}) \Rightarrow \oint \underset{\sim}{E} \cdot d\underset{\sim}{l} = 0$$

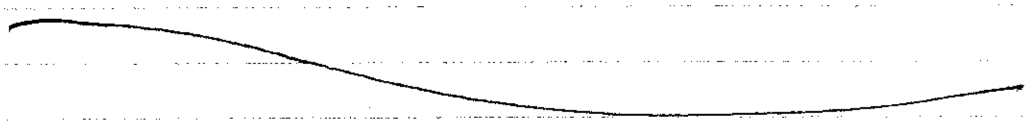
represents a complete solution ~~for~~ for the electric field given the charge density. Problem is we do not always know the charge density before hand, but we must sometimes determine it as part of the solution of the problem.

~~To do this we then require~~

ru

For example suppose we introduce an perfect electrical conductor into our problem. Charge on such a conductor is free to move and arrange it self such that $\underset{\sim}{E} = 0$ within the conductor. Once $\underset{\sim}{E} = 0$ there no longe will be any electrical force on a free charge <u>within</u> the conductor inducing the charge to move. (This does not imply that there is no force on the conductor as we shall see) vacuum

$$\underset{\sim}{E} \neq 0$$



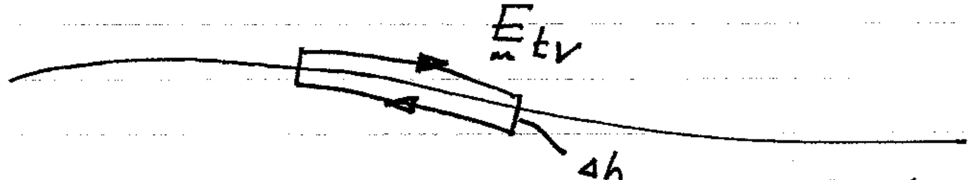
conductor

$$\underset{\sim}{E} = \underset{\sim}{0}$$

In conductor $\quad \nabla \cdot \underset{\sim}{E} = \nabla \cdot \underset{\sim}{0} = 0$

no net charge density inside conductor

At boundary



conductor

$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{l} = 0$$

closed loop

tangential component

This implies $\underset{\sim}{E}_{tv} - \underset{\sim}{E}_{tc} = 0$

but $\underset{\sim}{E}_{tc} = 0$ ($\underset{\sim}{E} = 0$ in cond.

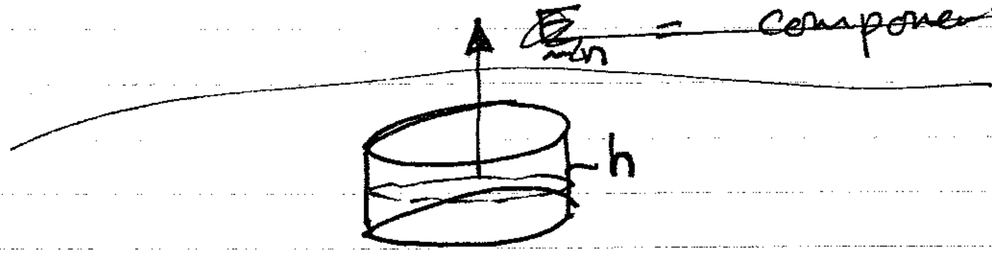
thus $\underset{\sim}{E}_{tr} = 0$ (two components)

$$\phi(1) - \phi(2) = -\int_1^2 \underset{\sim}{E} \cdot d\underset{\sim}{l}$$

taking path to lie in the surface implies that $\boxed{\phi = \text{const}}$ on surface of conductor.

~~this will be used to determine surface d~~

Normal Component



$$\epsilon_0 \int da\, \underset{\sim}{n} \cdot \underset{\sim}{E} = \cancel{4\pi}\, q_{enclosed}$$

$\underset{\sim}{E} = 0$ inside

$h \to 0$

$da \to 0$

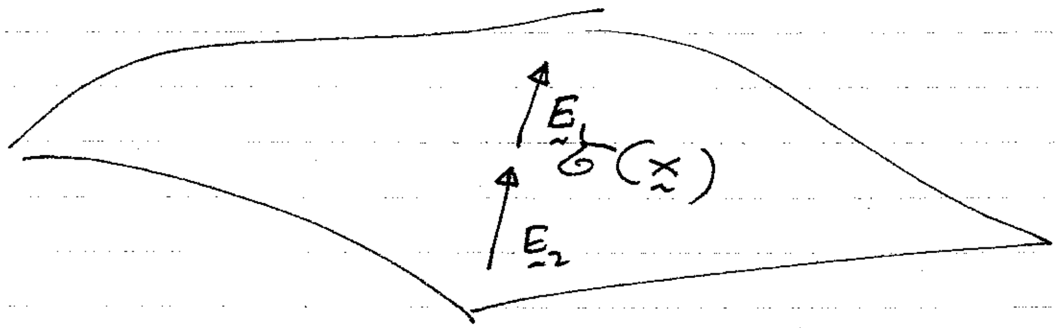
outward

$$\underset{\sim}{n} \cdot \underset{\sim}{E} = \cancel{4\pi} \left(\frac{q_{enclosed}}{\epsilon_0\, da} \right)$$

surface charge density

$\sigma =$ surface charge density

~~THUS,~~



in general there is a discontinuity in normal component of E field at a surface with surface charge density

$$(E_1 - E_2)_{tangent} = 0$$

normal pointing from $2 \to 1$

$$(E_1 - E_2)\cdot n = \cancel{4\pi}\,\sigma/\epsilon_0$$

conductor

for surface charge density at a $E_2 = 0$

Skip

Potential due to surface Charge density

$$\phi(x) = \int_S \frac{\sigma(x')\, da'}{|\underset{\sim}{x} - \underset{-}{x}|}$$

~~Dipo~~ continuous

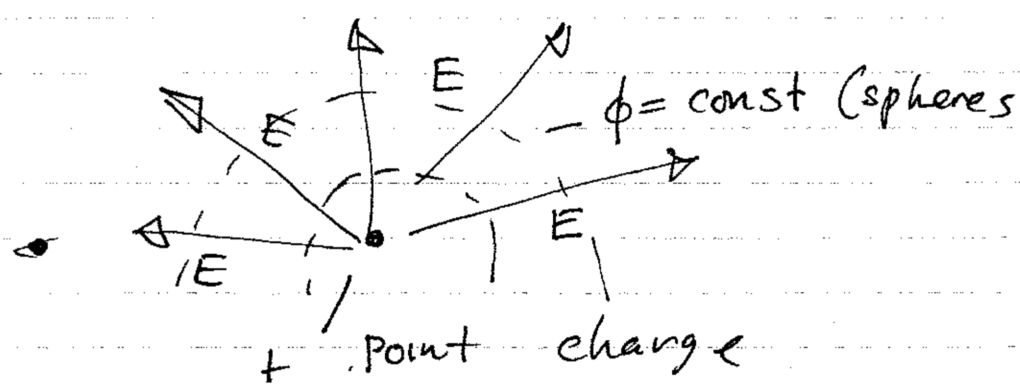
~~Other special charge distributions~~

OTHER DISTRIBUTIONS dipole we will come to this,

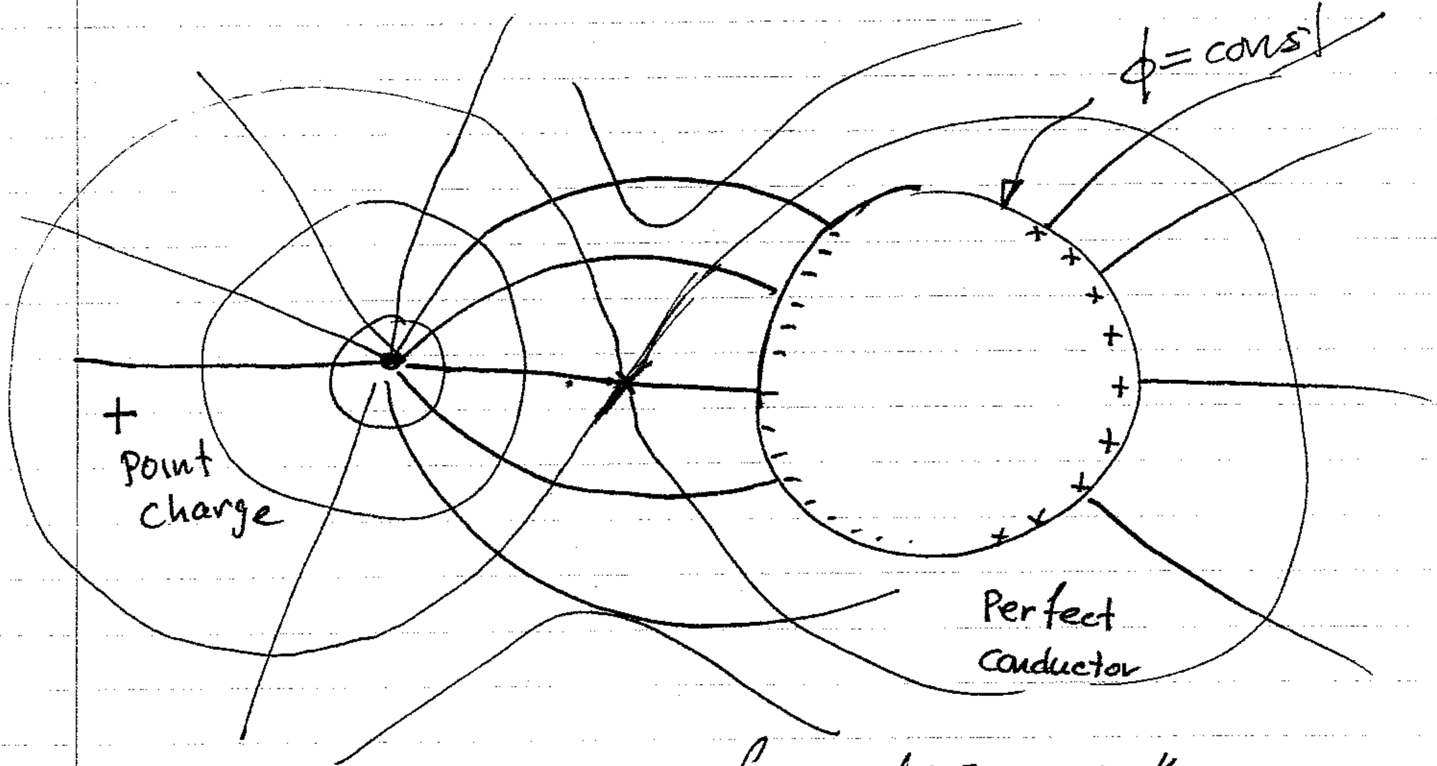
Caution while above is true you almost never know $\sigma(x')$ apriori

We have already seen that with conductors present ~~char~~ surface charge density appears on the conductors so that $\underset{\sim}{E} = 0$ in the conductor. The form of this surface charge density is not known apriori but must be determined ~~along~~ as part of the solution

1 Schematic Example



Suppose we move a ~~eo~~ perfect conductor into the picture



free charge within conductor arranges itself so that $\underset{\sim}{E} = 0$ inside conductor

Thus what we want to solve is

$$\epsilon_0 \nabla^2 \phi = -\cancel{4\pi}\rho(\underset{\sim}{x})$$

everywhere but in conductor
in vacuum

~~with ϕ = const on conductor~~

~~total charge on~~

~~CONSIDER~~

~~with~~ eil

Thus, the kind of problem we want to solve is

$$\epsilon_0 \nabla^2 \phi = - \cancel{4\pi} \rho(\underset{\sim}{x})$$

specified charge

does not include surface charge on boundary

along with certain boundary conditions on ϕ or normal derivative on given surfaces

ϕ specified on surface "DIRICHILET" (usual case if conductors) present

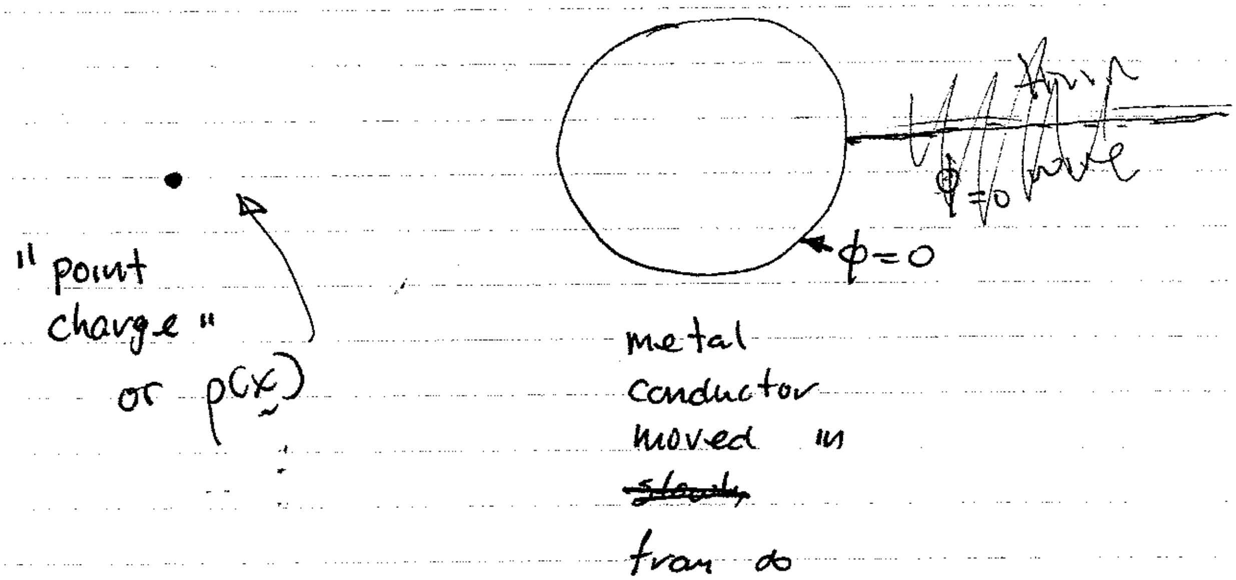
$$\underset{\sim}{n} \cdot \nabla \phi = \frac{\partial \phi}{\partial n} \text{ specified "Neumann"}$$

if surface charge

Caution

FOR POISSON can not specify both on all surfaces.

switch to uniqueness



What boundary conditions ?

$\phi =$ cons potential held ~~at~~ fixed

$$\int_S da\, \underset{\sim}{n} \cdot \nabla\phi = \text{specified}$$

~~(no net~~ surface charge)

fixed charge on sphere

Uniqueness of solutions

consider

$$\nabla^2 \phi = -4\pi\rho/\epsilon_0$$

subject to boundary conditions on enclosing surface

Suppose we have found a solution satisfying the equation and all B.C's. Is that solution unique? ans. yes

Proof by contradiction

suppose ϕ_1 and ϕ_2 are two different solutions then $U = \phi_1 - \phi_2$

satisfies

$$\nabla^2 U = 0$$

with B.C's $\begin{cases} U = 0 & \text{if } \phi \text{ satisfies Dirichlet} \\ \dfrac{\partial U}{\partial n} = 0 & \text{if } \phi \text{ satisfies Neumann B.C.} \end{cases}$

$$\int d^3x\, U \underbrace{\nabla^2 U}_{0} = \int_V d^3x \left[(\nabla \cdot U \nabla U) - |\nabla U|^2 \right] = 0$$

THUS $\int_V d^3x\, |\nabla U|^2 = \int_S da\, (\mathbf{n} \cdot \nabla U) U$

if either U or $\cancel{\partial U/\partial n}$ $n \cdot \nabla U$ $= 0$ on S then we must have

$$\int_V d^3x\, |\nabla U|^2 = 0$$

$U = \text{const}$

for dirichlet $U = 0$

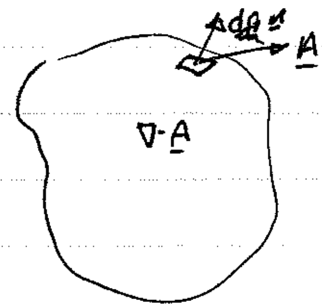
for neumann constant is an important

From this it follows that we can't specify both ϕ and $n \cdot \nabla \phi$

Green's Theorem

Follows from divergence theorem

$$\int_V d^3x\, \nabla\cdot \underset{\sim}{A} = \int_S \underset{\sim}{A}\cdot \underset{\sim}{n}\, da$$



Let $\underset{\sim}{A} = \phi\,\nabla\psi$

ϕ & ψ are two arbitrary scalars

$$\nabla\cdot\underset{\sim}{A} = \nabla\cdot(\phi\nabla\psi) = \phi\,\nabla^2\psi + \nabla\phi\cdot\nabla\psi$$ product rule

$$\int_V \left[\phi\,\nabla^2\psi + \nabla\phi\cdot\nabla\psi\right] d^3x = \int_S \phi\, n\cdot\nabla\psi\, da$$

interchange $\phi \leftrightarrow \psi$

$$\int_V \left(\psi\,\nabla^2\phi + \nabla\psi\cdot\nabla\phi\right) d^3x = \int_S \psi\, n\cdot\nabla\phi\, da$$

subtract

$$\boxed{\int_V \left[\phi\,\nabla^2\psi - \psi\,\nabla^2\phi\right] d^3x = \int_S \left[\phi\, n\cdot\nabla\psi - \psi\, n\cdot\nabla\phi\right] da}$$

~~change~~

TRUE FOR any choice of ϕ & ψ

NOW LET ϕ be the solution to the problem we are interested.

$$-\nabla^2\phi = \frac{\rho(\underline{x})}{\epsilon}$$ (haven't specified BC's

LET $\psi(\underline{x})$ be the solution to

$$\nabla^2\psi = -4\pi\,\delta(\underline{x}-\underline{x}')$$ (haven't specified BC's)

(example $\psi = \frac{1}{|\underline{x}-\underline{x}'|}$)

$$\int_V d^3x \left[\phi(x)(-4\pi\delta(x-\underline{x}')) + \psi(\underline{x},\underline{x}')\frac{\rho(\underline{x})}{\epsilon_0} \right]$$

$$= \int_S da\left[\phi\, n\cdot\nabla\psi - \psi\, n\cdot\nabla\phi\right]$$

FIRST TERM $= -4\pi\,\phi(\underline{x}')$ if x' is in V

$= 0$ otherwise

assume $\underline{x}'$ in V

$$\phi(\underline{x}') = \int \frac{d^3x\ \rho(x)}{4\pi\epsilon_0} \psi(\underline{x}, \underline{x}')$$

$$- \frac{1}{4\pi} \int_S da \left[\phi\, n\cdot\nabla\psi - \psi\, n\cdot\nabla\phi \right]$$

~~Boundary condit~~

Boundary Conditions

(dirichlet problem)

suppose $\phi(\underline{x})$ is known on boundary
(then $n\cdot\nabla\phi$ is unknown)

pick $\psi = G(\underline{x}, \underline{x}')$
so that $G\big|_{x = \text{Boundary}} = 0$
call G_D

$$\phi(x') = \underbrace{\int \frac{d^3x\ \rho(\underline{x})}{4\pi\epsilon_0} G_D(\underline{x}, \underline{x}')}$$

particular solution satisfying $\phi = 0$ on Boundary

$$\underbrace{- \frac{1}{4\pi} \int_S da\ \phi(\underline{x})\, n\cdot\nabla G_D(\underline{x}, \underline{x}')}$$

Homogeneous term from potential on boundary

Prove Symmetry of G

symmetry of

$$+\nabla^2 G_D = -4\pi\,\delta(\underline{x}-\underline{x}')$$

$$G_D(\underline{x},\underline{x}')\Big|_{\underline{x}=\text{Boundary}} = 0$$

Consider the problem

$$-\nabla^2\phi = \frac{\rho(\underline{x})}{\epsilon_0} \qquad \phi=0 \quad \text{on } B$$

$$\boxed{\phi(\underline{x}') = \frac{1}{4\pi\epsilon_0}\int_V d^3x\, \rho(\underline{x})\, G_D(\underline{x},\underline{x}')}$$

NOW suppose $\rho(\underline{x}) = 4\pi\epsilon_0\,\delta(\underline{x}-\underline{x}'')$

THEN $\phi(\underline{x}) = G_D(\underline{x},\underline{x}'')$

$$\phi(\underline{x}') = G_D(\underline{x}'',x')$$

eliminate prime

$$\boxed{\begin{aligned}\phi(\underline{x}) &= G_D(\underline{x}'',x)\\ &= G_D(x,x''\end{aligned}}$$

arguments reversed (reciprocity)

* Poisson Equation with Neumann boundary conditions

$$\nabla^2 \phi = -\frac{\rho(\underline{x})}{\epsilon} \quad \text{in Volume } V$$

$$n \cdot \nabla\phi \Big|_S \quad \text{specified on surface of } V$$

First: do solutions exist?

Ans: not necessarily.

~~From diverg~~

From Gauss's Law

$$\epsilon_0 \int_S da\, \underset{\sim}{n} \cdot \underline{E} = -\epsilon_0 \int_S da (n \cdot \nabla\phi) = q = \int_V d^3x\, \rho(\underline{x})$$

Thus, in order for a solution to exist one must specify values of $n \cdot \nabla\phi|_S$ that are consistent with Gauss's Law

Suppose this is the case. Then are solutions unique?

Ans: No they are not.

Suppose $\phi(x)$ satisfies equatons + BC.

Then $\phi(x)$ + Const also satisfies equation & BC's.

So to make solution unique we must add an additional condition.

Example $\int_S da\, \phi = 0$

The average value of ϕ on the surface is zero.

How to use Green's Theorem

$$-\nabla^2 \phi = \frac{\rho(x)}{\epsilon_0} \qquad n \cdot \nabla \phi \text{ specified}$$

$$\nabla^2 \psi = -4\pi \delta(\underline{x} - \underline{x}')$$

$$\phi(\underline{x}') = \int \frac{d^3x\, \rho(\underline{x})\, \psi(\underline{x}, \underline{x}')}{4\pi\epsilon_0} - \frac{1}{4\pi} \int_S da \left[\phi\, n \cdot \nabla \psi - \psi\, n \cdot \nabla \phi \right]$$

Now to pick the BC's for ψ equation.

Let's pick $n \cdot \nabla \psi = K$ a constant

What is K?

$$\int_S da \underbrace{n \cdot \nabla \psi}_{K} = -4\pi \int_V d^3x\, \delta(\underline{x} - \underline{x}') = -4\pi$$

$$K = \frac{-4\pi}{A} \leftarrow \text{area of bounding surface}$$

$$\Phi(\underline{x}') = \int \frac{d^3x\,\rho(x)\,\psi}{4\pi\epsilon_0} - \frac{1}{4\pi}\int_S da\left[\phi \overset{\text{const.}}{K} - \psi\, \underbrace{n\cdot\nabla\phi}_{\text{known}}\right]$$

But $\int_S da\,\phi = 0$ to make solution unique

$$\phi(x') = \int \frac{d^3x\,\rho(x)\,\psi}{4\pi\epsilon_0} + \frac{1}{4\pi}\int da\, n\cdot\nabla\phi\,\psi(x,x')$$