ENEE381

Lecture-19

Simple Antennas

Previously on ENEE381 …

We assumed time and space dependent, localized charge and current density distributions.

Expressed **E** and **H** in terms of potentials **A** and Φ. Solved for potentials.

Assumed sinusoidal time dependence. Evaluated potentials far from sources.

Calculated **E** and **H** from potentials.

Found radiated power density $\mathbf{S} = \langle \mathbf{E} \times \mathbf{H} \rangle_{time}$

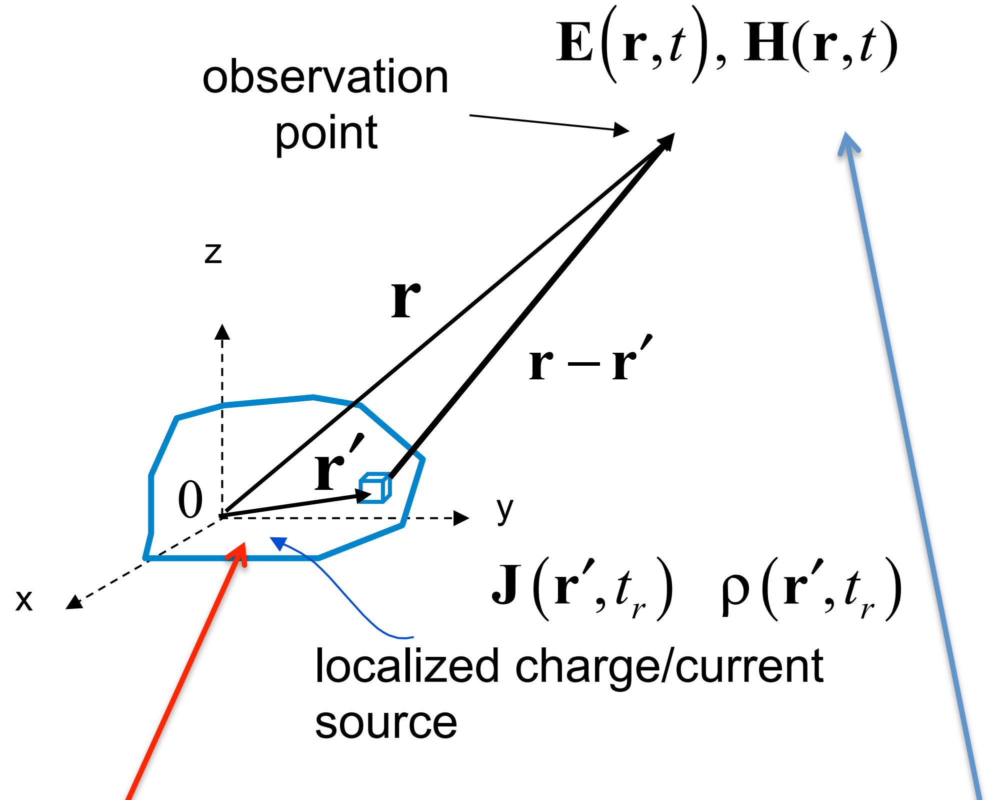
Review

$$\nabla \cdot \vec{\mathbf{B}} = 0$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$



Assume current and charge densities are sinusoidal and given. Calculate E and H far away. Find radiated power density S = E x H.

Introduce Potentials

$$\nabla \cdot \vec{\mathbf{B}} = 0$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Introduce Vector Potential

$$\mathbf{B} = \nabla \times \mathbf{A}$$

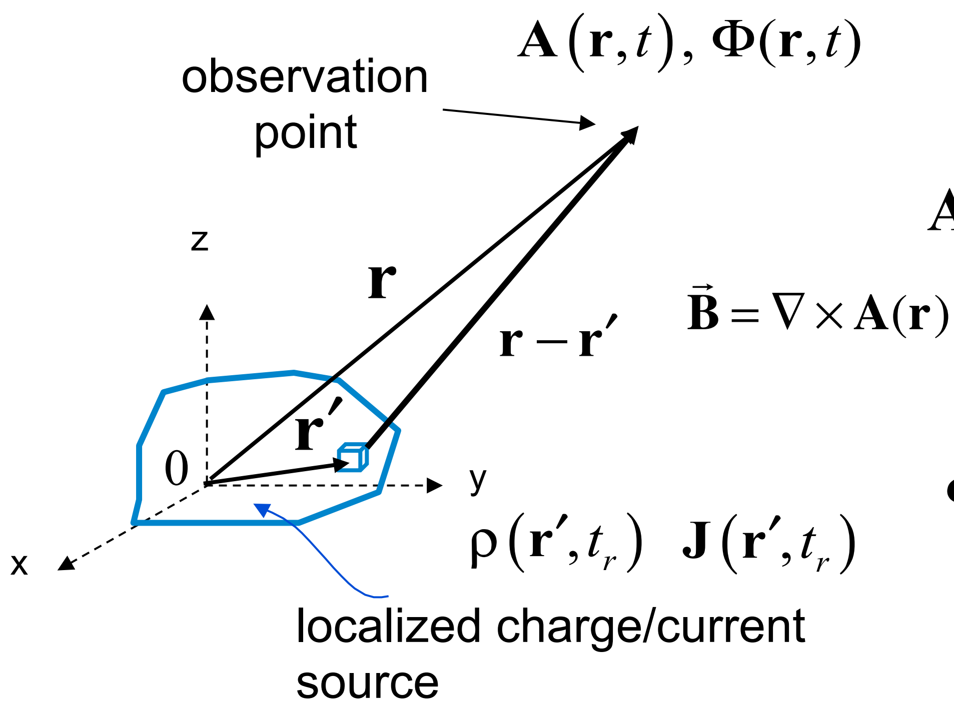
Insert in Faraday's Law

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} = -\nabla \times \frac{\partial}{\partial t}\mathbf{A}$$

$$\nabla \times \left(\mathbf{E} + \frac{\partial}{\partial t}\mathbf{A} \right) = 0, \quad \mathbf{E} + \frac{\partial}{\partial t}\mathbf{A} = -\nabla \phi$$

$$\mathbf{E} = -\nabla \phi - \frac{\partial}{\partial t}\mathbf{A}$$

Time-retarded potentials



$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_{Vol} d\mathbf{r}' \left.\frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}\right|_{t_r = t - |\mathbf{r}-\mathbf{r}'|/c}$$

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int_{Vol} d\mathbf{r}' \left.\frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}\right|_{t_r = t - |\mathbf{r}-\mathbf{r}'|/c}$$

where $t_r = t - |\mathbf{r}-\mathbf{r}'|/c$ is the retarded time (earlier time)

$$dr' = dx'\,dy'\,dz'$$

Sinusoidal Dependence on Time

If we assume harmonic (sinusoid dependence on time)

for all the fields and sources

$$\mathbf{A}(\mathbf{r},t) = \mathrm{Re}\left[\hat{\mathbf{A}}(\mathbf{r})e^{-i\omega t}\right] \qquad \Phi(\mathbf{r},t) = \mathrm{Re}\left[\hat{\Phi}(\mathbf{r})e^{-i\omega t}\right]$$

$$\mathbf{J}(\mathbf{r}',t_r) = \mathrm{Re}\left[\hat{\mathbf{J}}(\mathbf{r}')e^{-i\omega t_r}\right] \qquad \rho(\mathbf{r}',t_r) = \mathrm{Re}\left[\hat{\rho}(\mathbf{r}')e^{-i\omega t_r}\right]$$

$t_r = t - |\mathbf{r}-\mathbf{r}'|/c$ where $k = \omega / c = 2\pi / \lambda$ is the wavenumber

$$e^{-i\omega t_r} = e^{-i\omega t}e^{ik|\mathbf{r}-\mathbf{r}'|}$$

In phasor notation

$$\hat{\mathbf{A}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{Vol} d\mathbf{r}'\frac{\hat{\mathbf{J}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,e^{ik|\mathbf{r}-\mathbf{r}'|} \qquad \hat{\Phi}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{Vol} d\mathbf{r}'\frac{\hat{\rho}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,e^{ik|\mathbf{r}-\mathbf{r}'|}$$

Far Field Potentials

Using $\left|\mathbf{r}-\mathbf{r}'\right| \simeq r - \hat{\mathbf{n}}\cdot\mathbf{r}'$

$$\hat{\mathbf{A}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{Vol} d\mathbf{r}'\,\frac{\hat{\mathbf{J}}(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|}e^{ik\left|\mathbf{r}-\mathbf{r}'\right|} \simeq \frac{\mu_0}{4\pi r}e^{ikr}\int_{Vol} d\mathbf{r}'\,\hat{\mathbf{J}}(\mathbf{r}')\,e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

$$\hat{\Phi}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{Vol} d\mathbf{r}'\,\frac{\hat{\rho}(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|}e^{ik\left|\mathbf{r}-\mathbf{r}'\right|} \simeq \frac{e^{ikr}}{4\pi\varepsilon_0 r}\int_{Vol} d\mathbf{r}'\,\hat{\rho}(\mathbf{r}')\,e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

where $\mathbf{k} = k\,\hat{\mathbf{n}}$

for $r >> r'$, $\dfrac{1}{\left|\mathbf{r}-\mathbf{r}'\right|} \simeq \dfrac{1}{r}$ and $k\left|\mathbf{r}-\mathbf{r}'\right| \simeq k\,r - k\,\hat{\mathbf{n}}\cdot\mathbf{r}'$ in exponent

Summary

In the far zone $k\,r = 2\pi\,r\,/\,\lambda >> 1$

Define the Fourier transform of the current density

$$\hat{\mathbf{C}}(\mathbf{k}) = \int\limits_{Vol} d\tau'\,\hat{\mathbf{J}}(\mathbf{r}')\,e^{-i\,\mathbf{k}\cdot\mathbf{r}'} \qquad d\tau' = dx'\,dy'\,dz'$$

The fields in terms of the $F-T$ of the current density are

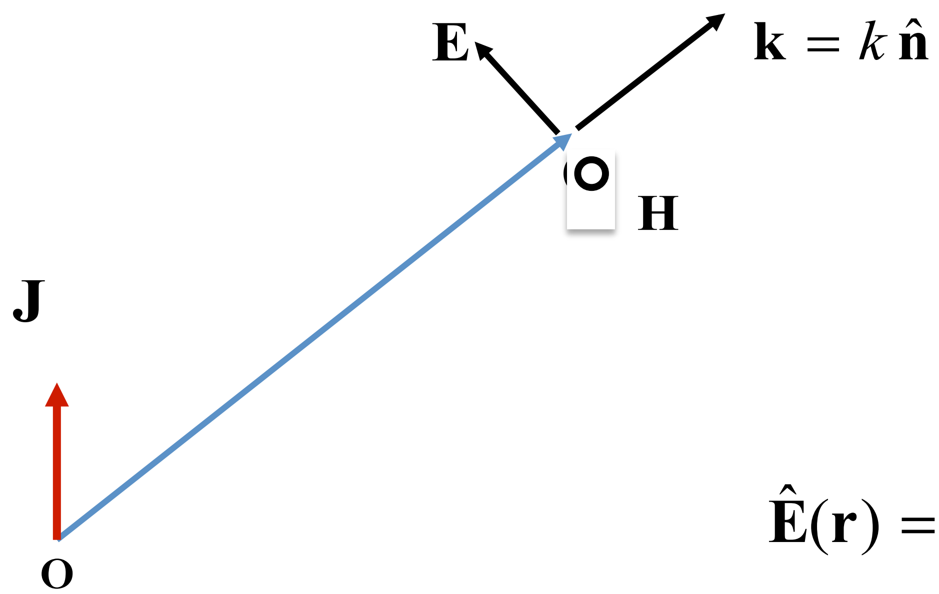
$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi\,r}\,e^{i\,k\,r}\,\hat{\mathbf{C}}(\mathbf{k}) \qquad \hat{\mathbf{H}}(\mathbf{r}) = i\,\frac{1}{4\pi\,r}\,e^{i\,k\,r}\,\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})$$

$$\hat{\mathbf{E}}(\mathbf{r}) = -\,i\,\frac{e^{i\,k\,r}}{4\pi\,r}\,\frac{1}{\varepsilon_0\,\omega}\,\mathbf{k}\times\left(\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})\right)$$

Radiation

In the far field zone

Direction of energy flow, Poynting's vector



$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr}\, \hat{\mathbf{C}}(\mathbf{k})$$

$$\hat{\mathbf{H}}(\mathbf{r}) = i \frac{1}{4\pi r} e^{ikr}\, \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k})$$

$$\hat{\mathbf{E}}(\mathbf{r}) = -\, i \frac{e^{ikr}}{4\pi r} \frac{1}{\varepsilon_0\, \omega} \mathbf{k} \times \left(\mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right)$$

$\hat{\mathbf{H}}(\mathbf{r})$ is transverse to $\hat{\mathbf{J}}$, $\mathbf{k} = k\, \hat{\mathbf{n}}$ and $\hat{\mathbf{E}}$

$\mathbf{S} = \frac{1}{2} \mathrm{Re}\{ \hat{\mathbf{E}} \times \hat{\mathbf{H}}^* \}$ Poynting Flux

Results

Total Radiated Power

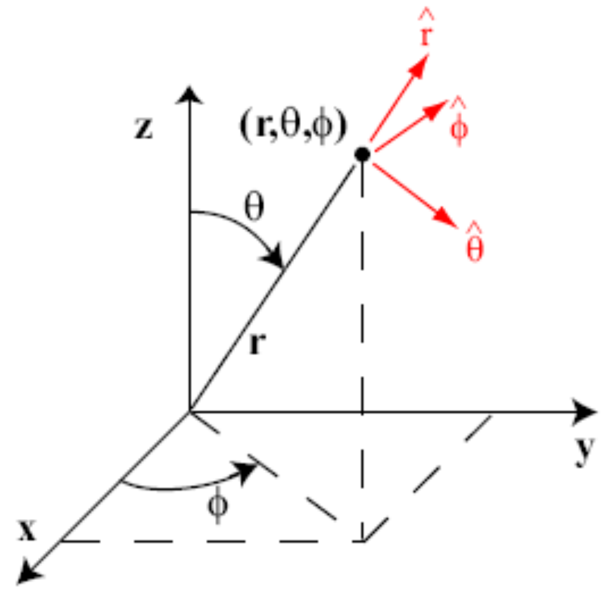
In spherical coordinates $da = r^2\, d\,\Omega$

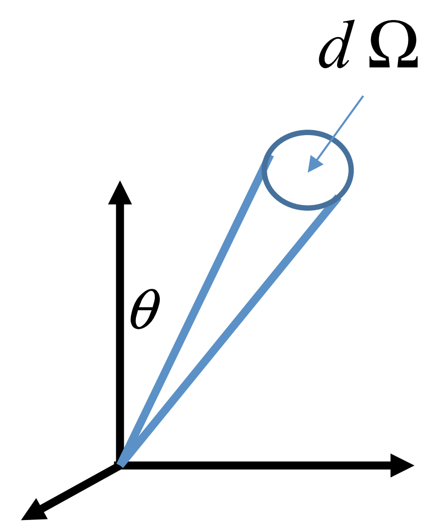
where $d\Omega = \sin\theta\, d\theta\, d\varphi$ is the solid angle

$$P_T = \oint_S \hat{\mathbf{n}}\cdot\langle\mathbf{S}\rangle r^2\, d\Omega = \oint_S \frac{dP_T}{d\Omega}\, d\Omega$$

Power radiated into the solid angle $d\Omega$

$$\frac{dP_T}{d\Omega} = \hat{\mathbf{n}}\cdot\langle\mathbf{S}\rangle r^2 = \frac{Z_0}{32\pi^2}\left|\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})\right|^2$$



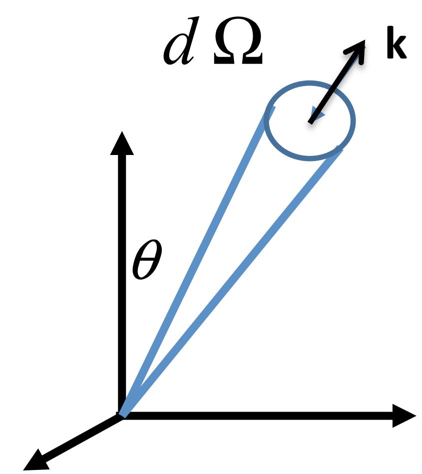


Radiated Power Flux

Fourier transform of the current density

$$\hat{\mathbf{C}}(\mathbf{k}) = \int_{Vol} d^3r' \, \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

$$P_T = \oint_S \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle \mathrm{r}^2 \, d\Omega = \oint_S \frac{dP_T}{d\Omega} d\Omega$$



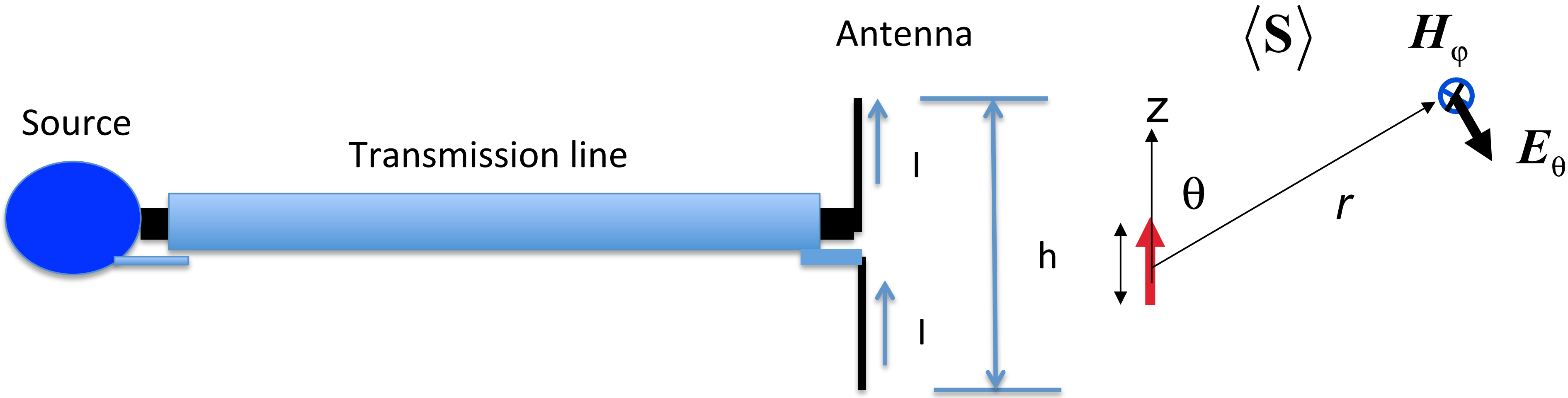
Power radiated into the solid angle $d\Omega$

$$\frac{dP_T}{d\Omega} = \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle \mathrm{r}^2 = \frac{Z_0}{32\pi^2} \left| \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right|^2$$

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{1}{c\,\varepsilon_0} = 377\,\Omega \text{ impedance of vacuum}$$

$$c = \frac{1}{\sqrt{\varepsilon_0 \, \mu_0}}$$

Simple Dipole Antenna



$$\hat{\mathbf{J}} = \hat{\mathbf{z}} I\, \delta(x)\delta(y) \quad \text{for } |z| < h/2, \text{ otherwise } 0$$

$$\hat{\mathbf{C}}(\mathbf{k}) = \int dx\, dy\, dz\, \hat{\mathbf{J}} \exp\left[ik_x x + ik_y y + ik_z z\right]$$

$$\hat{\mathbf{C}}(\mathbf{k}) = \hat{\mathbf{z}} I \int_{-h/2}^{h/2} dz \exp\left[ik_z z\right] = \hat{\mathbf{z}} I h \frac{\sin(k_z h/2)}{k_z h/2}$$

Remember: $|k| = \dfrac{2\pi}{\lambda}, \quad k_z = k \cos\vartheta$

suppose h<<λ

$$\hat{\mathbf{C}}(\mathbf{k}) = \hat{\mathbf{z}} I \int_{-h/2}^{h/2} dz \exp\left[ik_z z\right] = \hat{\mathbf{z}} I h \frac{\sin(k_z h/2)}{k_z h/2} \simeq \hat{\mathbf{z}} I h$$

Radiation Pattern

Remember: $|k| = \frac{2\pi}{\lambda}$,

$k_z = k\cos\theta$

suppose h<<λ

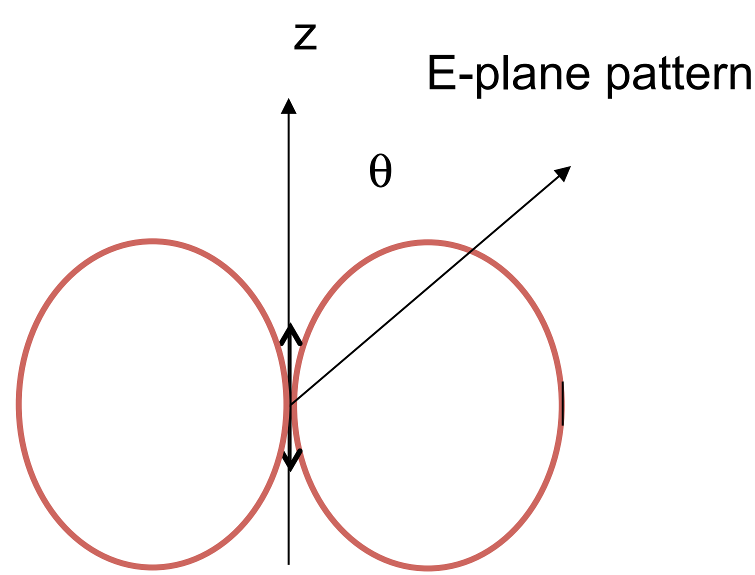
$\hat{\mathbf{C}}(\mathbf{k}) \simeq \hat{\mathbf{z}} Ih$

Power radiated into the solid angle $d\Omega$

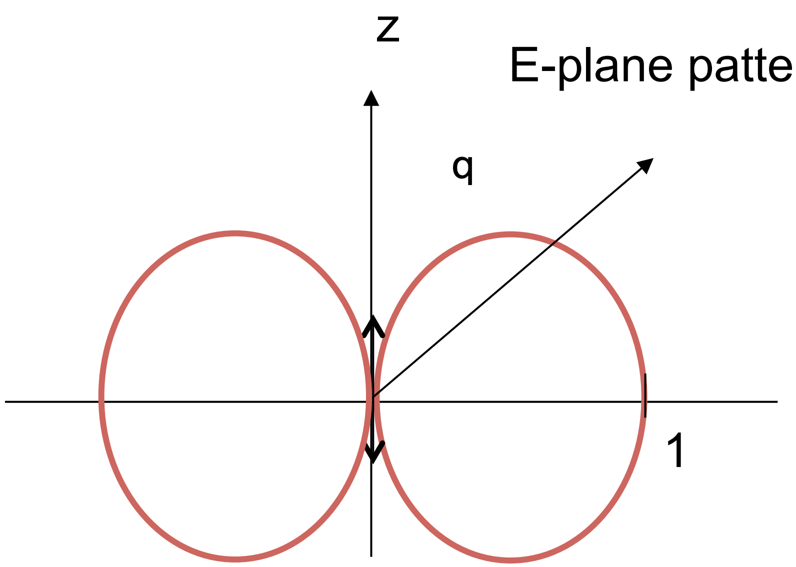
$$\frac{dP_T}{d\Omega} = \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle \mathrm{r}^2 = \frac{Z_0}{32\pi^2} \left| \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right|^2$$

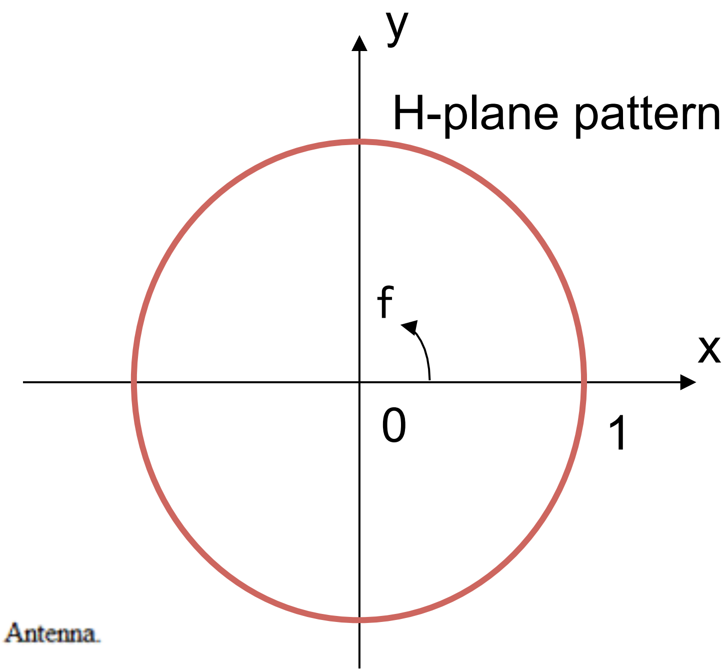
$$\left| \mathbf{k} \times \hat{\mathbf{z}} \right|^2 = k_y^2 + k_x^2 = k^2 \sin^2\theta$$

$$\frac{dP_T}{d\Omega} = \frac{Z_0}{32\pi^2} (khI)^2 \sin^2\theta$$

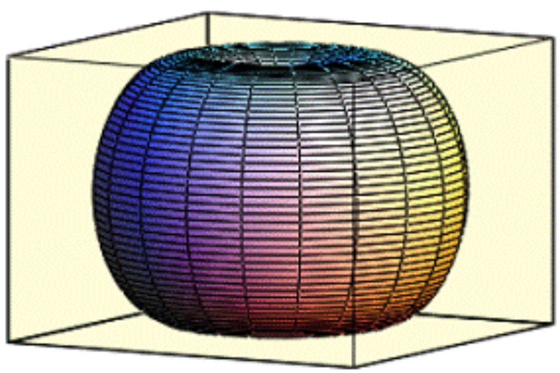


Antenna Pattern of a Hertzian Dipole





Radiation Pattern of the Dipole Antenna.



For a dipole of length
$L = 0.01\ (\lambda_0)$.

Total Power Radiated

Power radiated into the solid angle $d\Omega$

$$\frac{dP_T}{d\Omega} = \frac{Z_0}{32\pi^2}\left(khI\right)^2 \sin^2\theta$$

$$P_T = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \frac{dP_T}{d\Omega} = \frac{Z_0}{32\pi^2}\left(khI\right)^2 2\pi \int_0^{\pi} \sin^3\theta\, d\theta = \frac{Z_0}{12\pi}\left(kh\right)^2 I^2$$

$$P_T = \frac{1}{2} R_{rad} I^2 \quad \text{Radiation Resistance: } R_{rad} = \frac{Z_0}{6\pi}\left(kh\right)^2$$

$$\int_0^{\pi} \sin^3\theta\, d\theta : \text{ let } \mu = \cos\theta \quad d\mu = -\sin\theta d\theta \quad \sin^2\theta = 1-\mu^2$$

$$\int_0^{\pi} \sin^3\theta\, d\theta = \int_{-1}^{1} \left(1-\mu^2\right) d\mu = \left(\mu - \frac{\mu^3}{3}\right)\Bigg|_{-1}^{1} = \frac{4}{3}$$

Radiation Resistance and Antenna Gain

Power radiated into the solid angle $d\Omega$

$$\frac{dP_T}{d\Omega} = \frac{Z_0}{32\pi^2}\left(khI\right)^2 \sin^2\theta$$

$$P_T = \frac{1}{2} R_{rad} I^2 \quad \text{Radiation Resistance:} \quad R_{rad} = \frac{Z_0}{6\pi}\left(kh\right)^2$$

$$R_{rad} = \frac{Z_0}{6\pi}\left(kh\right)^2 \propto \left(\frac{h}{\lambda}\right)^2$$

Power radiated small for h<<λ

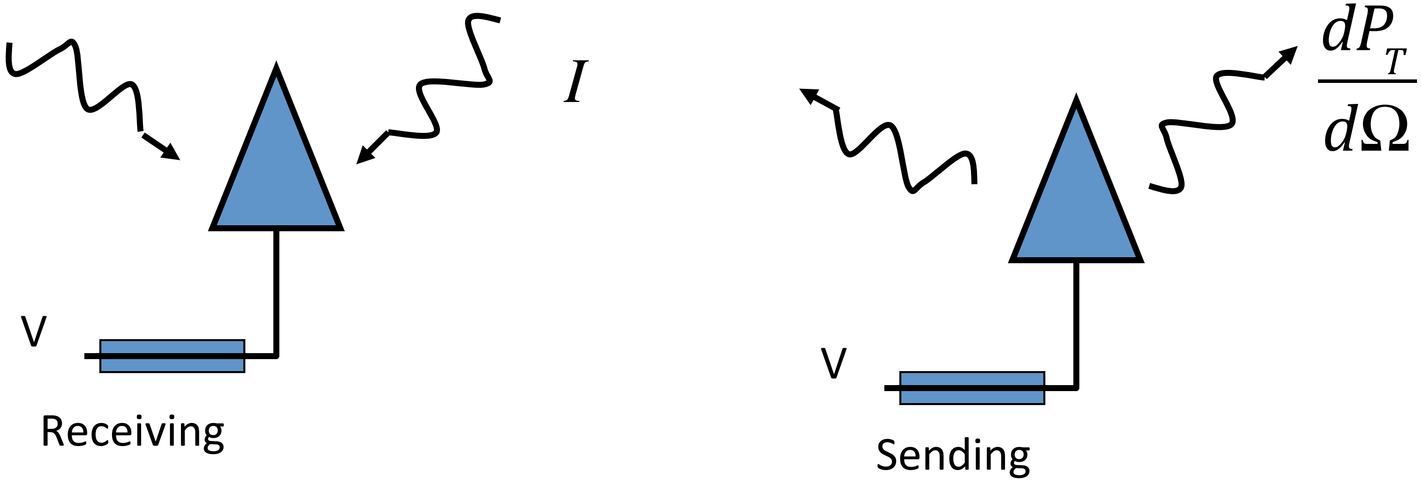
Antenna Gain - directivity

$$G(\theta,\phi) = \frac{dP_T}{d\Omega} \Big/ \left\langle \frac{dP_T}{d\Omega} \right\rangle = \frac{3}{2}\sin^2\theta,$$

$$\left\langle \frac{dP_T}{d\Omega} \right\rangle = \frac{1}{4\pi}\int d\Omega \frac{dP_T}{d\Omega}$$

$$G_{\max} = 1.5$$

Effective Area – Antenna Gain



Power received → $P_R = A_e(\Omega) I$ ← Incident intensity

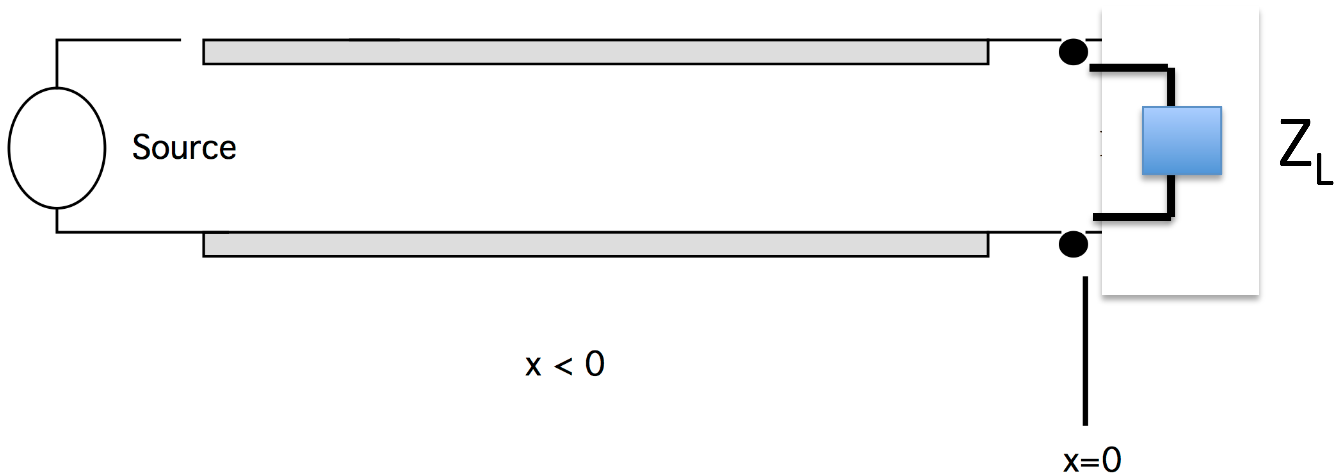
Effective area → $A_e(\Omega) = \frac{\lambda^2 G(\Omega)}{4\pi}$ ↔ gain

$$G(\Omega) = 4\pi \frac{dP_T}{d\Omega} / P_T$$

← Power per unit solid angle

$$P_T = \int \frac{dP_T}{d\Omega} d\Omega$$

Antenna Impedance



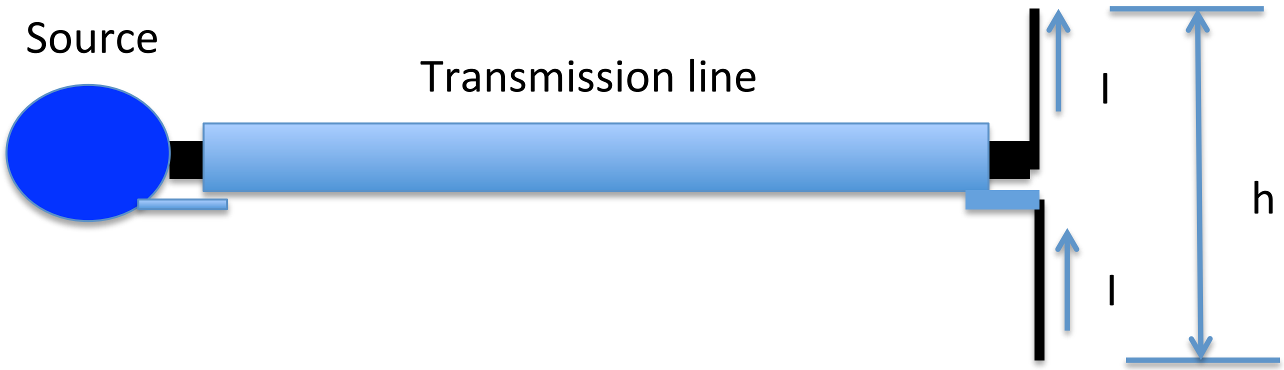
$Z_L = jX_{rad} + R_{rad}$

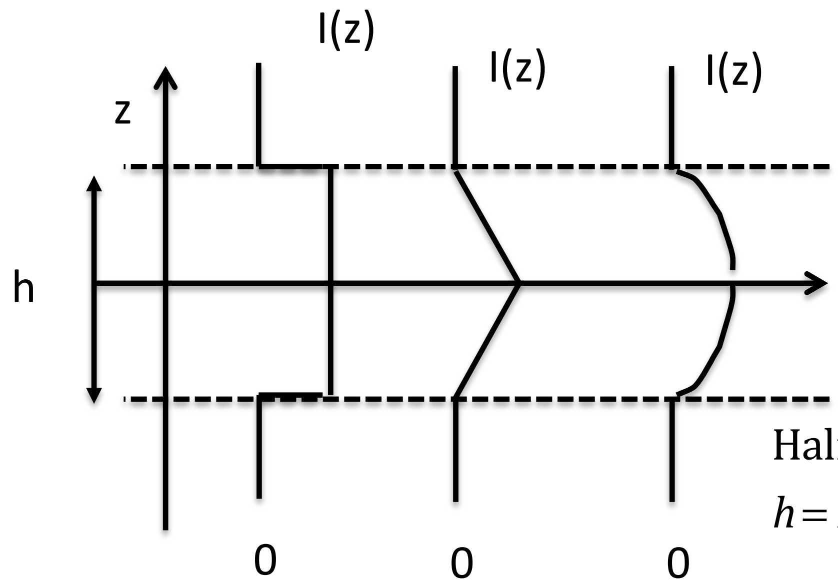
R_{rad} due to radiation

X_{rad} due to near fields.

Calculation of near fields must be done numerically

Current Distribution on Antenna?





$$\hat{\mathbf{J}} = \hat{\mathbf{z}} \quad I(z)\delta(x)\delta(y) \quad \text{for } |z| < h/2,$$

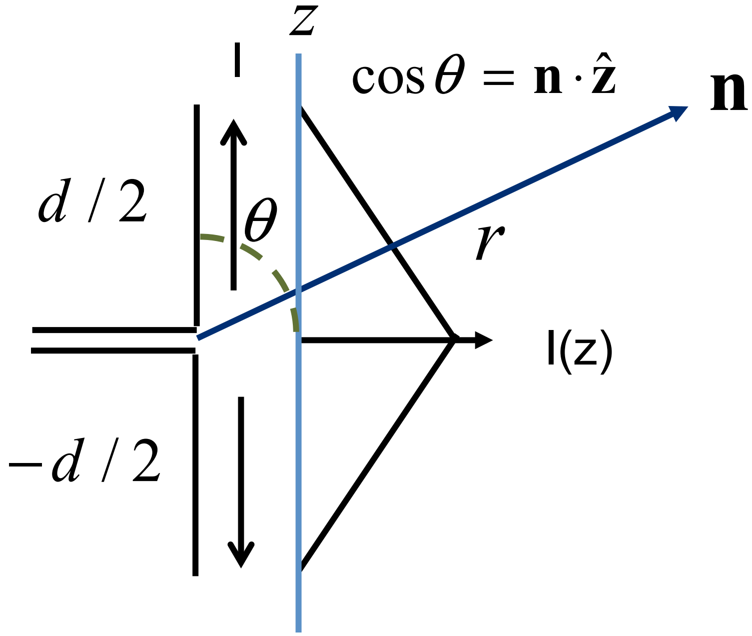
otherwise 0

$$\hat{\mathbf{C}}(\mathbf{k}) = \hat{\mathbf{z}} \int_{-h/2}^{h/2} dz I(z) \exp\left[ik_z z\right] = \hat{\mathbf{z}} C(k_z h)$$

Half wave dipole

$h = \lambda / 2$

Center Fed Linear Antenna



In short antennas current varies ~ linearly with z

$$\text{Current density}\quad \hat{\mathbf{J}}(\mathbf{r}) = I_0\,\delta(x)\delta(y)\left(1-2\frac{|z|}{d}\right)\hat{\mathbf{z}} \qquad |z| \le \frac{d}{2}$$

for $r >> r'$

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r}e^{ikr}\int_{Vol} d\tau'\,\hat{\mathbf{J}}(\mathbf{r}')e^{-i\mathbf{k}\cdot\mathbf{r}'} = \frac{\mu_0}{4\pi r}e^{ikr}I_0\,\hat{\mathbf{z}}\int_{-d/2}^{d/2} dz'\left(1-2\frac{|z'|}{d}\right)e^{-ikz'\cos\theta}$$

Center Fed Linear Antenna

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq I_0\,\hat{\mathbf{z}} \int_{-d/2}^{d/2} dz'\,(1-2\frac{|z'|}{d})\,e^{-ikz'\cos\theta}$$

use Euler's equ.

even

odd

$$= I_0\,\hat{\mathbf{z}} \int_{-d/2}^{d/2} dz' \left(1-2\frac{|z'|}{d}\right)\left(\cos(kz'\cos\theta) - i\sin(kz'\cos\theta)\right)$$

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq 2I_0\,\hat{\mathbf{z}} \int_{0}^{d/2} dz' \left(1-2\frac{z'}{d}\right)\cos(kz'\cos\theta)$$

Center Fed Linear Antenna

To carry out the integration, let $\rho' = k\,z'\cos\theta$ and $\rho_0 = \dfrac{k\,d\cos\theta}{2}$

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq 2\,I_0\,\hat{\mathbf{z}}\frac{d}{\rho_0}\int_0^{\rho_0} d\rho'\left(1-\frac{\rho'}{\rho_0}\right)\cos\rho'$$

using $\int x\cos x\,dx = \cos x + x\sin x$

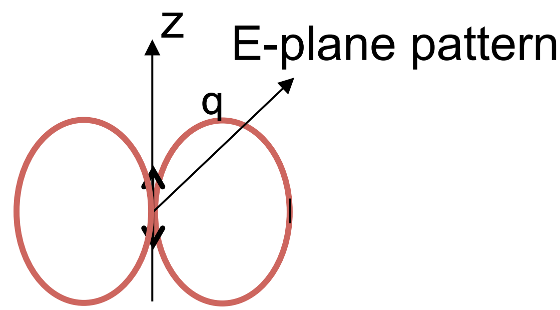
$$\hat{\mathbf{A}}(\mathbf{r}) \simeq I_0\,\hat{\mathbf{z}}\frac{d}{\rho_0^2}(1-\cos\rho_0) \quad \text{where } \rho_0 = \frac{k\,d\cos\theta}{2} = \frac{\pi\,d\cos\theta}{\lambda}$$

Antenna in the Dipole Limit

In the dipole limit $\lambda >> d \quad (\rho_0 << 1)$

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} I_0 \frac{d}{2} \hat{\mathbf{z}} \qquad \hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \hat{\mathbf{C}}(\mathbf{k})$$

hence, $\mathbf{C}(\mathbf{k}) = I_0 \frac{d}{2} \hat{\mathbf{z}}$



Power radiated into the solid angle $d\Omega$

$$\frac{dP_T}{d\Omega} = \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle \mathrm{r}^2 = \frac{Z_0}{32\pi^2} \left| \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right|^2 = \frac{Z_0}{32\pi^2} k^2 \frac{I_0^2 d^2}{4} \sin^2\theta$$

Total Power Radiated and Radiation Resistance

The total power radiated is $P_T = \oint_S \frac{dP_T}{d\Omega} d\Omega$

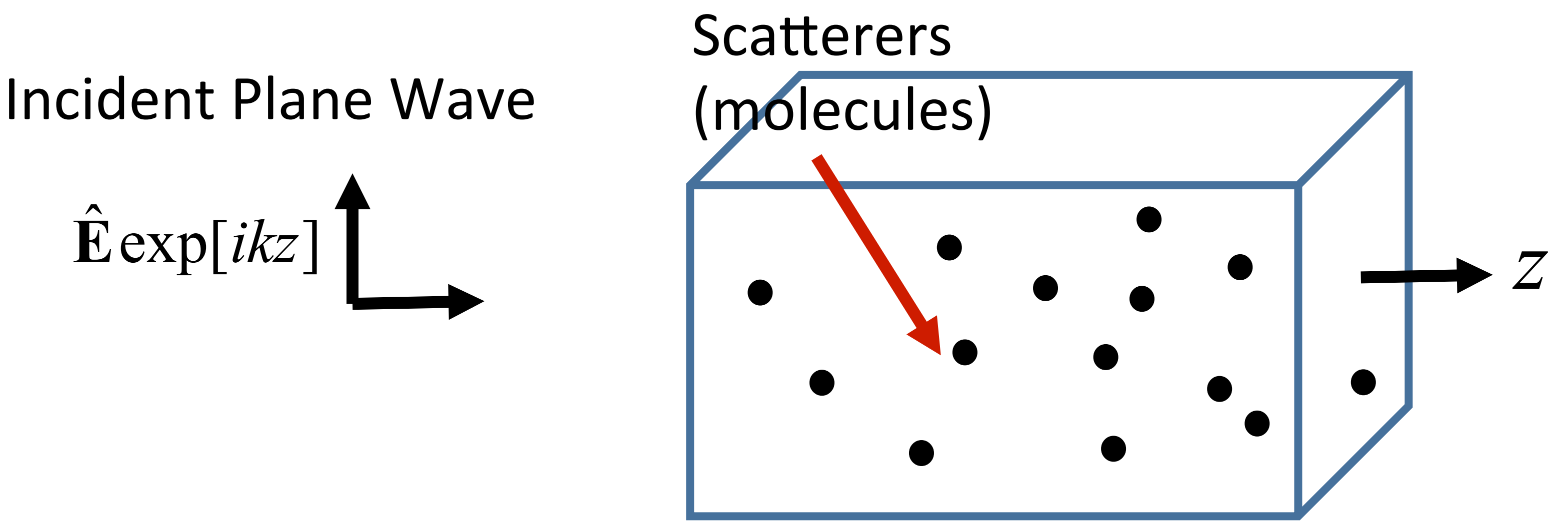
where $d\Omega = \sin\theta\, d\theta\, d\varphi$ is the solid angle

$$\frac{dP_T}{d\Omega} = \frac{Z_0}{32\pi^2} k^2 \frac{I_0^2 d^2}{4} \sin^2\theta \qquad \int_0^{2\pi} d\varphi \int_0^{\pi} \sin^3\theta\, d\theta = 2\pi \frac{4}{3}$$

$$\boxed{P_T = \frac{Z_0}{48\pi} k^2 d^2 I_0^2 = \frac{1}{2} Z_{rad} I_0^2}$$

where the radiation resistance is $Z_{rad} = \frac{Z_0}{24\pi} k^2 d^2$ [Ω]

Scattering



Amplitude due to ensemble of spatially distributed scatterers

$$\hat{\mathbf{C}}(\mathbf{k}) = \int_{Vol} d\tau' \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'} \simeq -i\omega \sum_i \hat{\mathbf{p}}_i e^{-i\mathbf{k}\cdot\mathbf{r}_i}$$

Dipole moment proportional to local electric field

$$\hat{\mathbf{p}}_i = \gamma \hat{\mathbf{E}} \exp[ikz_i]$$

Electric Dipole Radiation

The dipole moment is

$$\hat{\mathbf{p}} = \int\limits_{Vol} \mathbf{r}'\,\hat{\rho}(\mathbf{r}')\,d\tau' = -\frac{i}{\omega}\int\limits_{Vol} \mathbf{r}'\,\nabla'\cdot\hat{\mathbf{J}}(\mathbf{r}')\,d\tau'$$

Integrating by parts gives $(\nabla'\cdot\mathbf{r}' = 1)$

$$\hat{\mathbf{p}} = \frac{i}{\omega}\int\limits_{Vol} \hat{\mathbf{J}}(\mathbf{r}')\,d\tau'$$

Electric Dipole Radiation

Power radiated into the solid angle $d\Omega$ $\quad (da = r^2 d\Omega)$

$$\frac{dP_T}{d\Omega} = \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle \mathrm{r}^2 = \frac{Z_0}{32\pi^2} \left| \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right|^2 \qquad P_T = \oint_S \frac{dP_T}{d\Omega} d\Omega$$

where $\hat{\mathbf{C}}(\mathbf{k}) = \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'}$ is F-T of the current density

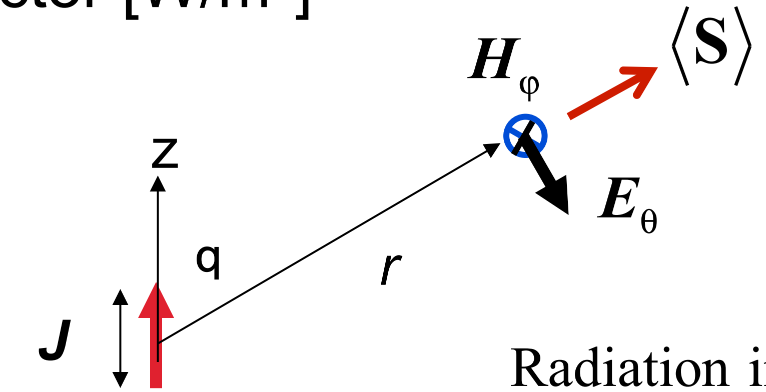
If the wavelength is large compared to the dimensions of the dipole

$\left| \mathbf{k} \cdot \mathbf{r}' \right| << 1, \quad k = 2\pi / \lambda \qquad \hat{\mathbf{C}}(\mathbf{k}) \approx \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') \qquad \hat{\mathbf{C}}(\mathbf{k}) \approx -i\omega \hat{\mathbf{p}}$

$$\frac{dP_T}{d\Omega} = \frac{Z_0}{32\pi^2} \frac{\omega^4}{c^2} \left| \hat{\mathbf{n}} \times \hat{\mathbf{p}} \right|^2 = \frac{Z_0}{32\pi^2} \frac{\omega^4}{c^2} \left| \hat{\mathbf{p}} \right|^2 \sin^2\theta \;\sim\; \omega^4$$

Electric Dipole Radiation

Poynting vector [W/m^2]



$$\langle \mathbf{S} \rangle = \frac{1}{2} \mathrm{Re}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}^* \right)$$

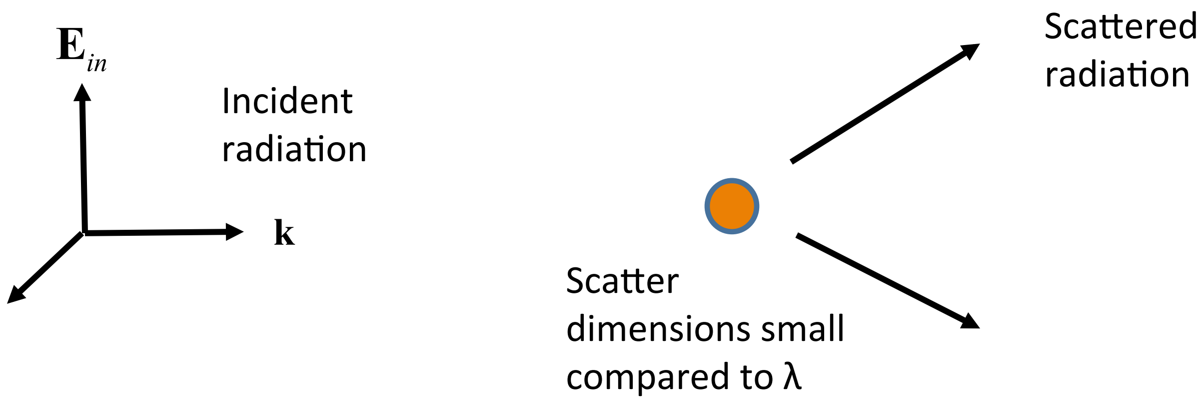
Radiation intensity is $\left| \langle \mathbf{S} \rangle \right|$

The dipole moment is $\hat{\mathbf{p}} = \int_{Vol} \mathbf{r}' \hat{\rho}(\mathbf{r}') \, d\tau'$

The charge density is $\rho(\mathbf{r},t) = \mathrm{Re}\left[\hat{\rho}(\mathbf{r}) \, e^{-i\omega t} \right]$

Conservation of charge, $\partial \rho / \partial t + \nabla \cdot \mathbf{J} = 0$, gives $\nabla \cdot \hat{\mathbf{J}}(\mathbf{r}) = i\omega \hat{\rho}(\mathbf{r})$

Scattering at Long Wavelengths



- The incident radiation induces an oscillating electric and magnetic dipole moment in the scatter

- The induced dipole moments radiate (scattered radiation)

- The scattered radiation is a function of the polarization and direction of both incident and scattered radiation

- If the wavelength is large compared to the size of the scatter the induced electric and magnetic dipole moments are sufficient to describe the scattered radiation   (opposite case is called Mie scattering)

Radiated Power

Sum over dipoles

$$\frac{dP_T}{d\Omega} = \frac{Z_0}{32\pi^2}\frac{\omega^4}{c^2}\left|\hat{\mathbf{n}}\times\gamma\hat{\mathbf{E}}\sum_i \exp[ikz_i - i\mathbf{k}\cdot\mathbf{r}_i]\right|^2$$

$$\frac{dP_T}{d\Omega} = \frac{dP_{T1}}{d\Omega} f(\mathbf{k}, N)$$

Radiation due to single dipole

$$\frac{dP_{T1}}{d\Omega} = \frac{Z_0}{32\pi^2}\frac{\omega^4}{c^2}\left|\hat{\mathbf{n}}\times\gamma\hat{\mathbf{E}}\right|^2$$

Form factor – sum over dipoles

$$f = \left|\sum_i \exp[ikz_i - i\mathbf{k}\cdot\mathbf{r}_i]\right|^2$$

Three cases

$$f = \left| \sum_i \exp[ikz_i - i\mathbf{k}\cdot\mathbf{r}_i] \right|^2$$

Dipoles localized to a volume smaller than a wavelength

$f \approx N^2$

Dipoles distributed randomly in a volume larger than a wavelength

$f \approx N$

Dipoles in an ordered array - grating

Cases

$$f = \left| \sum_i \exp[ikz_i - i\mathbf{k}\cdot\mathbf{r}_i] \right|^2$$

Dipoles localized to a volume smaller than a wavelength - coherent

$$f = \left| \sum_i \quad \right|^2 = N^2$$

Dipoles distributed randomly in a volume larger than a wavelength - incoherent

$$f = \left| \sum_{i,j} \exp[ik\left(z_i - z_j\right) - i\mathbf{k}\cdot\left(\mathbf{r}_i - \mathbf{r}_j\right)] \right|$$

Terms with i an j different average to zero. Only j=i survive

$$f = \left| \sum_i \quad \right| = N$$


Chincoteague 10/23/20

Ordered array 1D

$$f = \left| \sum_i \exp[ikz_i - i\mathbf{k} \cdot \mathbf{r}_i] \right|^2$$

$$\mathbf{r}_i = \mathbf{d}i$$

$$f = \left| \sum_i \exp[i\left(kd\cos\theta - \mathbf{k} \cdot \mathbf{d}\right)i] \right|^2$$

f peaks when

$$\left(kd\cos\theta - \mathbf{k} \cdot \mathbf{d}\right) = 2\pi n$$

Higher Order Moments of the Fields

Include both electric and magnetic dipole contributions

In the far field zone

$$\hat{\mathbf{A}}(\mathbf{r})\ ;\ \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') \, e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

$$\hat{\mathbf{A}}(\mathbf{r})\ ;\ \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') \left(1 - ik\,\hat{\mathbf{n}}\cdot\mathbf{r}' - \frac{k^2}{2}(\mathbf{n}\cdot\mathbf{r}')^2 + \ldots \right)$$

terms fall off rapidly

Electric and Magnetic Dipole Radiation

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'} \simeq \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}')(1 - ik\,\hat{\mathbf{n}}\cdot\mathbf{r}')$$

electric dipole   magnetic dipole

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \left\{ -i\omega\hat{\mathbf{p}} - ik\,\hat{\mathbf{m}}\times\hat{\mathbf{n}} \right\}$$

The F-T of the current density including both the **electric** and **magnetic dipole** contributions is

$$\hat{\mathbf{C}}(\mathbf{k}) = \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'} \simeq -i\omega(\hat{\mathbf{p}} + \hat{\mathbf{m}}\times\hat{\mathbf{n}}/c)$$

Electric and Magnetic Dipole Radiation

Power radiated into the solid angle $d\Omega$

$$\frac{dP_T}{d\Omega} = \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle \mathrm{r}^2 = \frac{Z_0}{32\pi^2} \left| \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right|^2$$

$$\frac{dP_T}{d\Omega} = \frac{Z_0}{32\pi^2} \frac{\omega^4}{c^2} \left| \hat{\mathbf{n}} \times (\hat{\mathbf{p}} + \hat{\mathbf{m}} \times \hat{\mathbf{n}} / c) \right|^2$$

Far field zone

$\hat{\mathbf{p}}$ and $\hat{\mathbf{m}}$ are the electric and magnetic dipole moments

Shorter wavelengths scatter more (blue sky)