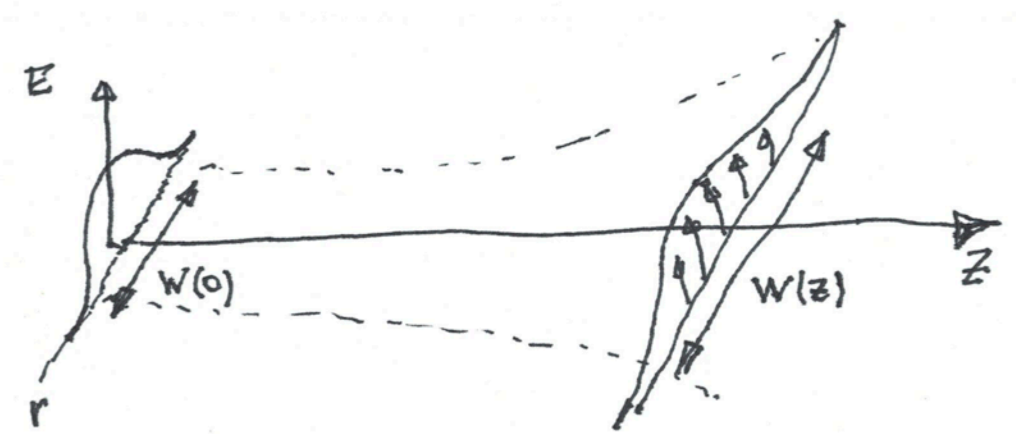
ENEE381

Lecture 09

Diffraction, Interference

Diffraction

Waves launched from a finite size source spread out as they propagate.

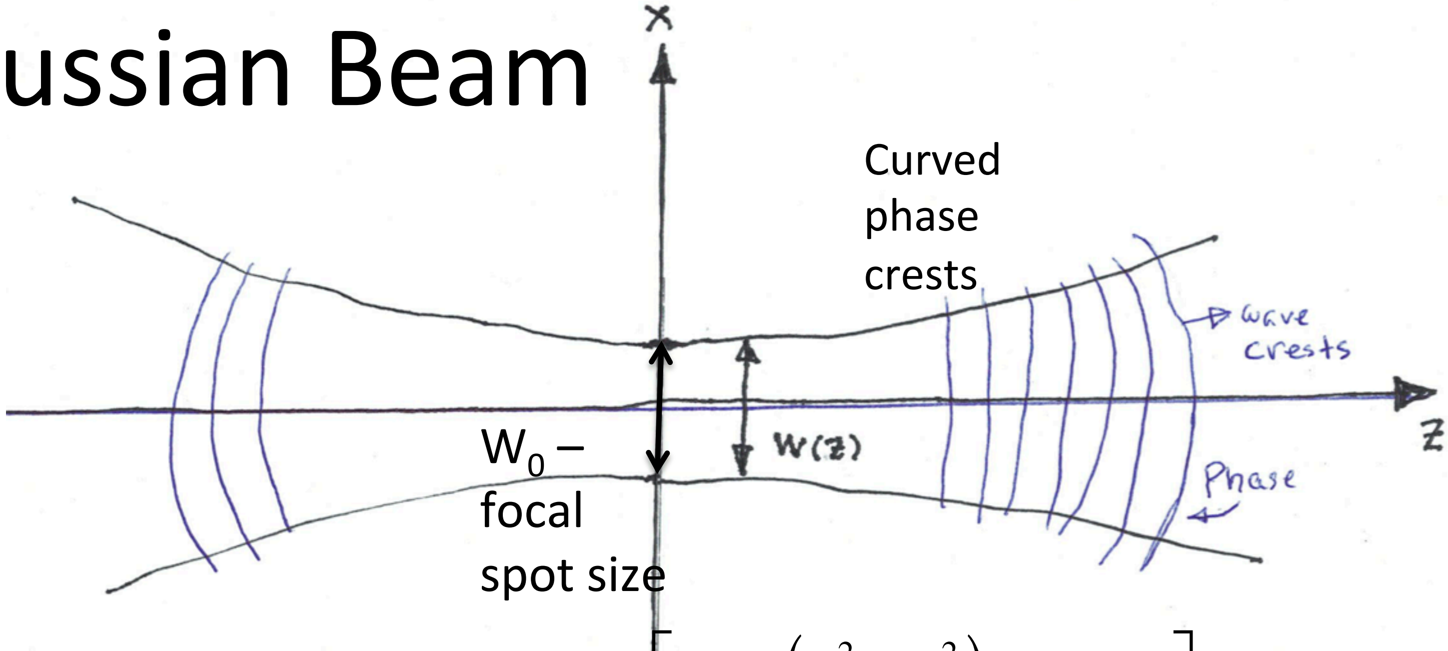


$$W(z) = W(0)\sqrt{1 + z^2 / Z_R^2}$$

$$Z_R = \frac{1}{2} k W^2(0)$$

Rayleigh Length

Gaussian Beam



$$E_{x,y}(x,y,z) = \frac{E_0}{1+iz/Z_R}\exp\left[-\frac{\left(x^2+y^2\right)}{W_0^2\left(1+iz/Z_R\right)}+ikz\right]$$

$$W(z) = W_0\sqrt{1+z^2/Z_R^2} \qquad Z_R = \frac{1}{2}kW_0^2$$ Rayleigh Length

$$E_{x,y}(0,0,z) = \frac{E_0}{1+iz/Z_R}\exp[+ikz] = \frac{E_0}{\sqrt{1+\left(z/Z_R\right)^2}}\exp[+ikz+i\phi(z)]$$

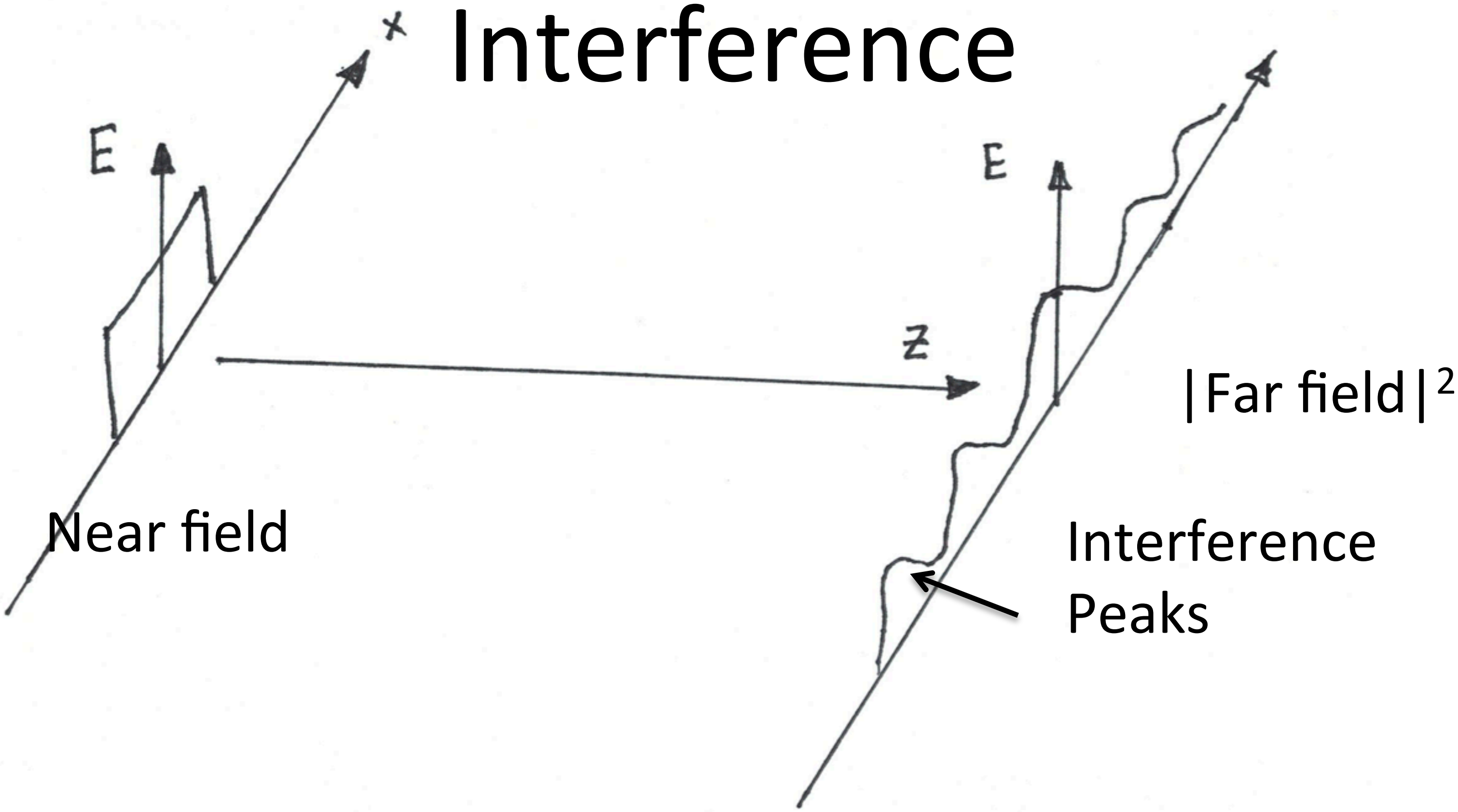
Guoy Phase $\tan\phi = -z/Z_R$

$$E_{x,y}(x,y,z)=\frac{E_0}{1+iz/Z_R}\exp\left[-\frac{\left(x^2+y^2\right)}{W_0^2\left(1+iz/Z_R\right)}+ikz\right]$$

$$=\underbrace{\frac{E_0}{\sqrt{1+z^2/Z_R^2}}\exp\left[-\frac{\left(x^2+y^2\right)}{W_0^2\left(1+z^2/Z_R^2\right)}\right]}_{\text{Amplitude}}\underbrace{\exp\left\{i\left[kz+\frac{z\left(x^2+y^2\right)}{Z_RW_0^2\left(1+z^2/Z_R^2\right)}+\phi_G\right]\right\}}_{\text{Phase}}$$

As $z\to\infty$ $\qquad ik\left[z+\frac{\left(x^2+y^2\right)}{2z}\right]+\phi_G=ikr+\phi_G$

Interference



Far field is Fourier transform of near field.

$$E_{x,y}(x,y,z)=\frac{k}{2i\pi z}\bar{E}_{x,y}\left(\frac{kx}{z},\frac{ky}{z},0\right)\exp\left[ik\left(z+\frac{x^2+y^2}{2z}\right)\right]$$

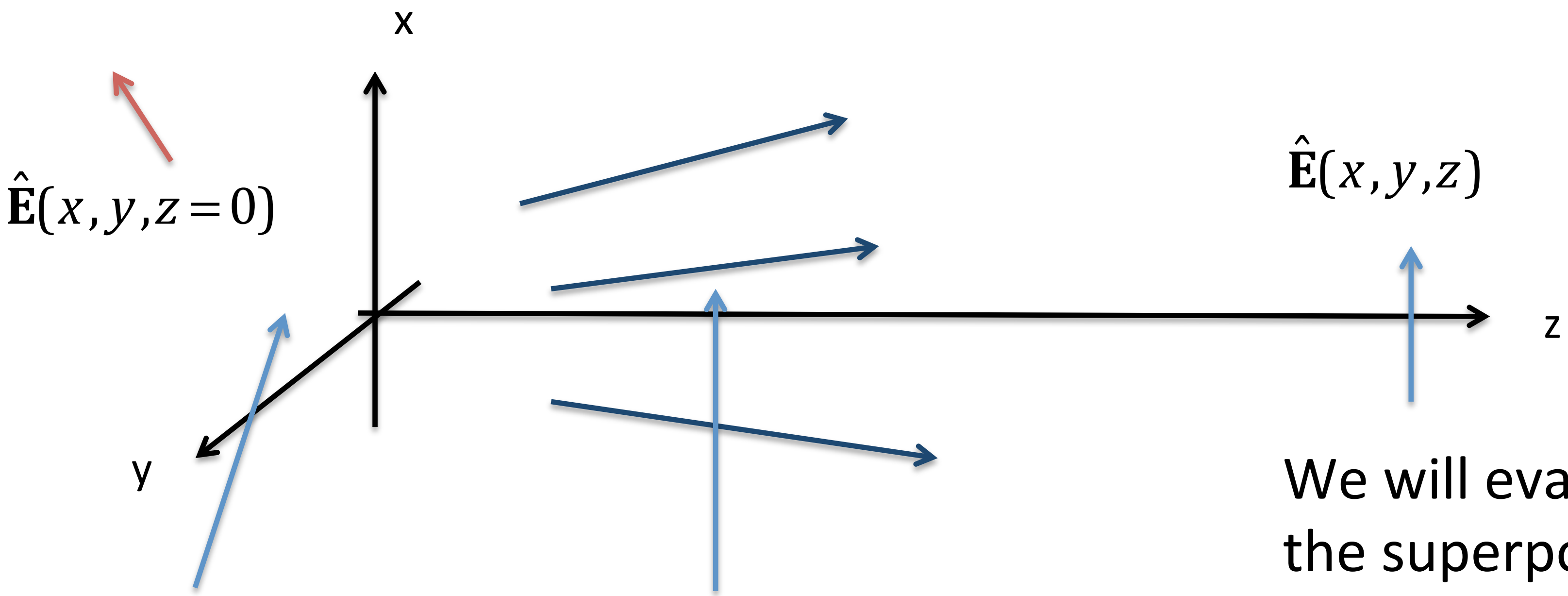
Problem

A certain infrared (wavelength 1 micrometer) laser beam can be focused to a spot size (W_0 = 15 micrometers).

1. What is the Rayleigh distance?

2. Suppose the central intensity at the focal point is 10^{18} W/cm^2, What is the central intensity 3 meters from the focus? What is the RMS electric field?

Approach



We will assume we know E_x and E_y in plane z=0

Fourier Transform E(z=0). Construct a superposition of plane waves giving E_x and E_y in plane z=0

We will evaluate the superposition of plane waves as a function of z. Inverse Fourier transform

Wave Equation

$$\nabla \cdot \vec{\mathbf{E}} = 0 \qquad \nabla \cdot \vec{\mathbf{H}} = 0 \qquad \nabla \times \vec{\mathbf{E}} = -\mu \frac{\partial \vec{\mathbf{H}}}{\partial t} \qquad \nabla \times \vec{\mathbf{H}} = \varepsilon \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

To get equations for phasor amplitudes

$$\vec{\mathbf{E}}, \vec{\mathbf{H}} \Rightarrow \mathrm{Re}\left\{ \left(\hat{\mathbf{E}}(\mathbf{x}), \hat{\mathbf{H}}(\mathbf{x}) \right) e^{-i\omega t} \right\}$$

$$\frac{\partial}{\partial t}, \nabla \Rightarrow -i\omega, \nabla$$

$$\nabla \cdot \hat{\mathbf{E}} = 0 \qquad \nabla \cdot \hat{\mathbf{H}} = 0 \qquad \nabla \times \hat{\mathbf{E}} = i\omega\mu \hat{\mathbf{H}} \qquad \nabla \times \hat{\mathbf{H}} = -i\omega\varepsilon \hat{\mathbf{E}}$$

Combine

$$\nabla \times \left(\nabla \times \hat{\mathbf{E}} \right) = i\omega\mu \nabla \times \hat{\mathbf{H}} = \omega^2 \varepsilon\mu \hat{\mathbf{E}} = k^2 \mathbf{E}$$

$$\nabla \left(\nabla \cdot \hat{\mathbf{E}} \right) - \nabla^2 \hat{\mathbf{E}} = \boxed{-\nabla^2 \hat{\mathbf{E}} = k^2 \hat{\mathbf{E}}}$$

$$k^2 = \frac{\omega^2}{v^2}, \quad \lambda = \frac{2\pi}{k}$$

Consider transverse components, E_x E_y

$$-\nabla^2\hat{\mathbf{E}} = k^2\hat{\mathbf{E}} \quad \rightarrow \quad -\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_{x,y} = k^2 E_{x,y} \quad k^2 = \omega^2\varepsilon\mu$$

Take spatial Fourier transform in x and y

$$-\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \exp\left[-ik_x x - ik_y y\right]\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_{x,y} = k^2 \bar{E}_{x,y}(k_x,k_y,z)$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \exp\left[-ik_x x - ik_y y\right]E_{x,y} = \bar{E}_{x,y}(k_x,k_y,z)$$

$$\boxed{\left(k_x^2 + k_y^2 - \frac{\partial^2}{\partial z^2}\right)\bar{E}_{x,y} = k^2 \bar{E}_{x,y}(k_x,k_y,z)}$$

Solutions

$$\left(k_x^2 + k_y^2 - \frac{\partial^2}{\partial z^2}\right)\bar{E}_{x,y} = k^2 \bar{E}_{x,y}(k_x, k_y, z)$$

Second order DEQ- 2 solutions

$$\bar{E}_{x,y} = A\exp\left(ik_z z\right) + B\exp\left(-ik_z z\right), \qquad k_z = \sqrt{k^2 - \left(k_x^2 + k_y^2\right)}, \qquad k^2 > \left(k_x^2 + k_y^2\right)$$

$$\bar{E}_{x,y} = A\exp\left(-\kappa_z z\right) + B\exp\left(+\kappa_z z\right), \qquad \kappa_z = \sqrt{\left(k_x^2 + k_y^2\right) - k^2}, \qquad k^2 < \left(k_x^2 + k_y^2\right)$$

Boundary condition as z goes to infinity B=0

$$\bar{E}_{x,y} = \bar{E}_{x,y}(k_x, k_y, z = 0)\exp\left(ik_z z\right), \qquad k_z = \sqrt{k^2 - \left(k_x^2 + k_y^2\right)}, \qquad k^2 > \left(k_x^2 + k_y^2\right)$$

$$\bar{E}_{x,y} = \bar{E}_{x,y}(k_x, k_y, z = 0)\exp\left(-\kappa_z z\right), \qquad \kappa_z = \sqrt{\left(k_x^2 + k_y^2\right) - k^2}, \qquad k^2 < \left(k_x^2 + k_y^2\right)$$

We know this

Divergence E = 0

$\nabla \cdot \hat{\mathbf{E}} = 0$ Fourier Transform in x and y $\Rightarrow$ $\frac{\partial}{\partial z}\bar{E}_z = -i\left(k_x\bar{E}_x + k_y\bar{E}_y\right)$

$$\bar{E}_z = \frac{-1}{k_z}\left(k_x\bar{E}_x + k_y\bar{E}_y\right)\exp\left(ik_z z\right), \qquad k^2 > \left(k_x^2 + k_y^2\right)$$

$$\bar{E}_{x,y} = \frac{i}{\kappa_z}\left(k_x\bar{E}_x + k_y\bar{E}_y\right)\exp\left(-\kappa_z z\right), \qquad k^2 < \left(k_x^2 + k_y^2\right)$$

We can find E_z after the problem for Ex and Ey is solved

Fourier Inversion

$$E_{x,y}(x,y,z) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_x \bar{E}_{x,y}(k_x,k_y,z)\exp\left[ik_x x + ik_y y\right]$$

$$\bar{E}_{x,y} = \bar{E}_{x,y}(k_x,k_y,z=0)\exp\left(ik_z z\right), \qquad k_z = \sqrt{k^2 - \left(k_x^2 + k_y^2\right)}, \qquad k^2 > \left(k_x^2 + k_y^2\right)$$

$$\bar{E}_{x,y} = \bar{E}_{x,y}(k_x,k_y,z=0)\exp\left(-\kappa_z z\right), \qquad \kappa_z = \sqrt{\left(k_x^2 + k_y^2\right) - k^2}, \qquad k^2 < \left(k_x^2 + k_y^2\right)$$

Let's assume $\bar{E}_{x,y}(k_x,k_y,z=0) \rightarrow 0$ for $k^2 < \left(k_x^2 + k_y^2\right)$

$$E_{x,y}(x,y,z) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_x \bar{E}_{x,y}(k_x,k_y,0)\exp\left[ik_x x + ik_y y + iz\sqrt{k^2 - \left(k_x^2 + k_y^2\right)}\right]$$

Consider Gaussian Dependence on x,y in plane z=0

Gaussian E $$E_{x,y} = E_0 \exp\left[-\frac{x^2+y^2}{W_0^2}\right]$$

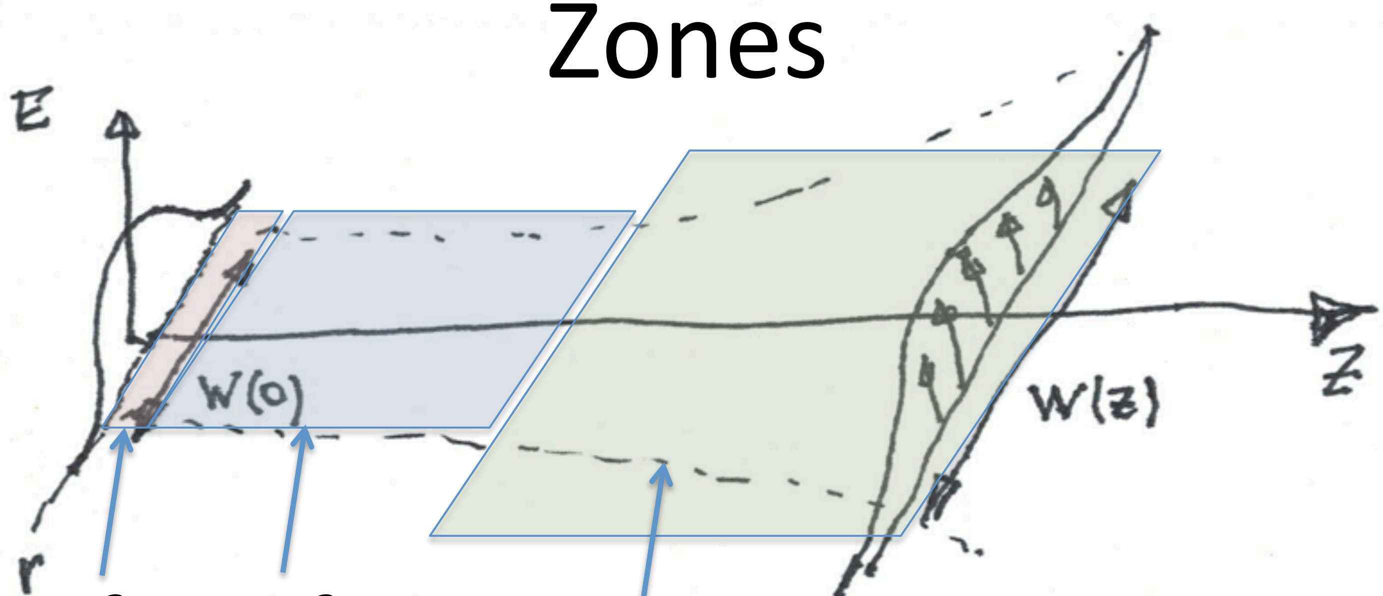
Fourier Transform $$\bar{E}_{x,y}(k_x,k_y,0) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \exp\left[-ik_x x - ik_y y\right] E_0 \exp\left[-\frac{x^2+y^2}{W_0^2}\right]$$

Complete Square $$\frac{x^2}{W_0^2} + ik_x x = \frac{\left(x + ik_x W_0^2/2\right)^2}{W_0^2} + \frac{k_x^2 W_0^2}{4}$$

K-space E $$\bar{E}_{x,y}(k_x,k_y,0) = \pi W_0^2 E_0 \exp\left[-\frac{\left(k_x^2+k_y^2\right)W_0^2}{4}\right]$$

W_0 - width in space $2/W_0$ - Width in k_x, k_y

Zones



$z \simeq \lambda = \frac{2\pi}{k}$ $\lambda = \frac{2\pi}{k} < z < Z_R$ $\lambda = \frac{2\pi}{k} < Z_R << z$

Near field Fresnel Fraunhoffer

Fresnel: Invented the Fresnel lens. Installed in lighthouses around the world. Saved many lives.

Fraunhoffer: Orphaned at age 11. Worked for glass maker. Buried in collapsed building. Rescued by a Prince. Invented the spectroscope.

1 . Fourier transform $\bar{E}_{x,y}(k_x,k_y,0) = \pi W_0^2 E_0 \exp\left[-\frac{\left(k_x^2+k_y^2\right)W_0^2}{4}\right]$

2 . Inverse transform

$$E_{x,y}(x,y,z) = \int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\int_{-\infty}^{\infty}\frac{dk_y}{2\pi}\bar{E}_{x,y}(k_x,k_y,0)\exp\left[ik_x x + ik_y y + iz\sqrt{k^2-\left(k_x^2+k_y^2\right)}\right]$$

Note: $\bar{E}_{x,y}(k_x,k_y,0) \to 0 \text{ for } k_x^2+k_y^2 > W_0^{-2} << k^2$

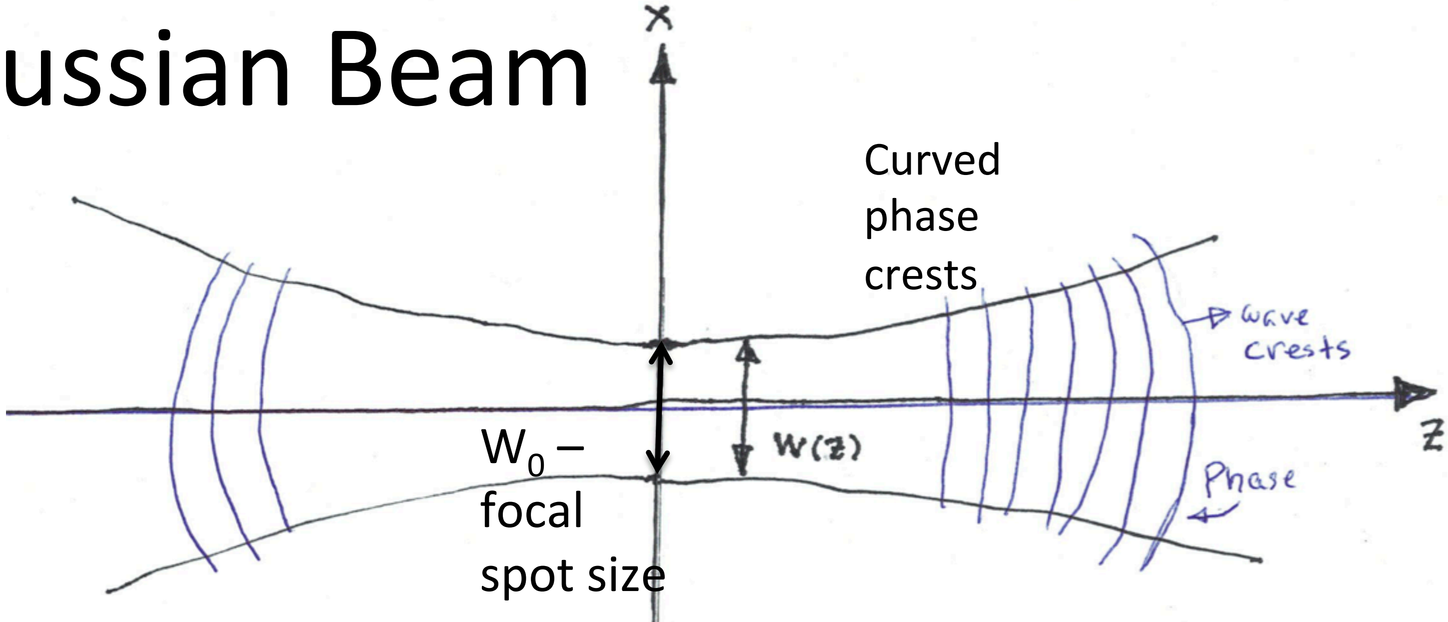
Expand: $\sqrt{k^2-\left(k_x^2+k_y^2\right)} \simeq k - \frac{k_x^2+k_y^2}{2k}$

$$E_{x,y}(x,y,z) = \pi W_0^2 E_0 \int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\int_{-\infty}^{\infty}\frac{dk_y}{2\pi}\exp\left[-\left(k_x^2+k_y^2\right)\left[\frac{W_0^2}{4}+\frac{iz}{2k}\right]+ik_x x+ik_y y+ikz\right]$$

Complete square:

$$E_{x,y}(x,y,z) = \frac{E_0}{1+iz/Z_R}\exp\left[-\frac{\left(x^2+y^2\right)}{W_0^2\left(1+iz/Z_R\right)}+ikz\right]$$

Gaussian Beam



$$E_{x,y}(x,y,z)=\frac{E_0}{1+iz/Z_R}\exp\left[-\frac{\left(x^2+y^2\right)}{W_0^2\left(1+iz/Z_R\right)}+ikz\right]$$

$$W(z)=W_0\sqrt{1+z^2/Z_R^2}$$

$$Z_R=\frac{1}{2}kW_0^2$$

Rayleigh Length

$$E_{x,y}(0,0,z)=\frac{E_0}{1+iz/Z_R}\exp[+ikz]=\frac{E_0}{\sqrt{1+\left(z/Z_R\right)^2}}\exp[+ikz+i\phi(z)]$$

Guoy Phase $\tan\phi=-z/Z_R$

Far Field Radiation Pattern

$$E_{x,y}(x,y,z) = \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{\infty} \frac{dk_y}{2\pi} \bar{E}_{x,y}(k_x, k_y, 0) \exp\left[ik_x x + ik_y y + iz\sqrt{k^2 - \left(k_x^2 + k_y^2\right)} \right]$$

$$\bar{E}_{x,y}(k_x, k_y, 0) \to 0 \text{ for } k_x^2 + k_y^2 << k^2 \qquad \sqrt{k^2 - \left(k_x^2 + k_y^2\right)} \simeq k - \frac{k_x^2 + k_y^2}{2k}$$

$$\lim_{z\to\infty} \exp\left[-\frac{iz}{2k}\left(k_x^2 + k_y^2\right) + ik_x x + ik_y y \right] =$$

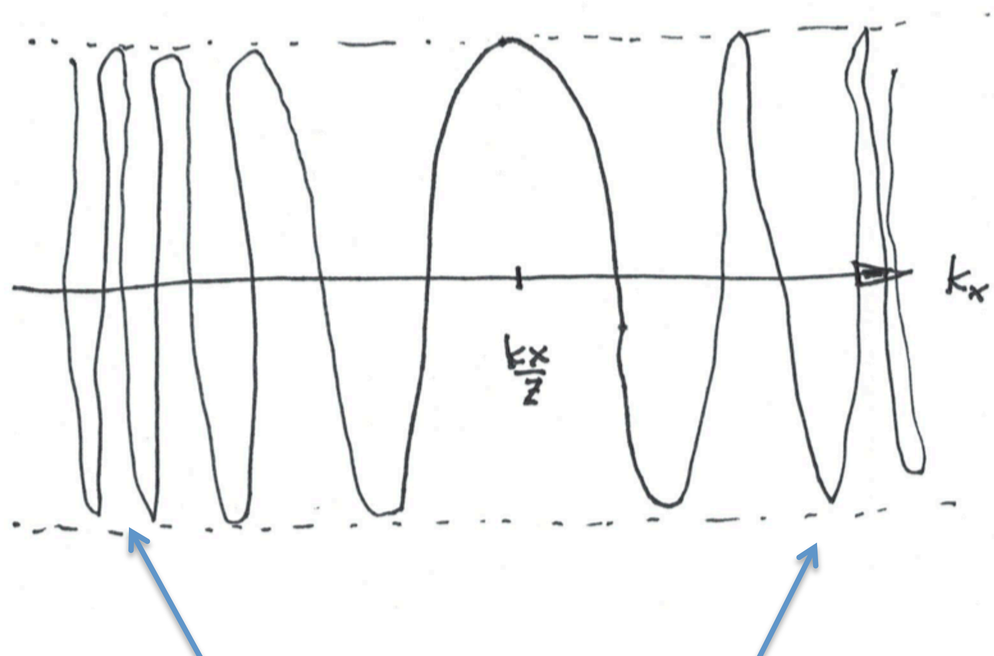
Delta functions

$$\frac{2\pi k}{iz} \exp\left[ik \frac{x^2 + y^2}{2z} \right] \delta\left(k_x - \frac{kx}{z} \right) \delta\left(k_y - \frac{ky}{z} \right)$$

$$E_{x,y}(x,y,z) = \frac{k}{2i\pi z} \bar{E}_{x,y}\left(\frac{kx}{z}, \frac{ky}{z}, 0\right) \exp\left[ik\left(z + \frac{x^2 + y^2}{2z} \right) \right]$$

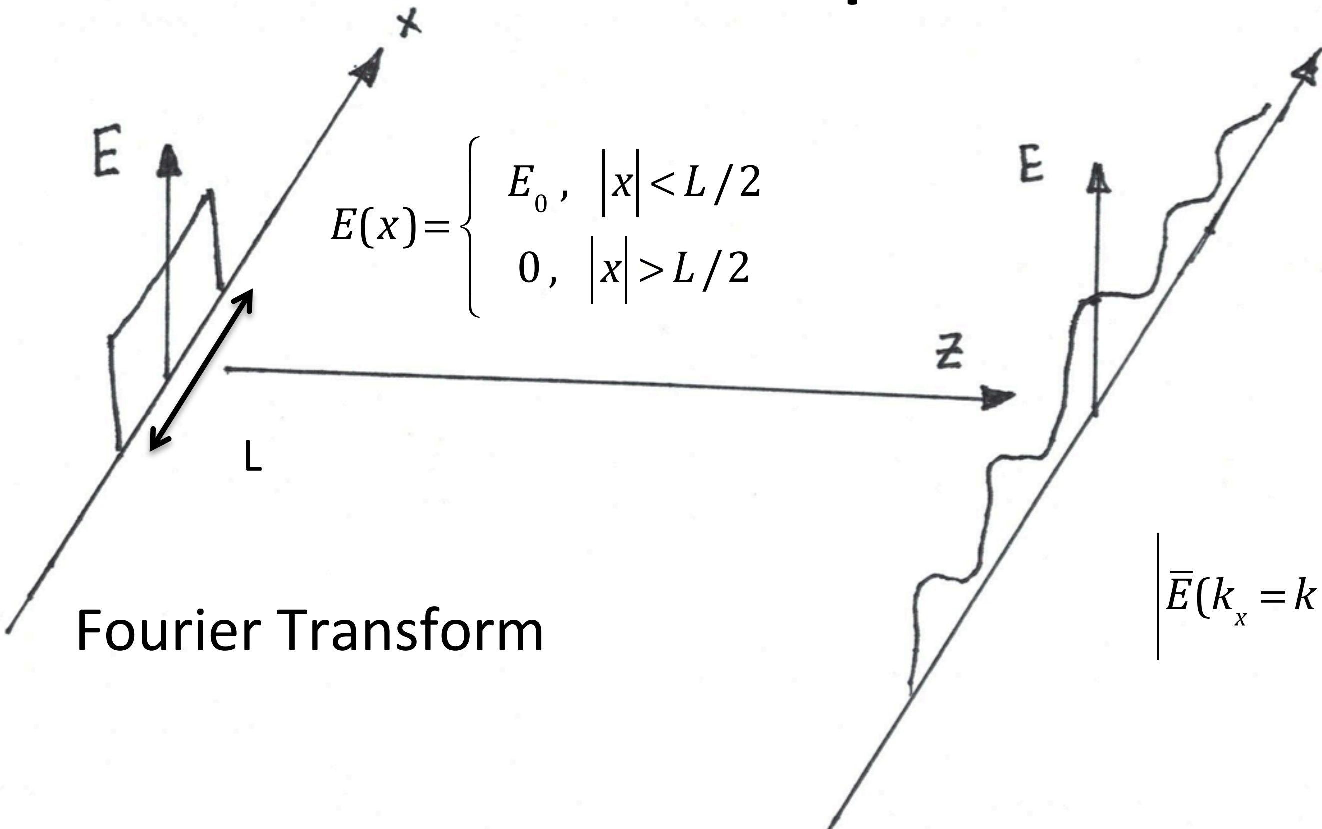
Stationary Phase

$$\lim_{z\to\infty} \exp\left[-\frac{iz}{2k}k_x^2 + ik_x x\right] = \lim_{z\to\infty} \exp\left[-\frac{iz}{2k}\left(k_x - \frac{kx}{z}\right)^2 + i\frac{kx^2}{2z}\right]$$



Rapid oscillations integrate to zero

Example



$$E(x) = \begin{cases} E_0, & |x| < L/2 \\ 0, & |x| > L/2 \end{cases}$$

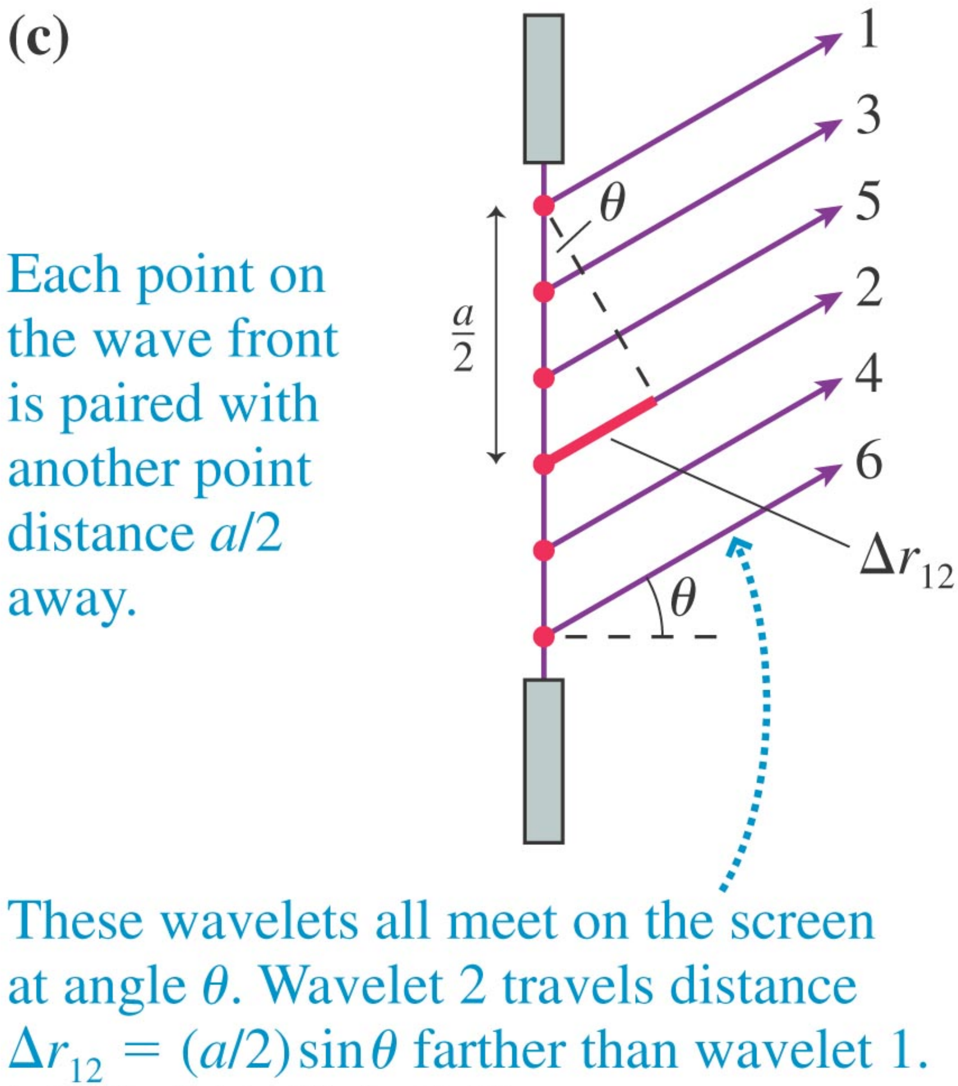
$$\left| \bar{E}(k_x = k\frac{x}{z}) \right|^2$$

Fourier Transform

$$\bar{E}(k_x) = \int_{-L/2}^{L/2} dx E_0 \exp\left[-ik_x x\right] = E_0 \frac{\sin\left(k_x L/2\right)}{k_x/2}$$

zero when $\frac{k_x L}{2} = p\pi \rightarrow \tan\theta = \frac{x}{z} = \frac{2\pi p}{L}$

When is there perfect destructive interference?



Destructive when

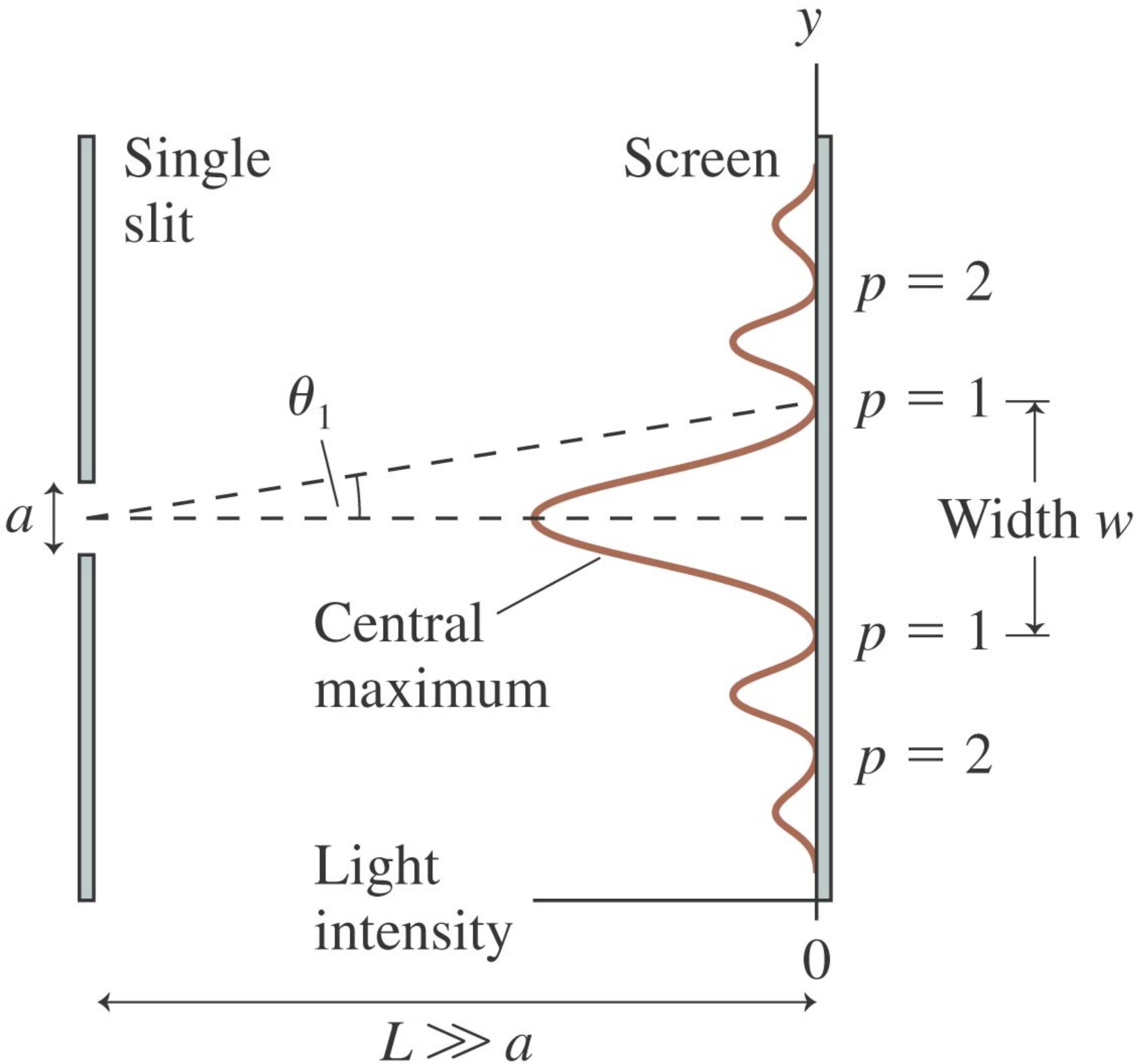
$$\Delta r_{12} = \frac{a}{2}\sin\theta = \frac{\lambda}{2}$$

1 cancels 2
3 cancels 4
5 cancels 6
Etc.

Also:

$$\frac{a}{2p}\sin\theta_p = \frac{\lambda}{2}$$

$$p = 1,2,3.....$$


y
Single slit
Screen
p = 2
θ1
p = 1
a
Width w
Central maximum
p = 1
p = 2
Light intensity
0
L ≫ a