ENEE381

Plane Electromagnetic Waves

In One Dimension

Impedance, Sinusoidal Waves, Phasors

Where We Stand

$$\nabla \cdot \vec{\mathbf{D}} = \rho_f$$

$$\nabla \cdot \vec{\mathbf{B}} = 0$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_f + \frac{\partial \vec{\mathbf{D}}}{\partial t}$$

Assume the following:

Linear, isotropic, instantaneous, media

$$\vec{\mathbf{D}} = \varepsilon \vec{\mathbf{E}}$$

$$\vec{\mathbf{B}} = \mu \vec{\mathbf{H}}$$

Propagation in free space, no free charge or current.

$$\rho_f = 0, \quad \vec{\mathbf{J}}_f = 0$$

Write out component by component

$$\varepsilon \nabla \cdot \vec{\mathbf{E}} = 0$$

$$\varepsilon \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) = 0$$

$$\mu \nabla \cdot \vec{\mathbf{H}} = 0$$

$$\mu \left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} \right) = 0$$

$$\nabla \times \vec{\mathbf{E}} = -\mu \frac{\partial \vec{\mathbf{H}}}{\partial t}$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu \frac{\partial H_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu \frac{\partial H_y}{\partial t}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu \frac{\partial H_z}{\partial t}$$

$$\nabla \times \vec{\mathbf{H}} = \varepsilon \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \varepsilon \frac{\partial E_x}{\partial t}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \varepsilon \frac{\partial E_y}{\partial t}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \varepsilon \frac{\partial E_z}{\partial t}$$

“Look” for solutions in the form

$$\vec{\mathbf{E}} = \left(E_x(z,t), E_y = 0, E_z = 0\right) \qquad \vec{\mathbf{H}} = \left(H_x = 0, H_y(z,t), H_z = 0\right)$$

Then verify that all 2 x 3 + 2 = 8 equations can be satisfied

$$\varepsilon \nabla \cdot \vec{\mathbf{E}} = 0 \qquad \mu \nabla \cdot \vec{\mathbf{H}} = 0$$

$$\varepsilon\left(\frac{\partial E_x}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial 0}{\partial z}\right) = 0 \qquad \mu\left(\frac{\partial 0}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial 0}{\partial z}\right) = 0$$

0

0

$$\vec{\mathbf{E}} = \left(E_x(z,t),\, E_y = 0,\, E_z = 0\right) \qquad \vec{\mathbf{H}} = \left(H_x = 0,\, H_y(z,t),\, H_z = 0\right)$$

Insert fields

$$\nabla \times \vec{\mathbf{E}} = -\mu \frac{\partial \vec{\mathbf{H}}}{\partial t}$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu \frac{\partial H_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu \frac{\partial H_y}{\partial t}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu \frac{\partial H_z}{\partial t}$$

$$\nabla \times \vec{\mathbf{E}} = -\mu \frac{\partial \vec{\mathbf{H}}}{\partial t}$$

$$\frac{\partial 0}{\partial y} - \frac{\partial 0}{\partial z} = -\mu \frac{\partial 0}{\partial t}$$

$$\boxed{\frac{\partial E_x}{\partial z} - \frac{\partial 0}{\partial x} = -\mu \frac{\partial H_y}{\partial t}}$$

$$\frac{\partial 0}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu \frac{\partial 0}{\partial t}$$

0

OK!

$$\vec{\mathbf{E}} = \left(E_x(z,t), E_y = 0, E_z = 0\right) \qquad \vec{\mathbf{H}} = \left(H_x = 0, H_y(z,t), H_z = 0\right)$$

Insert fields

$$\boxed{\begin{aligned} &\nabla \times \vec{\mathbf{H}} = \varepsilon \frac{\partial \vec{\mathbf{E}}}{\partial t} \\ &\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \varepsilon \frac{\partial E_x}{\partial t} \\ &\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \varepsilon \frac{\partial E_y}{\partial t} \\ &\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \varepsilon \frac{\partial E_z}{\partial t} \end{aligned}}$$

$$\nabla \times \vec{\mathbf{H}} = \varepsilon \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

$$\boxed{\frac{\partial 0}{\partial y} - \frac{\partial H_y}{\partial z} = \varepsilon \frac{\partial E_x}{\partial t}}$$

OK!

$$\frac{\partial 0}{\partial z} - \frac{\partial 0}{\partial x} = \varepsilon \frac{\partial 0}{\partial t}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial 0}{\partial y} = \varepsilon \frac{\partial 0}{\partial t}$$

0

When the dust settles

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \qquad -\frac{\partial H_y}{\partial z} = \varepsilon \frac{\partial E_x}{\partial t}$$

Combining

$$\frac{\partial}{\partial z}\frac{\partial E_x}{\partial z} = \frac{\partial^2 E_x}{\partial z^2} = -\mu \frac{\partial}{\partial z}\frac{\partial H_y}{\partial t} = -\mu \frac{\partial}{\partial t}\frac{\partial H_y}{\partial z} = \varepsilon\mu \frac{\partial^2}{\partial t^2} E_x$$

The 1D wave equation

$$\boxed{\frac{\partial^2 E_x}{\partial z^2} = \varepsilon\mu \frac{\partial^2}{\partial t^2} E_x = \frac{1}{v^2}\frac{\partial^2}{\partial t^2} E_x \qquad v^2 = \frac{1}{\varepsilon\mu}}$$

Once E_x is found you can go back and integrate to find H_y

General Solution of 1D W.E.

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}E_x \qquad E_x = E_+(z - vt) + E_-(z + vt)$$

E_+ is a pulse moving in the +z direction
E_- is a pulse moving in the –z direction

If you move with speed v such that your position is z =vt + z_0
Then E+ does not change.

Then using
$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}$$

$$H_y = \frac{1}{\eta}\left[E_+(z - vt) - E_-(z + vt)\right] \qquad v = \frac{1}{\sqrt{\varepsilon\mu}} \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}} - \text{medium's impedance}$$

The most important minus sign this semester

Plane Wave

$$E_x = E_+(z - vt) + E_-(z + vt)$$

$$H_y = \frac{1}{\eta}\left[E_+(z - vt) - E_-(z + vt)\right]$$

$$v = \frac{1}{\sqrt{\varepsilon\mu}} \quad \eta = \sqrt{\frac{\mu}{\varepsilon}} - \text{medium's impadance}$$

Units

[E] – volts/meter

[H] – Amps/meter

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} - [\text{Ohms}]$$

Poynting Flux

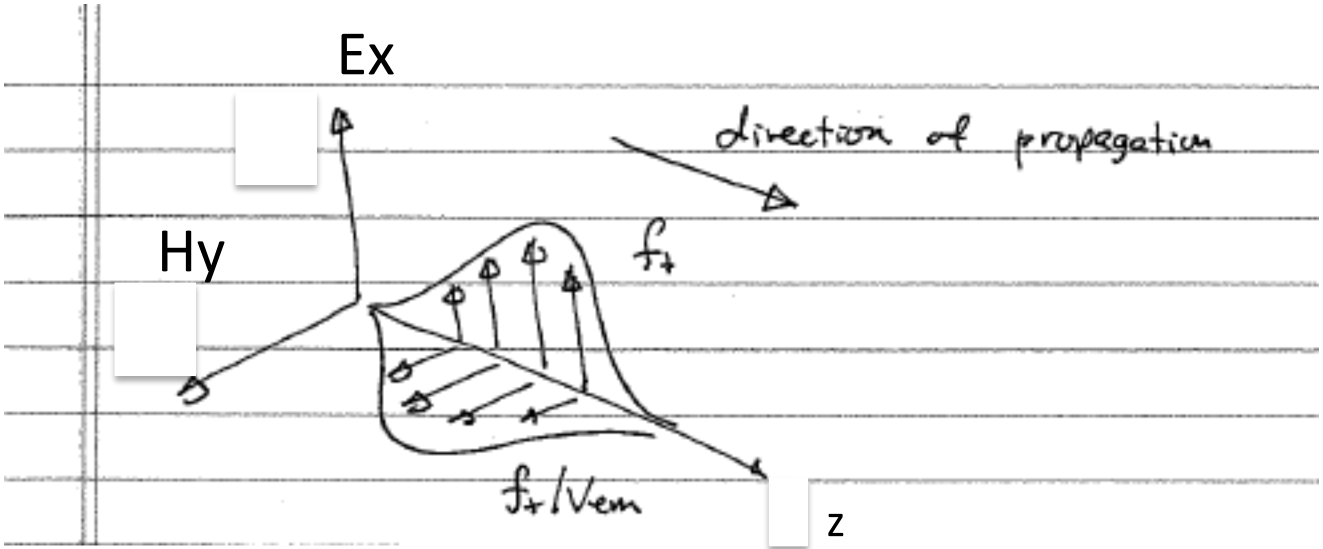
$$S_z = E_x H_y = \frac{1}{\eta}\left[\left(E_+(z - vt)\right)^2 - \left(E_-(z + vt)\right)^2\right]$$

Forward Backward

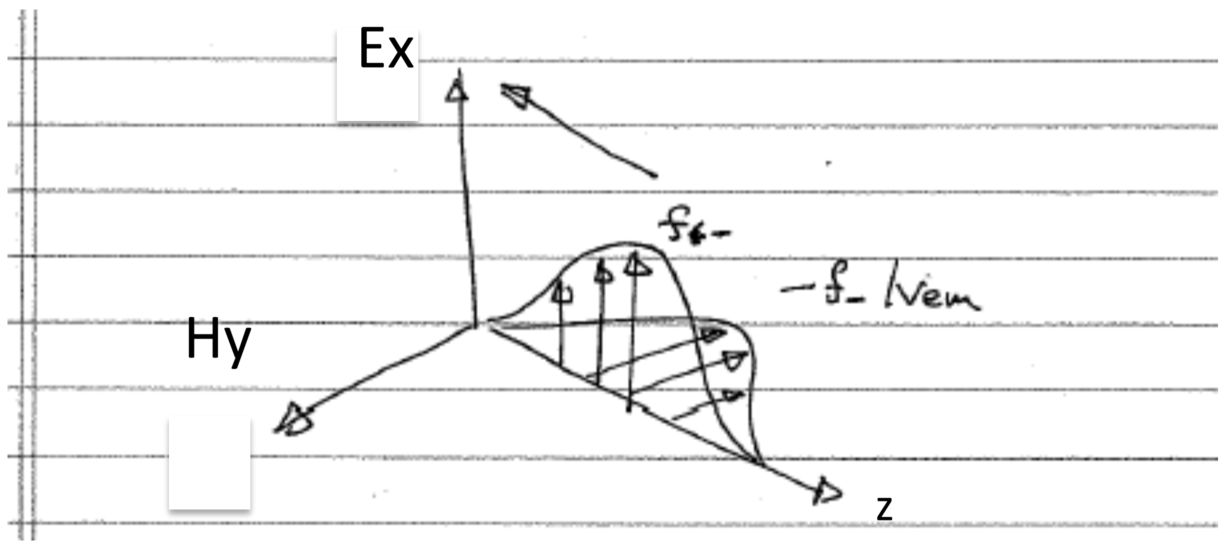
E and B fields in waves and Right Hand Rule:

f+ solution

Wave propagates in **E** x **B** direction



f- solution



Energy Density and Intensity of EM Waves

Energy density associated with electric and magnetic fields

$$u_E = \frac{\varepsilon_0 \left|\vec{\mathbf{E}}\right|^2}{2} \qquad u_B = \frac{\left|\vec{\mathbf{B}}\right|^2}{2\mu_0}$$

For a wave: $$\left|\vec{\mathbf{B}}\right| = \frac{1}{v_{em}}\left|\vec{\mathbf{E}}\right| = \sqrt{\varepsilon_0 \mu_0}\left|\vec{\mathbf{E}}\right|$$

Thus:

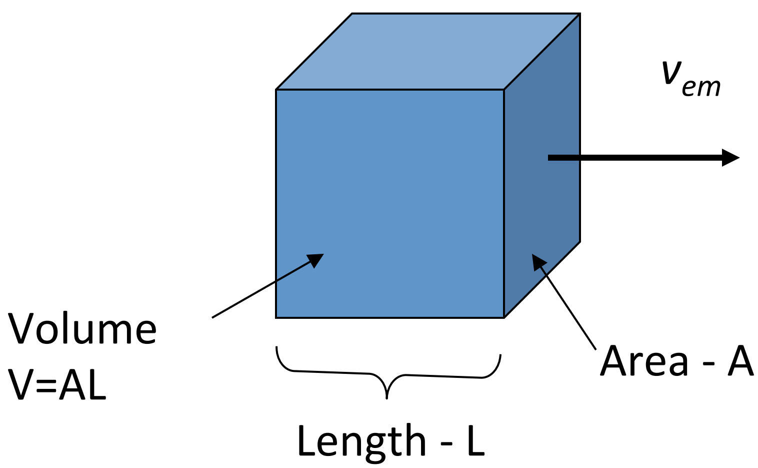
$$u_E = u_B$$ Units: J/m^3

Energy density in electric and magnetic fields are equal for a wave in vacuum.

Wave Intensity - Power/area

Energy density inside cube

$$u_E = \frac{\varepsilon_0 \left|\vec{\mathbf{E}}\right|^2}{2} = u_B = \frac{\left|\vec{\mathbf{B}}\right|^2}{2\mu_0}$$



In time *T=L/v*$_{em}$ an amount of energy

$$U = V(u_E + u_B) = AL\varepsilon_0 \left|\vec{\mathbf{E}}\right|^2$$

comes through the area A.

Intensity

I=Power/Area

$$I = \frac{U}{TA} = \sqrt{\frac{\varepsilon_0}{\mu_0}} \left|\vec{\mathbf{E}}\right|^2$$

Poynting Vector

The power per unit area flowing in a given direction

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}}$$

$$\left|\vec{\mathbf{S}}\right| = I = \sqrt{\frac{\varepsilon_0}{\mu_0}}\left|\vec{\mathbf{E}}\right|^2$$

What are the units of $\sqrt{\frac{\mu_0}{\varepsilon_0}}$

Ans: Ohms

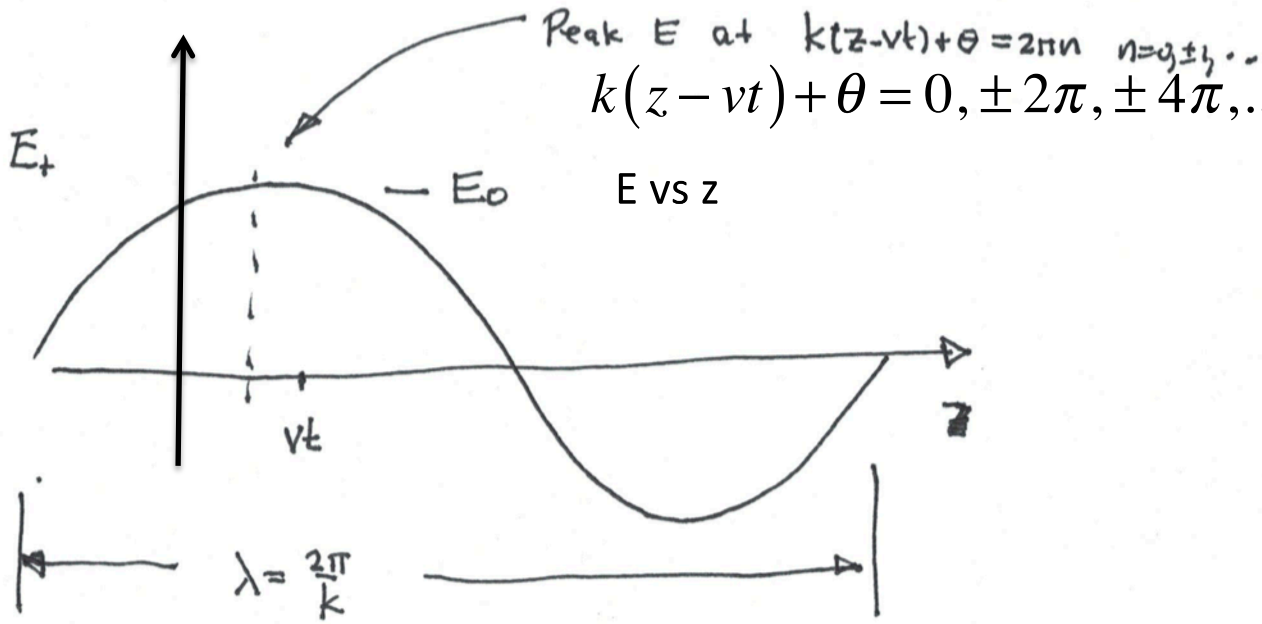
I - W/m^2, E - V/m

$$\sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\Omega$$

Sinusoidal Waves

$$E_x = E_+(z - vt) + E_-(z + vt)$$

$$E_+(z - vt) = E_0 \cos\left[k\left(z - vt\right) + \theta\right]$$



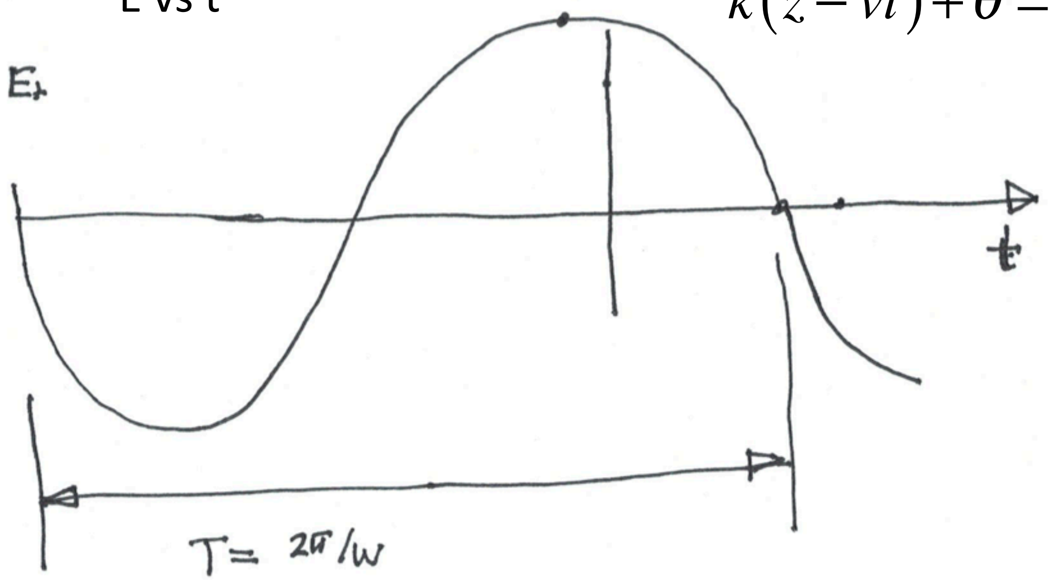
$$k\left(z - vt\right) + \theta = 0, \pm 2\pi, \pm 4\pi, \ldots$$

E vs z

Peak E at $k(z-vt)+\theta = 2\pi n$

E vs t

$$k(z - vt) + \theta = 0, \pm 2\pi, \pm 4\pi, \ldots$$

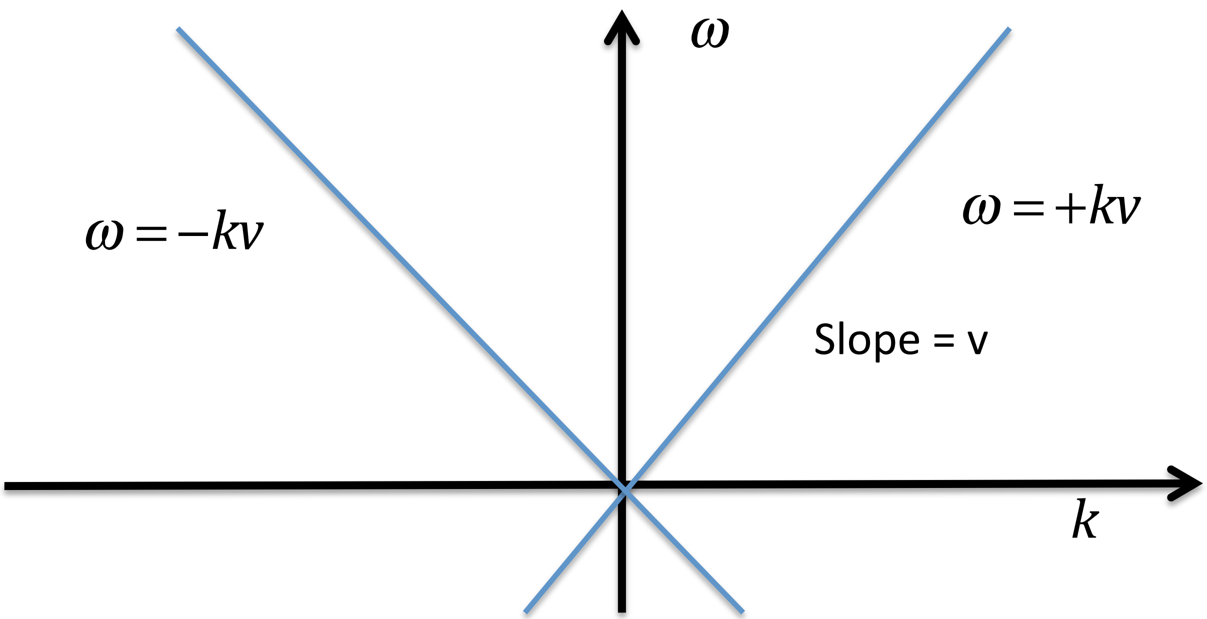


$\omega = kV$

Dispersion relation given k what is ω

Dispersion Relation

Plane waves in a nondispersive medium

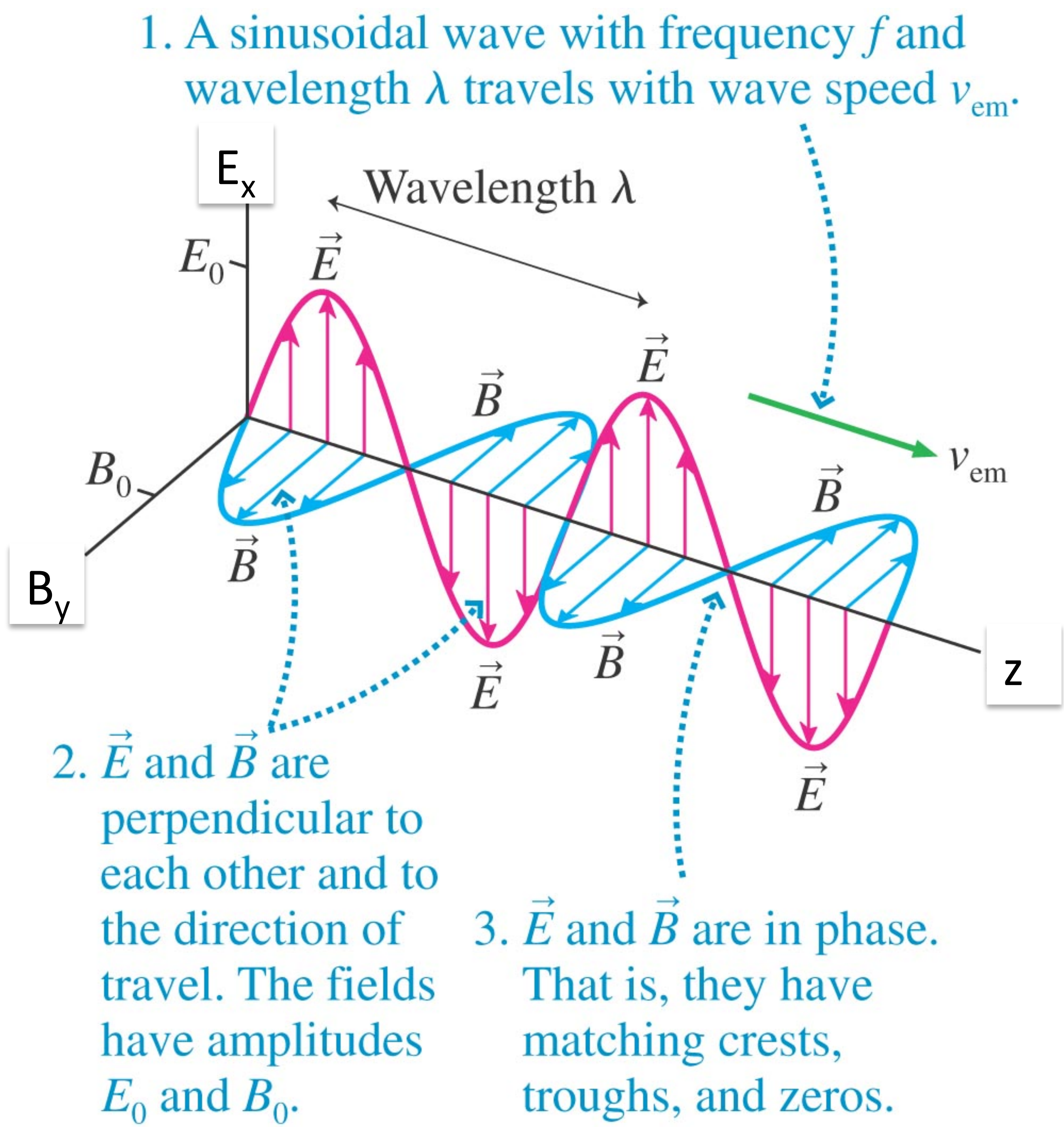


$$\omega = +kv$$
$$\lambda = v / f$$

Slope of omega vs k is independent of k or omega

$$\lambda = 2\pi / k$$
$$\omega = 2\pi f$$

Pulses maintain their shape while propagating


1. A sinusoidal wave with frequency f and wavelength λ travels with wave speed v_{em}.
E_x
Wavelength λ
E_0
$\vec{E}$
$\vec{B}$
B_0
B_y
v_{em}
z
2. $\vec{E}$ and $\vec{B}$ are perpendicular to each other and to the direction of travel. The fields have amplitudes E_0 and B_0.
3. $\vec{E}$ and $\vec{B}$ are in phase. That is, they have matching crests, troughs, and zeros.

Complex Phasors

Sinusoidal Wave Parameters:

Properties of the medium

$$v = \frac{1}{\sqrt{\varepsilon\mu}}, \quad \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

Determined by the source

ω or k

usually ω

But $\omega = kv$

Local amplitude and phase

E_0

$H_0 = E_0 / \eta$

θ

Three Equivalent representations

$$E_x = E_0 \cos\left[kz - \omega t + \theta\right] \quad \text{or} \quad k$$

Physicists

$$E_x = \mathrm{Re}\left[E_0 \exp\left[i\left(kz - \omega t + \theta\right)\right]\right]$$

Engineers

$$E_x = \mathrm{Re}\left[E_0 \exp\left[j\left(\omega t - kz - \theta\right)\right]\right]$$

$$j = -i = \sqrt{-1}$$

Griffiths uses the Physics convention

$$E_x = \mathrm{Re}\left[E_0 \exp\left[i\left(kz - \omega t + \theta\right)\right]\right]$$
$$= \mathrm{Re}\left[\left(E_0 e^{i\theta}\right)\exp\left[i\left(kz - \omega t\right)\right]\right]$$

$$\hat{E}_x = \left(E_0 e^{i\theta}\right) \quad \text{complex phasor amplitude}$$

Quick Review of Phasors

Phasors - a way of representing amplitude and phase

Imaginary number $$i = \sqrt{-1}$$

Engineers use *j*
Physicists and mathematicians use *i*

Complex number $$Z = X + iY$$

X is the real part
Y is the imaginary part

Complex numbers follow the same rules of algebra as regular numbers

$$Z_1 = X_1 + iY_1 \qquad Z_2 = X_2 + iY_2$$

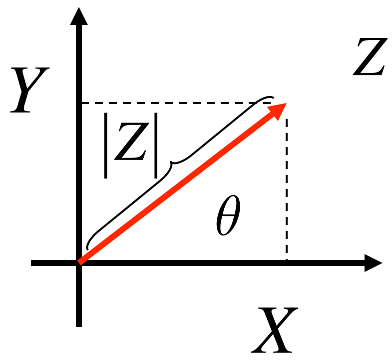
Addition: $$Z_1 + Z_2 = (X_1 + X_2) + i(Y_1 + Y_2)$$

-1

Multiplication:

$$Z_1 Z_2 = (X_1 + iY_1)(X_2 + iY_2) = X_1 X_2 + i^2 Y_1 Y_2 + i(X_1 Y_2 + X_2 Y_1)$$

A complex number is specified by two real numbers



$$Z = X + iY$$

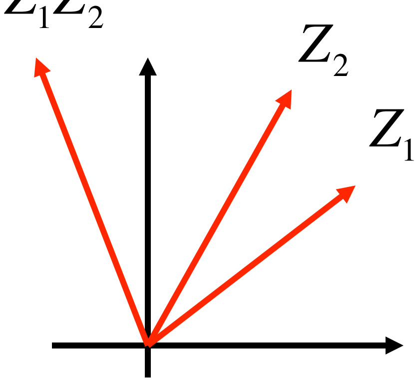
Instead of real and imaginary parts can give magnitude and phase

$$|Z| = \sqrt{X^2 + Y^2}$$

$$\tan\theta = Y / X$$

Multiplying complex numbers - part 2
Magnitudes multiply
Phases add

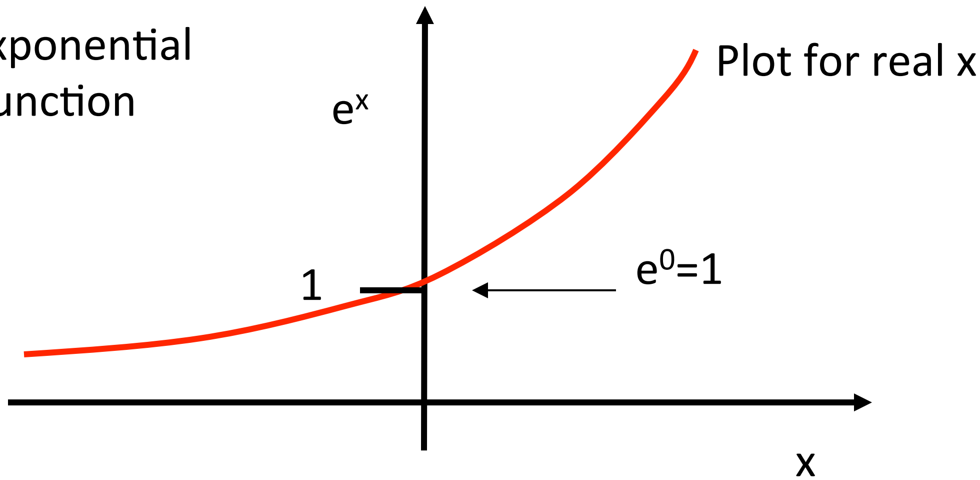
$$Z_3 = Z_1 Z_2$$



$$|Z_3| = |Z_1||Z_2|$$

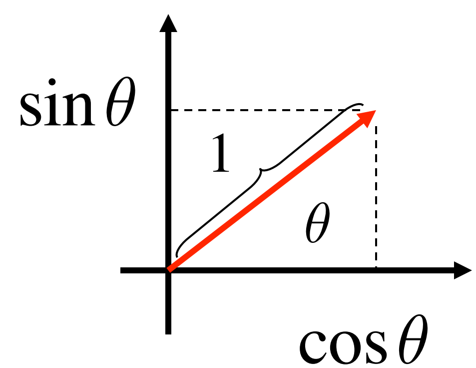
$$\theta_3 = \theta_1 + \theta_2$$

Exponential function



But, what if x is imaginary?

$$e^{i\theta} = \cos\theta + i\sin\theta$$



Let $$X = \cos\theta, \quad Y = \sin\theta$$

$$Z = \cos\theta + i\sin\theta$$

Then you can show:

$$\frac{dZ}{d\theta} = iZ$$

So: $$Z(\theta) = Z(0)e^{i\theta} = e^{i\theta}$$

$$Z(0) = 1 + i0 = 1$$

Phasors

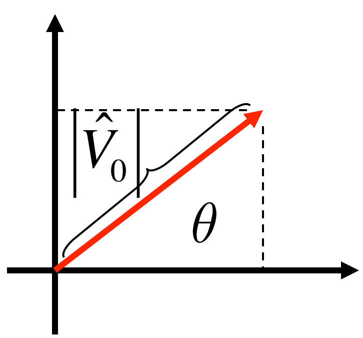
Suppose I have an oscillating voltage

$$V(t) = V_0 \cos[\omega t - \theta]$$
$$= V_0 \cos[\theta - \omega t]$$

I can write this as the real part of a complex number.

$$V(t) = \mathrm{Re}\left[\left(V_0 e^{i\theta}\right) e^{-i\omega t}\right]$$

Call this $\hat{V}_0$ a complex amplitude or "phasor"



In this class, hat means a complex #

$$\hat{V}_0 = V_0 e^{i\theta}$$

$$\left|\hat{V}_0\right| = V_0$$

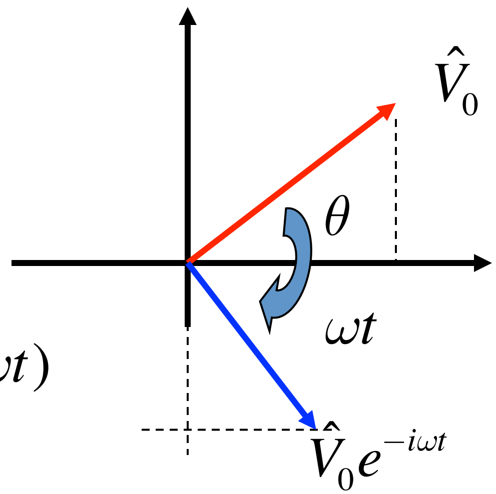
Magnitude of phasor gives peak amplitude of signal.
Angle give phase of signal.

$$V(t) = \mathrm{Re}\left[\hat{V}_0 e^{-i\omega t}\right]$$

Multiplying $\hat{V}_0$ by $e^{-i\omega t}$
rotates the angle of the product by
$-\omega t$

Remember: $|Z_3| = |Z_1||Z_2|$

$$\theta_3 = \theta_1 + \theta_2$$



$$V_0 \cos(\omega t - \theta)$$
$$= V_0 \cos(\theta - \omega t)$$

$$\frac{d}{dt} I(t) = \frac{d}{dt}\mathrm{Re}\left[\hat{I}_0 e^{-i\omega t}\right] = \mathrm{Re}\left[\frac{d}{dt}\hat{I}_0 e^{-i\omega t}\right] = \mathrm{Re}\left[-i\omega \hat{I}_0 e^{-i\omega t}\right]$$

Now suppose

$$V(t) = L\frac{d}{dt} I(t)$$

$$\hat{V}_0 = -i\omega L \hat{I}_0$$

$$\mathrm{Re}\left[\hat{V}_0 e^{-i\omega t}\right] = \mathrm{Re}\left[-i\omega L \hat{I}_0 e^{-i\omega t}\right]$$

Real parts equal

$$\hat{V}_0 e^{-i\omega t} = -i\omega L \hat{I}_0 e^{-i\omega t}$$

$$\frac{d}{dt} \Rightarrow -i\omega$$