Adjoint Methods in Charged Particle Dynamics

or

When the solution to your problem
is not your problem.

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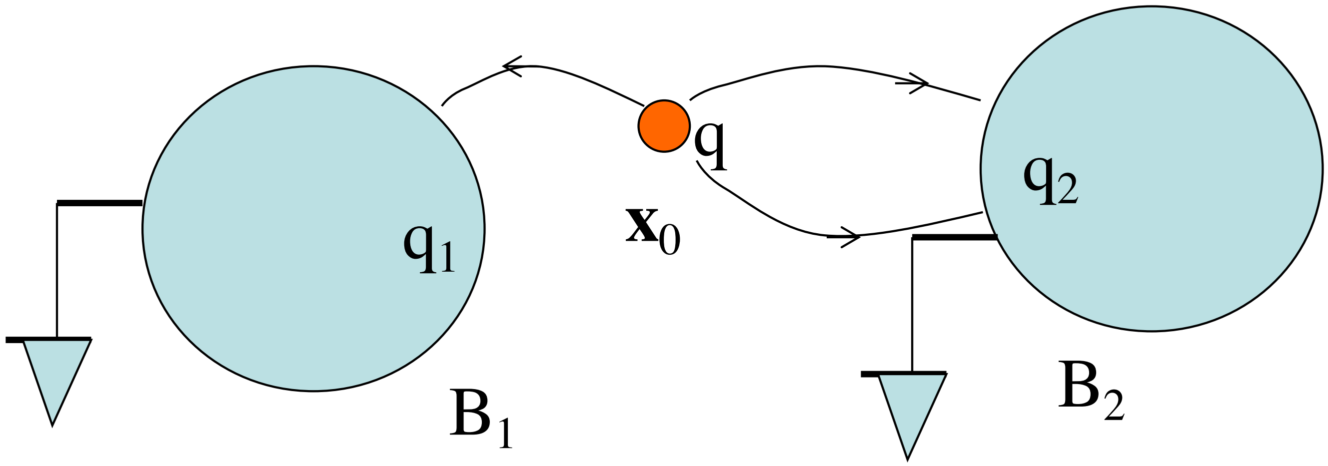
Example
Jackson, Classical Electrodynamics
Problems 1.12 and 1.13

A charge q is placed at an arbitrary point, $\mathbf{x}_o$, relative to two grounded, conducting electrodes.



What is the charge q_1 on the surface of electrode 1?

<u>Repeat for different $\mathbf{x_0}$</u>

Solution – Green's Reciprocation Theorem

<u>Prob #1</u>
Your Problem

$$\nabla^2\phi = -q\delta(\mathbf{x}-\mathbf{x}_0) \qquad \text{BC:} \quad \phi\big|_{B1} = \phi\big|_{B2} = \phi(x\to\infty) = 0$$

$$q_1 = \int_{B1} d^2x\, \mathbf{n}\cdot\nabla\phi$$

<u>Prob #2</u>
Adjoint (Not your) Problem

$$\nabla^2\psi = 0 \qquad \text{BC:} \quad \psi\big|_{B1} = 1, \quad \psi\big|_{B2} = \psi(x\to\infty) = 0$$

Green's Theorem

$$\int_V d^3x\left(\psi\nabla^2\phi - \phi\nabla^2\psi\right) = \int_S d^2x\, n\cdot\left(\psi\nabla\phi - \phi\nabla\psi\right)$$

0 $\qquad$ 0

When the dust settles:

$$-q\psi(\mathbf{x}_0) = q_1$$

George Green 1793-1841

The Green of Green Functions

In 1828, an English miller from Nottingham published a mathematical essay that generated little response. George Green's analysis, however, has since found applications in areas ranging from classical electrostatics to modern quantum field theory.

Lawrie Challis and Fred Sheard Physics Today Dec. 2003

his family built a house next to the mill, Green spent most of his days and many of his nights working and indeed living in the mill. When he was 31, Jane Smith bore him a daughter. They had seven children in all but never married. It was said that Green's father felt that Jane was not a suitable wife for the son of a prosperous tradesman and landowner and threatened to disinherit him.

- Born in Nottingham (Home of Robin Hood)
- Father was a baker
- At age 8 enrolled in Robert Goodacre's Academy
- Left after 18 months (extent of formal education pre age 40)
- Worked in bakery for 5 years
- Sent by father to town mill to learn to be a miller

- Fell in love with Jane, the miller's daughter.
- Green's father forbade marriage.
- Had 7 children with Jane.
- Self published work in 1828
- With help, entered Cambridge 1833, graduated 1837.
- Theory of Elasticity, refraction, evanescence
- "Discovered" by Lord Kelvin in 1840.
- Died of influenza, 1841

Green's Mill: still functions

Features of Problems Suited to an Adjoint Approach

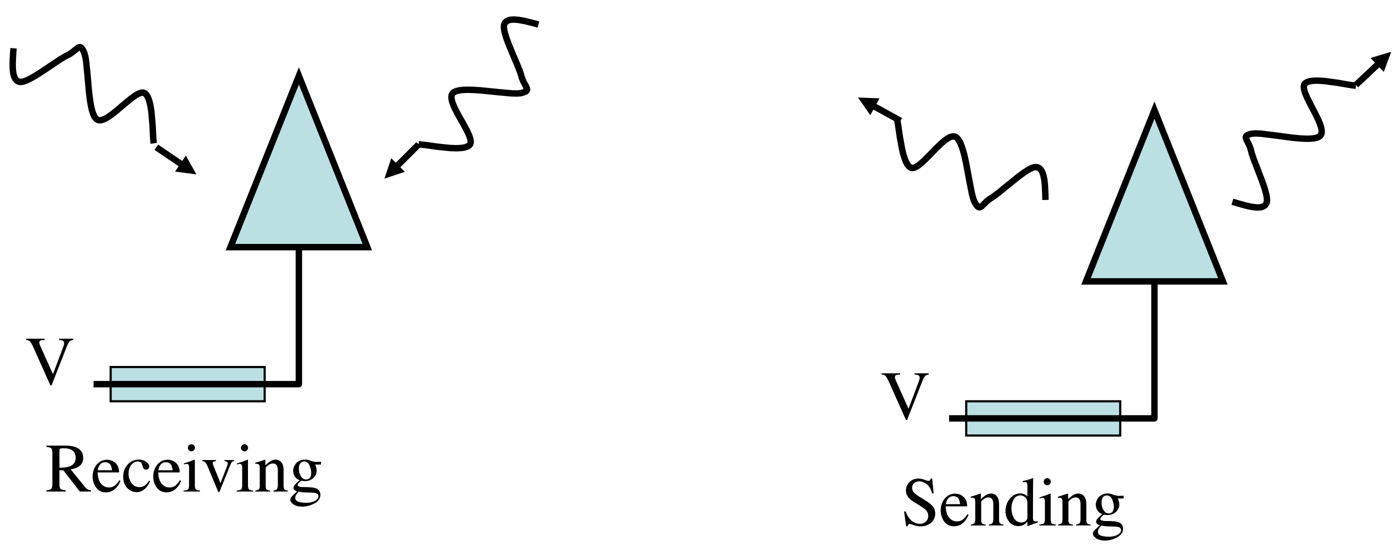
1. Many computations need to be repeated.
   (many different locations of charge, q)

2. Only a limited amount of information about the solution is required.
   (only want to know charge on electrode #1)

Relation to Reciprocity

Example:

- Antenna sending and receiving radiation patterns are equal due to time reversal symmetry of ME.
- Direct calculation of receiving pattern requires many simulations
- Instead, calculate sending pattern and invoke reciprocity



Other Examples of Reciprocity

Electrostatics	Symmetry of the Capacitance Matrix
Electromagnetics	Symmetry of the Inductance Matrix Symmetry of Scattering Matrix

Collisional Transport: Onsager Symmetry of off-diagonal elements of transport matrices.

Temperature gradient	→	Electric current
Electric field	→	Heat flux

<u>Neoclassical Tokamak Transport</u>

Pressure gradient	→	Bootstrap Current
Toroidal E-field	→	Ware particle flux

Adjoint Methods in Engineering

E3232 *Journal of The Electrochemical Society*, **164** (11) E3232-E3242 (2017)

JES Focus Issue on Mathematical Modeling of Electrochemical Systems at Multiple Scales in Honor of John Newman

Adjoint Method for the Optimization of the Catalyst Distribution in Proton Exchange Membrane Fuel Cells

James Lamb,[a,b] Grayson Mixon,[a,b] and Petru Andrei[a,b,*,z]

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[b]*Aero-Propulsion, Mechatronics and Energy Center, Florida State University, Tallahassee, Florida 32310, USA*

Adjoint method for the optimization of insulated gate bipolar transistors

Cite as: AIP Advances 9, 095301 (2019); doi: 10.1063/1.5113764
Submitted: 7 June 2019 • Accepted: 23 August 2019 •
Published Online: 3 September 2019

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Adjoint shape optimization applied to electromagnetic design

Christopher M. Lalau-Keraly,[1,*] Samarth Bhargava,[1] Owen D. Miller,[2] and Eli Yablonovitch[1]

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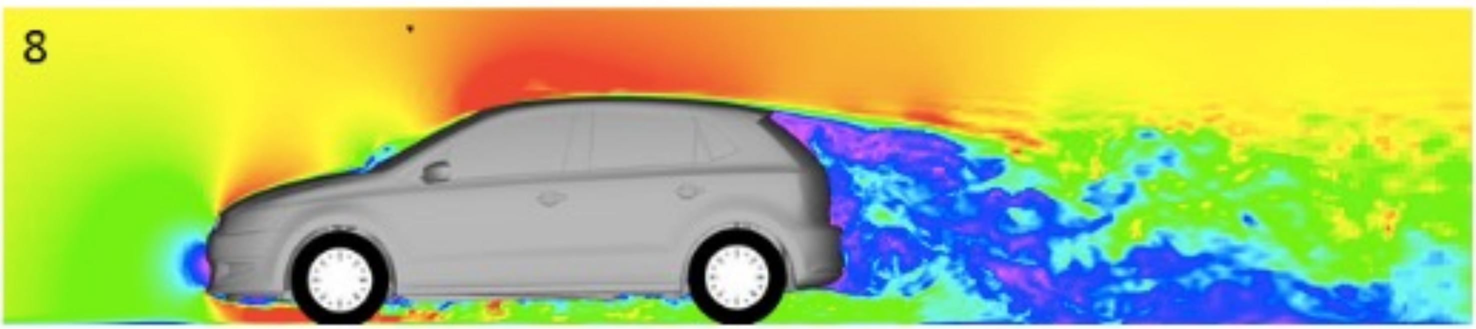
RESEARCH | OPEN ACCESS

Courtesy, Elizabeth Paul

Adjoint methods for car aerodynamics

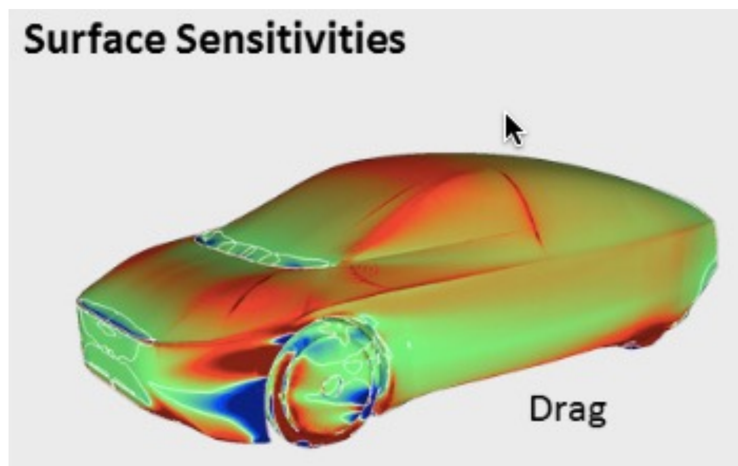
Carsten Othmer

Journal of Mathematics in Industry 2014 4:6 | DOI: 10.1186/2190-5983-4-6 | © Othmer; licensee Springer. 2014
Received: 30 March 2013 | **Accepted:** 5 March 2014 | **Published:** 3 June 2014



Optimize shape to minimize drag.

Super Computer



Result is also aesthetically appealing.

1985 Volvo 240 DL

Oops, coding error.

Adjoint Approach in Plasma and Beam Physics

- Neoclassical Transport, F. Hinton, and R. Hazeltine, Rev. Mod Phys, 48 (2) , 1976
- Calculation of beam driven currents in magnetized plasmas, S. Hirshman, PoF, 23, 1238 (1980).
- Calculation of RF current drive in magnetic confinement plasma configurations, TMA and K. Chu PoF 25, (1982)
- Calculation of RF induced transport in magnetic confinement plasmas, TMA and K. Yoshioka, PoF 29, (1986), Nucl. Fusion, 26 (1986).
- Shot noise on gyrotron beams, TMA, W. Manheimer and A. Fliflet, PoP (2001).

RF Current Drive in Fusion Plasmas

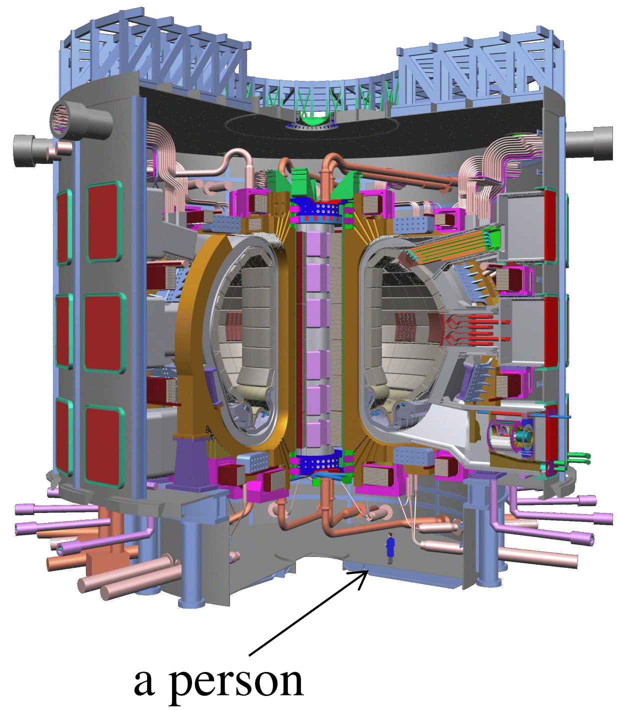
Magnetic Confinement: ITER

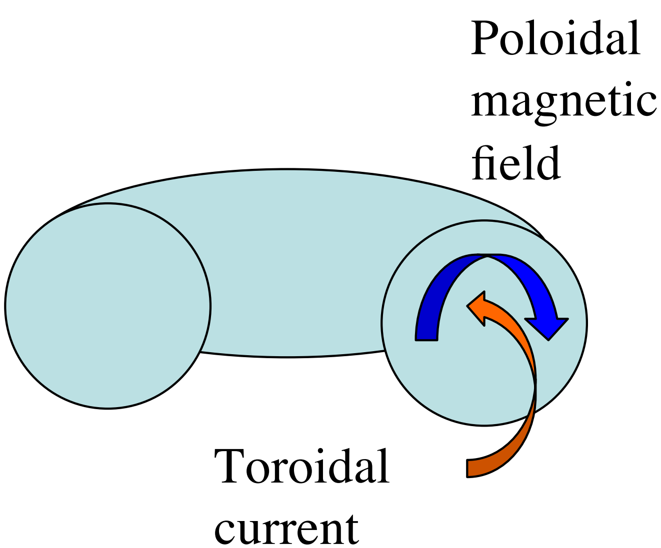
US-EU-Russia-Japan-India Collaboration

Will be built in Cardarache France

Completion 2016??

http://www.iter.org/





Injecting RF waves can drive a toroidal current. N. Fisch PRL (1978)

RF Current Drive Efficiency

Original Langevin Treatment: Nat Fisch, PRL (1978)

RF pushes particles in velocity space.

$\vec{\Gamma}$ = RF induced velocity space particle flux

Collisions relax distribution back to equilibrium.

What is the current generated per unit power dissipated? J/P_D

$$J = \int d^3v\, \vec{\Gamma} \cdot \frac{\partial}{\partial v}\Psi \qquad P_D = \int d^3v\, \vec{\Gamma} \cdot \frac{\partial}{\partial v}\varepsilon$$

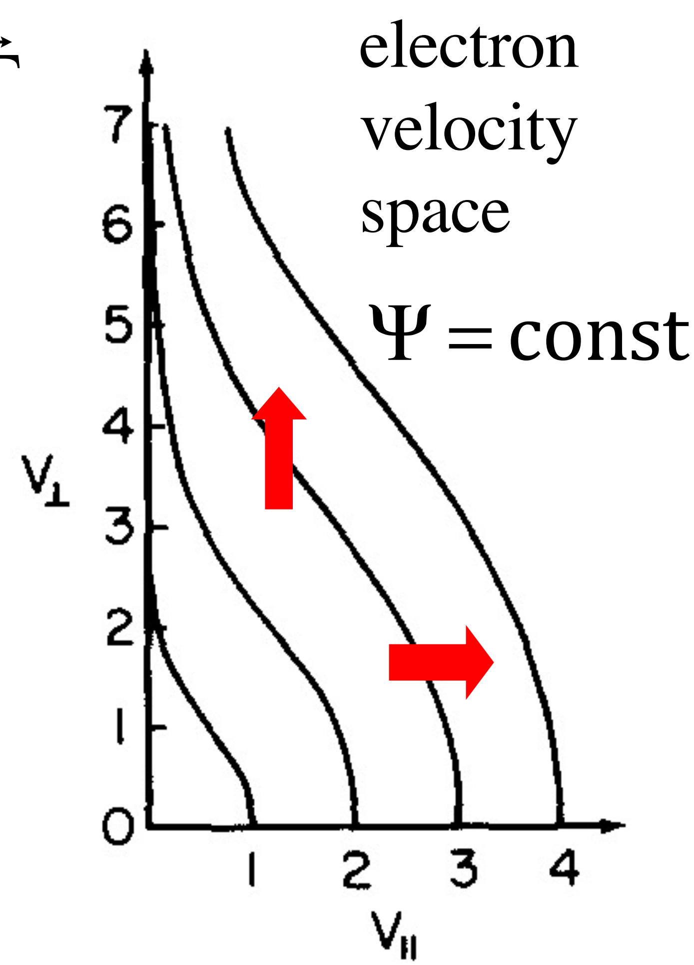
ψ inversely proportional to collision rate

RF Current Drive Efficiency

RF pushes particles in velocity space. $\vec{\Gamma}$

Collisions relax distribution back to equilibrium.

$$J = \int d^3v \, \vec{\Gamma} \cdot \frac{\partial}{\partial v} \Psi$$



N. Fisch: Current generated in parallel direction even if push is in perpendicular direction

Adjoint Approach: S. Hirshman, PoF, 23, 1238 (1980), TMA and KR Chu, PoF 25, (1982)

For a Homogeneous Plasma, we want to solve steady state kinetic equation

$$\frac{\partial f}{\partial t} = 0 = C(f) - \frac{\partial}{\partial \mathbf{v}} \cdot \Gamma$$

Problem #1

Linearized collision operator

RF induced velocity space flux

Then find parallel current

$$J_{\parallel} = -e\int d^3v \, v_{\parallel} f$$

Problem #2

Adjoint problem: Spitzer-Harm Distribution function driven by a DC electric field.

$$-ev_{\parallel} f_M = C(g)$$

Parallel current

$$J_{\parallel} = \int d^3v \, \Gamma \cdot \frac{\partial}{\partial \mathbf{v}} \left(\frac{g}{f_M} \right)$$

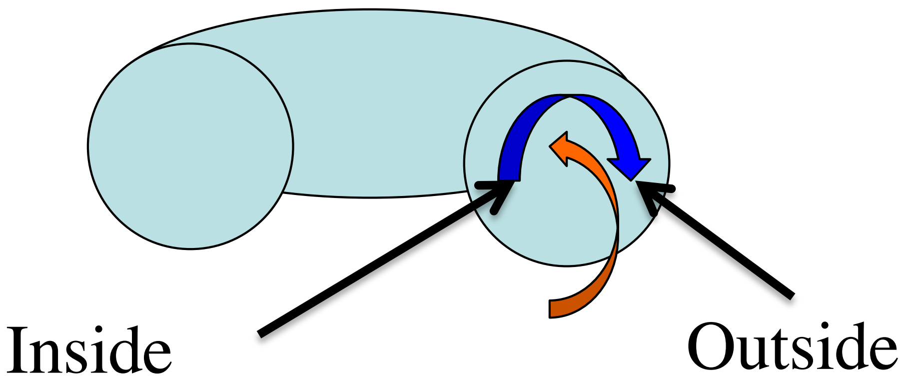
Toroidal Geometry Makes a Difference,

TMA and KR Chu, PoF 25, (1982)

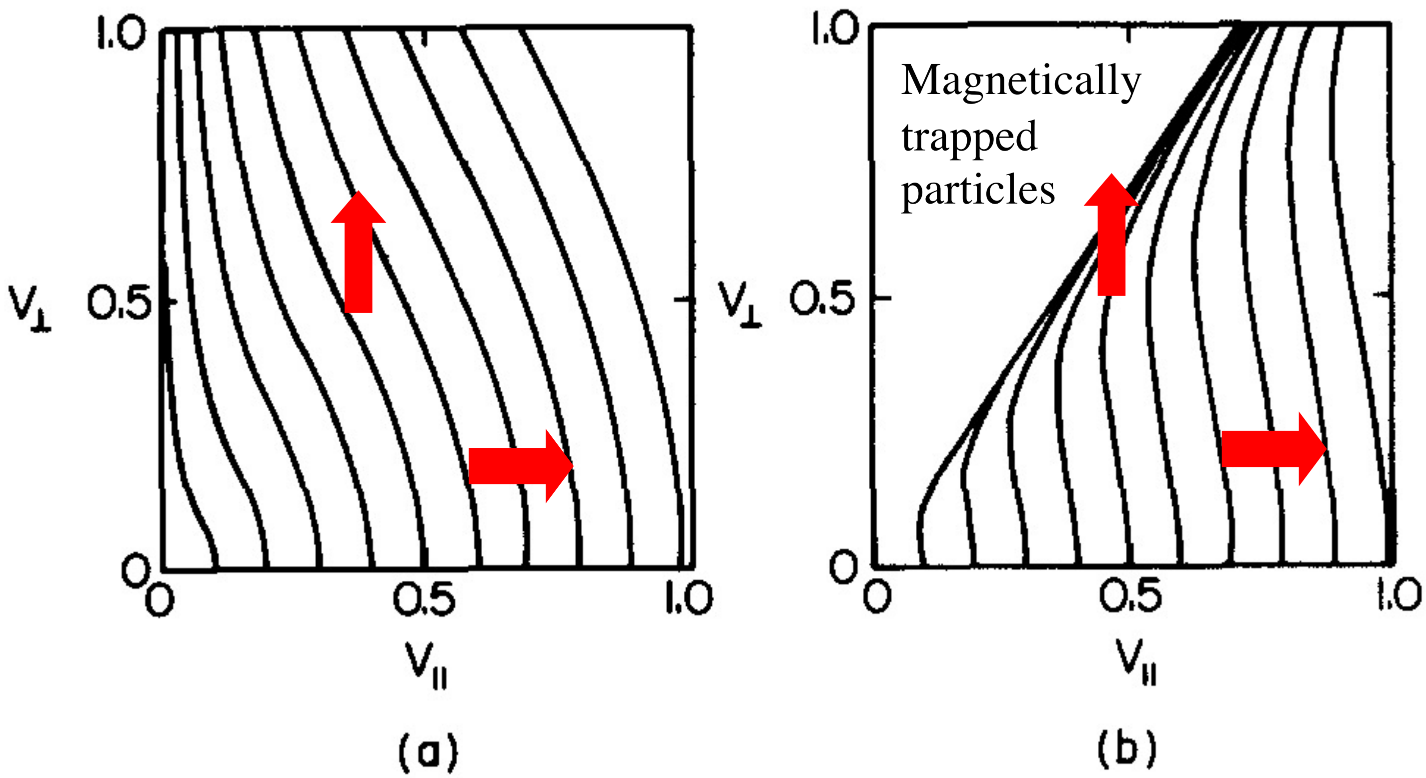
streaming

$$v_{\|}\mathbf{b}\cdot\nabla g - ev_{\|}f_M = C(g)$$

$$J = \int d^3v\, \Gamma \cdot \frac{\partial}{\partial v} \frac{g}{f_M}$$



$\Psi = \text{const.}$



Recent Adjoint Approaches

- Beam optics sensitivity function, TMA, D. Chernin, J. Petillo, Phys. Plasmas 26, 013109 (2019); doi: 10.1063/1.5079629
- Stellarator Optimization and Sensitivity, E. Paul, M. Landreman, TMA, *J. Plasma Phys*. (2019), v*ol*. 85, 905850207, *J. Plasma Phys*. (2021), v*ol*. 87, 905870214
- Optimization of Flat to Round Transformers in Particle Accelerators, L. Dovlatyan, B. Beaudoin, S. Bernal, I. Haber, D. Sutter and TMA, Phys Rev Accel and Beams V25, 044002 (2022).
- Adjoint Equations for Beam-Wave Interaction and Optimization of TWT Design, A. Vlasov, TMA, D. Chernin and I. Chernyavskiy, IEEE Trans. Plasma Sci. V. March (2022).

Give a child a hammer and everything becomes a nail.

Global Beam Sensitivity Function for Electron Guns

Goal

Derive and Calculate a function that gives the variation of specific beam parameters to

- variations in electrode potential/position
- variations in magnet current/position

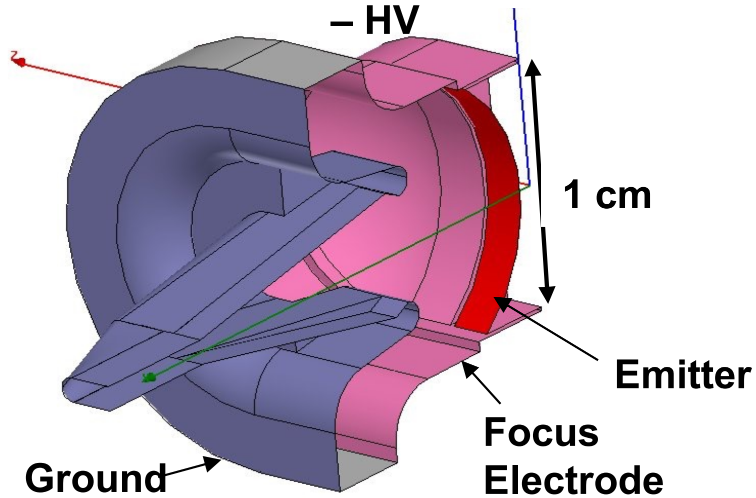
Can be used to

- establish manufacturing tolerances
- optimize gun designs

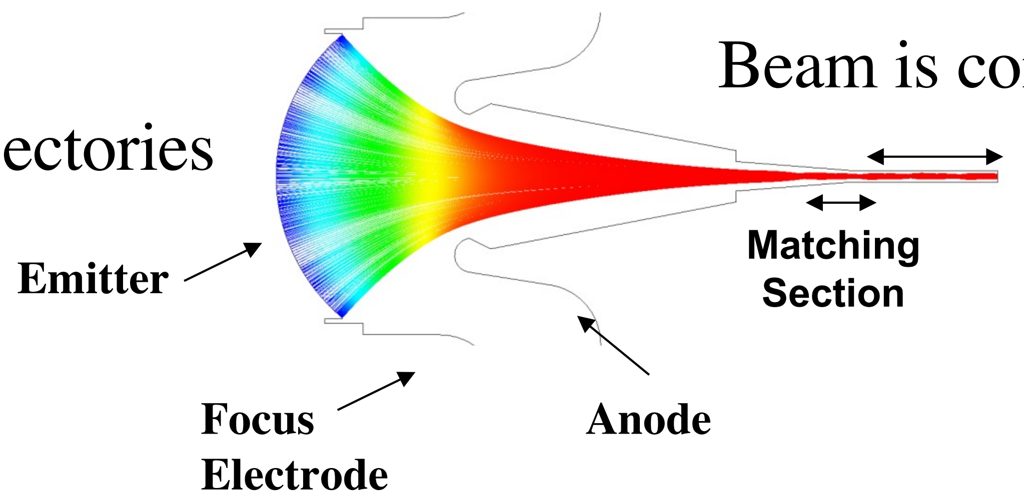
Should be embedded in gun code (e.g. Michelle)

Thermionic Cathode Electron Gun

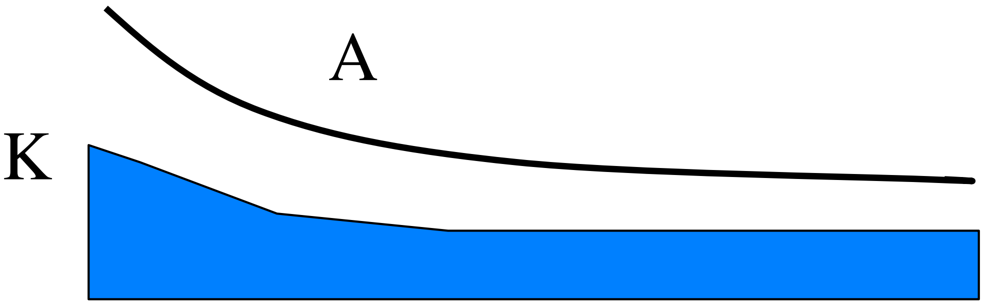
Solid Model of Electrodes



Cut away view of trajectories



What shape to make electrodes?



Michelle: Petillo, J; Eppley, K; Panagos, D; et al., IEEE TPS 30, 1238-1264 (2002).

Code (Michelle) solves the following equations:

Equations of motion for N particles j=1,N

Start with vacuum fields

$$\frac{d\boldsymbol{x}_j}{dt} = \frac{\partial H}{\partial \boldsymbol{p}} \qquad \frac{d\boldsymbol{p}_j}{dt} = -\frac{\partial H}{\partial \boldsymbol{x}}$$

Accumulates a charge density

$$\rho(\boldsymbol{x}) = \sum_j I_j \int_0^{T_j} dt\, \delta(\boldsymbol{x} - \boldsymbol{x}_j(t))$$

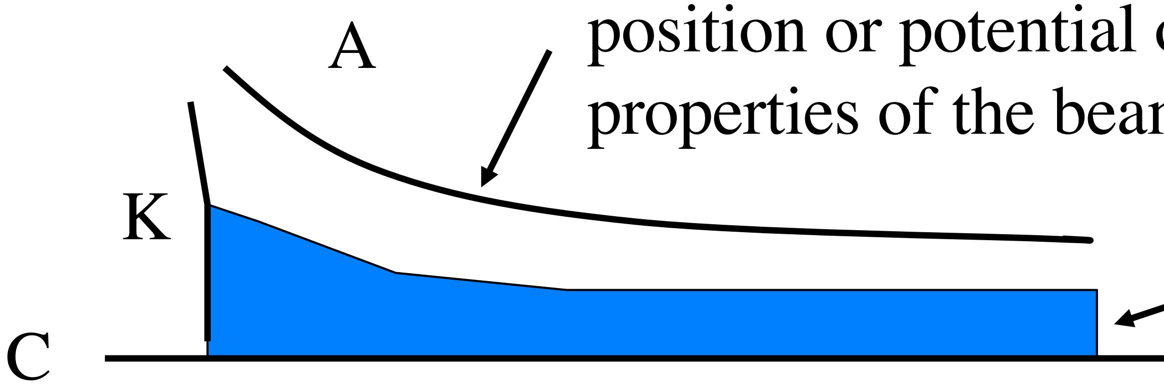
Solves Poisson Equation

$$-\nabla^2 \Phi = \rho / \varepsilon_0$$

Iterates until converged

Sensitivity Function

<u>Basic question:</u> How do small changes in position or potential of anode affect the properties of the beam leaving the gun?



Beam leaves here

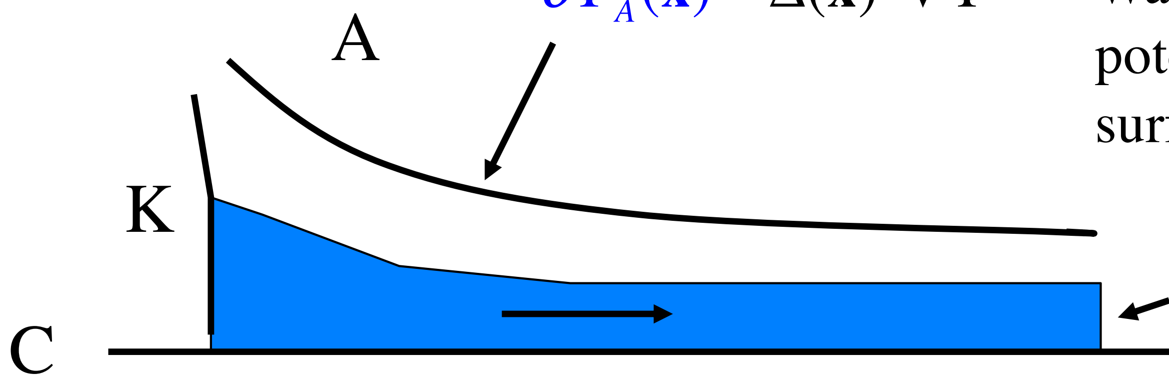
<u>Conventional approach</u>: trial and error. Do many simulations with different anode potentials or positions select the best based on some metric measured at the exit.

It will be shown …

Problem #1

$\delta\Phi_A(\mathbf{x}) = \Delta(\mathbf{x}) \cdot \nabla\Phi$

Wall displacement changes potential on unperturbed surface.



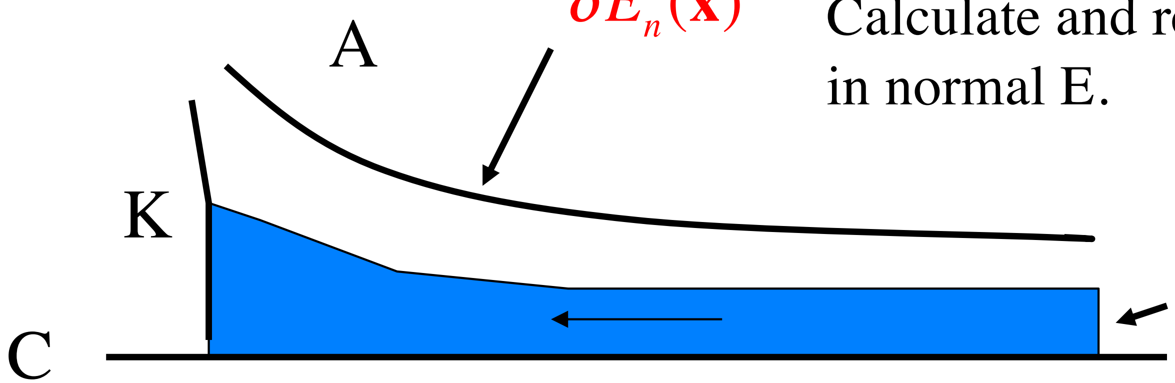
Leads to change in RMS beam radius ΔR_{RMS}

Problem #2

x

$\delta E_n(\mathbf{x})$

Calculate and record change in normal E.



Reverse and perturb electron coordinates

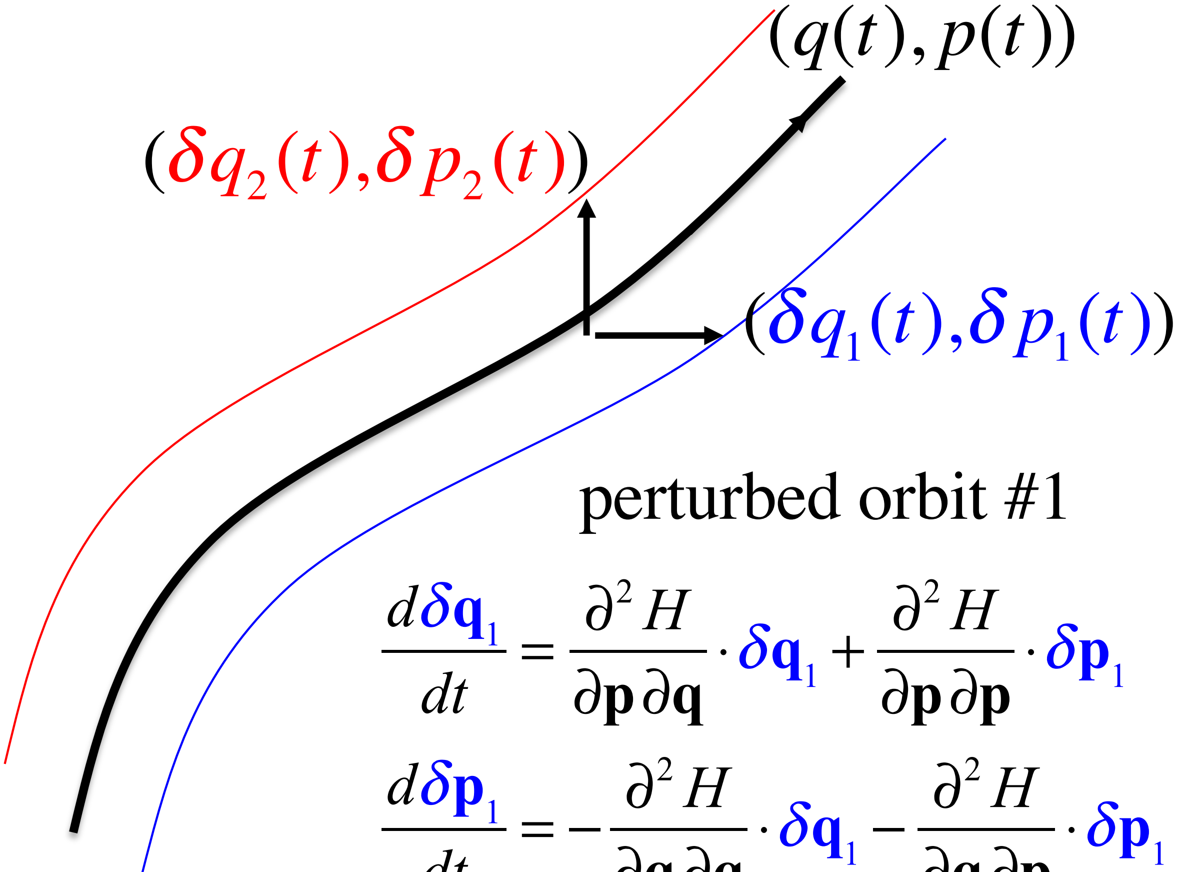
Electrons run backwards

Sensitivity function

δE_n Is the sensitivity function

$$\Delta R_{RMS} \propto \int_S da\, \delta\Phi_A(\mathbf{x}) \delta E_n(\mathbf{x})$$

Hamilton's Equations $H(\boldsymbol{p},\boldsymbol{q},t)$ Conserve Symplectic Area



$$\frac{d\mathbf{q}}{dt}=\frac{\partial H}{\partial \mathbf{p}}$$
$$\frac{d\mathbf{p}}{dt}=-\frac{\partial H}{\partial \mathbf{q}}$$

perturbed orbit #1

$$\frac{d\delta\mathbf{q}_1}{dt}=\frac{\partial^2 H}{\partial\mathbf{p}\,\partial\mathbf{q}}\cdot\delta\mathbf{q}_1+\frac{\partial^2 H}{\partial\mathbf{p}\,\partial\mathbf{p}}\cdot\delta\mathbf{p}_1$$
$$\frac{d\delta\mathbf{p}_1}{dt}=-\frac{\partial^2 H}{\partial\mathbf{q}\,\partial\mathbf{q}}\cdot\delta\mathbf{q}_1-\frac{\partial^2 H}{\partial\mathbf{q}\,\partial\mathbf{p}}\cdot\delta\mathbf{p}_1$$

perturbed orbit #2

$$\frac{d\delta\mathbf{q}_2}{dt}=\ldots$$
$$\frac{d\delta\mathbf{p}_2}{dt}=-\ldots$$

$$\frac{d}{dt}\left(\delta\mathbf{p}_1\cdot\delta\mathbf{q}_2-\delta\mathbf{p}_2\cdot\delta\mathbf{q}_1\right)=0$$

Area conserved for any choice of 1 and 2

Reference Solution + Two Linearized Solutions

$$\left(\mathbf{x}_j,\mathbf{p}_j\right)\rightarrow\left(\mathbf{x}_j,\mathbf{p}_j\right)+\left(\delta\mathbf{x}_j,\delta\mathbf{p}_j\right)$$

$$\rho(\mathbf{x})\rightarrow\rho(\mathbf{x})+\delta\rho(\mathbf{x})$$

$$\Phi(\mathbf{x})\rightarrow\Phi(\mathbf{x})+\delta\Phi(\mathbf{x})$$

Reference Solution Perturbation

Two Linearized Solutions

$[\delta x_j(t),\delta p_j(t)]$ true

$[\delta\hat{x}_j(t),\delta\hat{p}_j(t)]$ adjoint

subject to different BC's

Can show

$$\sum_j I_j\left(\delta\hat{\mathbf{p}}_j\cdot\delta\mathbf{x}_j-\delta\mathbf{p}_j\cdot\delta\hat{\mathbf{x}}_j\right)\Big|_0^{T_j}=-q\varepsilon_0\int_S da\,\mathbf{n}\cdot\left[\delta\Phi\nabla\delta\hat{\Phi}-\delta\hat{\Phi}\nabla\delta\Phi\right]$$

Generalized Green Theorem

Generalized Green's Theorem

$$\sum_j I_j \left(\delta\hat{\mathbf{p}}_j \cdot \delta\mathbf{x}_j - \delta\mathbf{p}_j \cdot \delta\hat{\mathbf{x}}_j \right)\Big|_0^{T_j} = -q\varepsilon_0 \int_S da\,\mathbf{n} \cdot \left[\delta\Phi \nabla \delta\hat{\Phi} - \delta\hat{\Phi} \nabla \delta\Phi \right]$$

Problem #1 (true problem) Unperturbed trajectories at cathode, Perturbed potential on boundary.

$$\delta p_j\big|_0 = 0, \quad \delta x_j\big|_0 = 0, \quad \delta\Phi(\mathbf{x}) \neq 0$$

Problem #2 (adjoint problem) Perturbed trajectories at exit, Unperturbed potential on boundary.

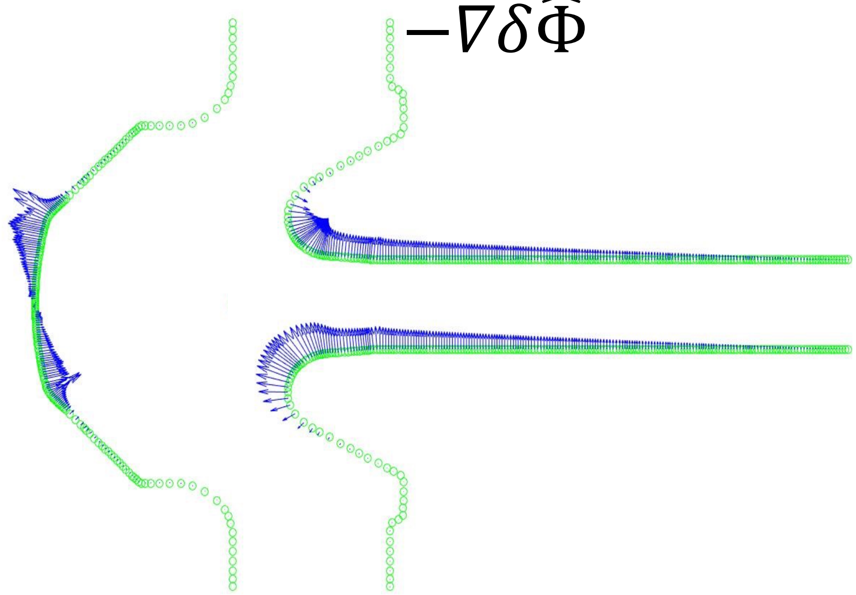
$$\delta \hat{p}_j\big|_T = \lambda \mathbf{x}_{\perp j}, \quad \delta x_j\big|_T = 0, \quad \delta\hat{\Phi}(\mathbf{x}) = 0$$

$$\lambda I R_{RMS} \delta R_{RMS} = \lambda \sum_j I_j \left(\mathbf{x}_j \cdot \delta\mathbf{x}_j \right)\Big|_{T_j} = -q\varepsilon_0 \int_S da\, \delta\Phi \left(\mathbf{n} \cdot \nabla \delta\hat{\Phi} \right)$$

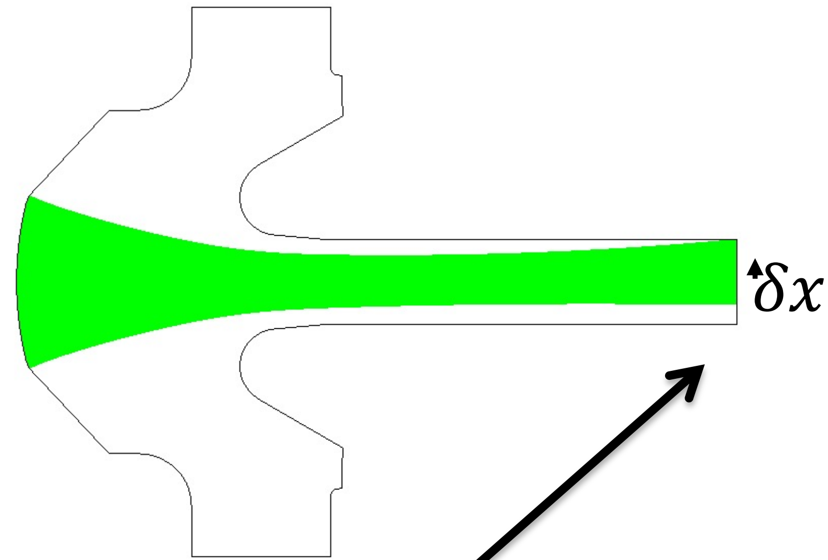
Sensitivity Function

Vertical Displacement of the Beam

Vector plot of the 'sensitivity' or Green's function



'Direct' MICHELLE Simulation with Perturbed Anode Voltages



$$\delta x = -\frac{q\varepsilon_0}{\lambda I}\int_S da\,\mathbf{n}\cdot\delta\Phi\nabla\delta\hat{\Phi}$$

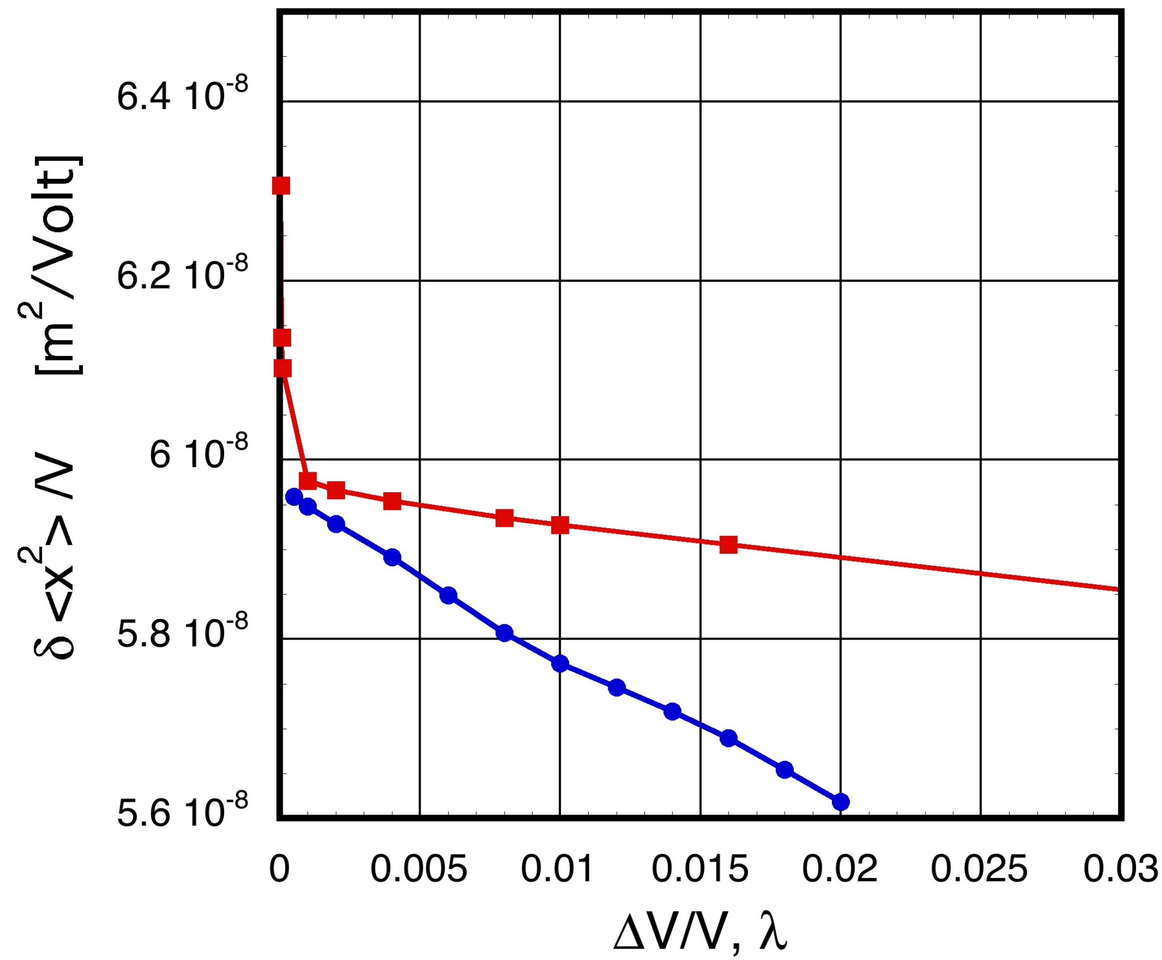
Predicted displacement / Calculated displacement = 0.9969

Numerical Accuracy

Problem #1 (true problem) Change anode voltage, find change in RMS radius.

Problem #2 (adjoint problem) Perturbed trajectories at exit, Unperturbed potential on boundary.

$$\delta \hat{p}_j \big|_T = \lambda \mathbf{x}_{\perp j},$$

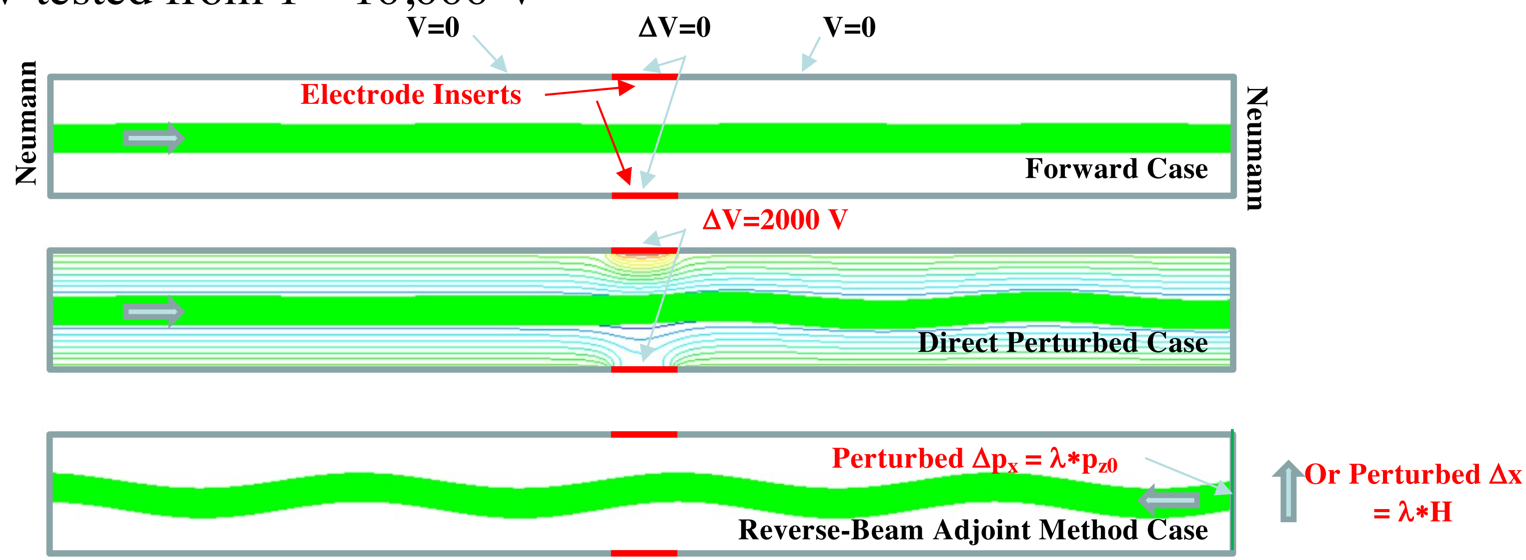


$$\lambda I R_{RMS} \delta R_{RMS} = \lambda \sum_j I_j \left(\mathbf{x}_j \cdot \delta \mathbf{x}_j \right) \Big|_{T_j} = -q\varepsilon_0 \int_S da\, \delta\Phi \left(\mathbf{n} \cdot \nabla \delta\hat{\Phi} \right)$$

AO 2.4-4: John Petillo, Serguei Ovtchinnikov, Aaron Jensen (Leidos), Philipp Borchard (Dymenso)
Kyle Kuhn, Heather Shannon, Brain Beaudoin TMA. (U. Maryland)

Application: 2D parallel plate sheet beam

- Forward Case: Grounded Inserts top and bottom
- Direct Perturbation Case: Electrode inserts on top and bottom set to ΔV
  ΔV tested from 1 – 10,000 V



DYMENSO

UNIVERSITY OF MARYLAND 18 56

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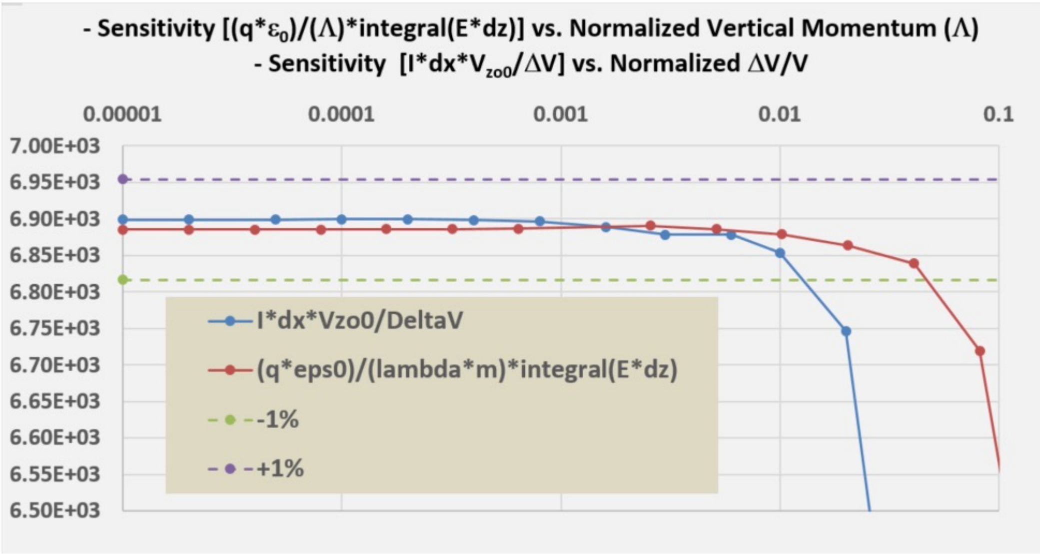
leidos

Mean Displacement: 2D parallel plate sheet beam

- Manufacturing sensitivity to beam centering offset

- Results of direct vs. adjoint methods agree to within **0.20%.**
- Verification: ***Hamiltonian Approach***
  Excellent first successful Adjoint method to beam transport in a magnetic field.
- Results:
  - As the perturbed-case voltage values became small enough it easily entered the linear regime.
  - There is very a broad range of both Λ and ΔV where the results are all in a linear regime.



Adjoint method predicted the deflection sensitivity to within 0.2%

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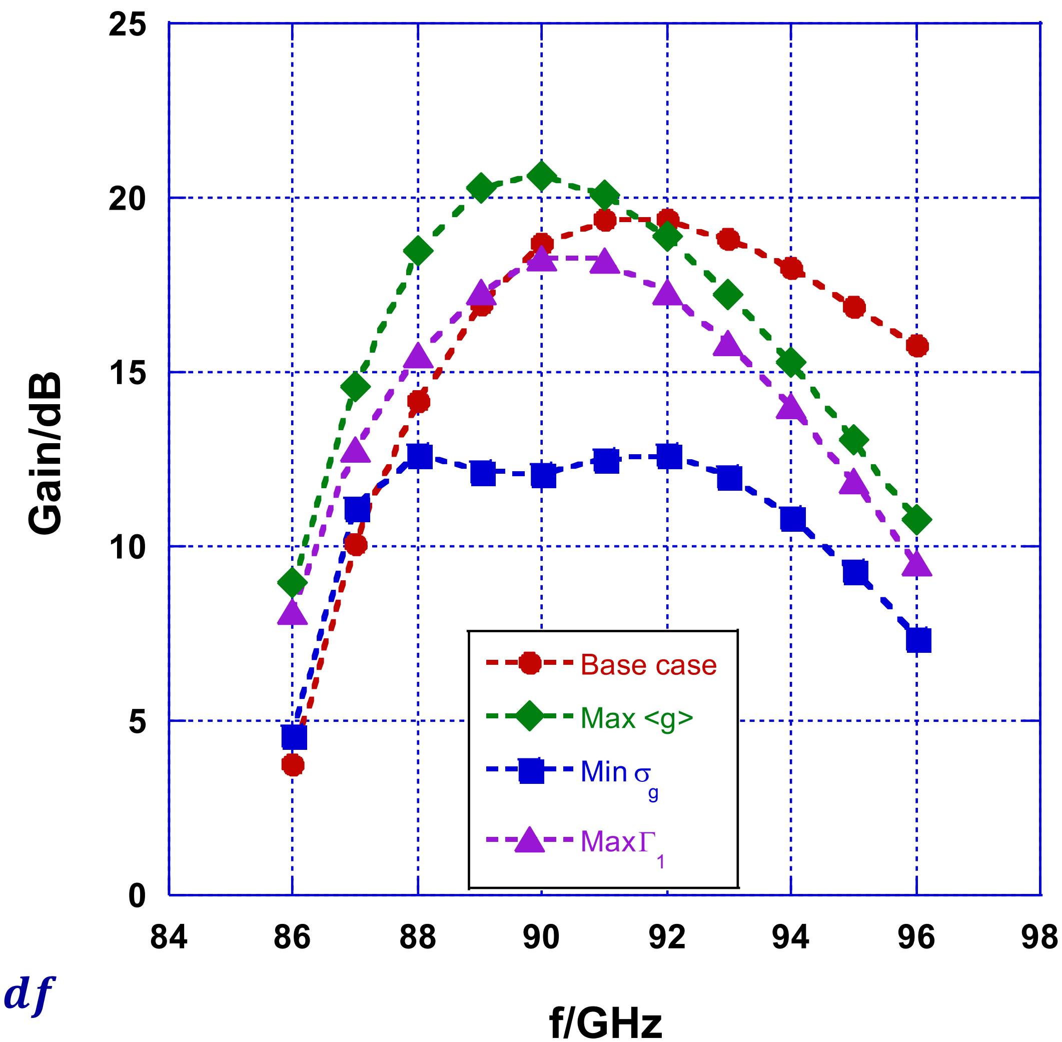
AO 2.4-2 – Optimization of TWT Design by Using Adjoint Approach

A. Vlasov, T. M. Antonsen Jr., D. Chernin, I. Chernyavskiy

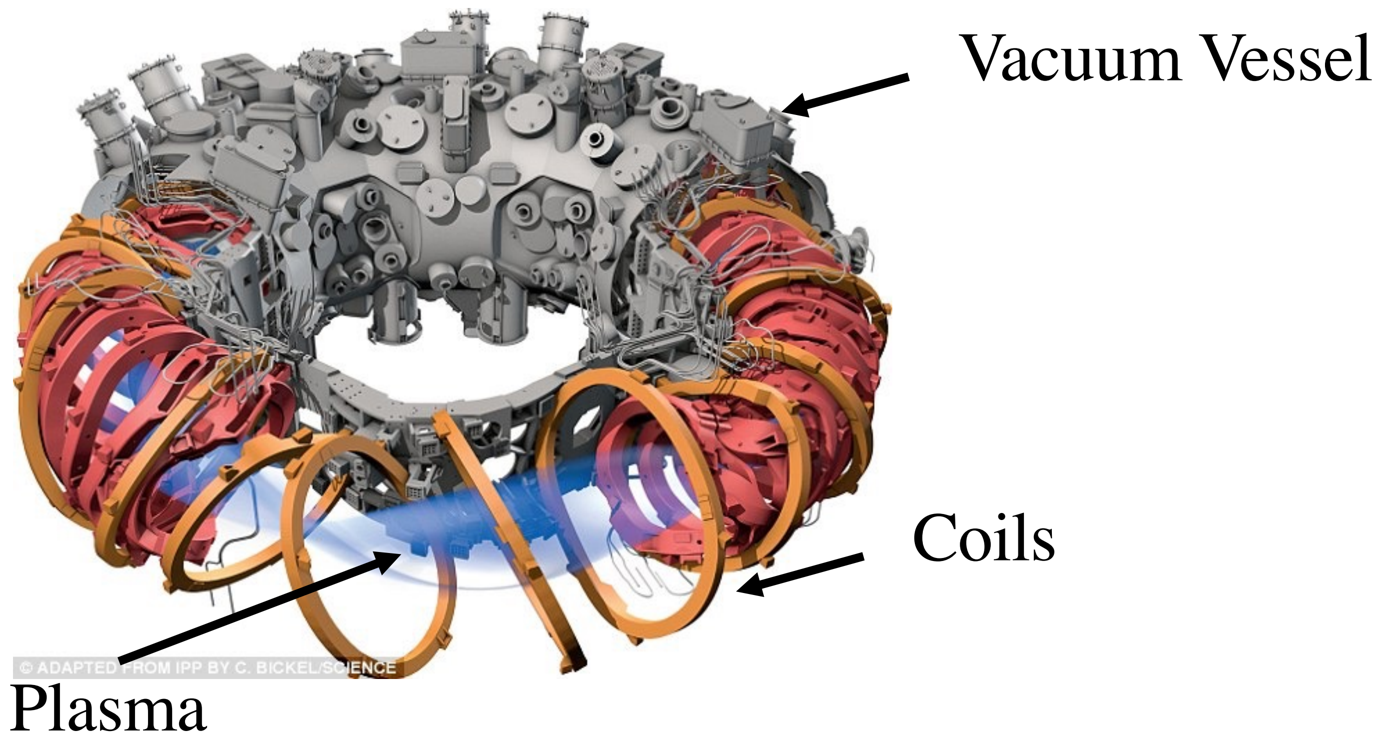
Optimization of Small Signal Gain

Distance between gaps (two sections).
3 different goal functions
2 optimization parameters

- $p = L_{g1},\ \ L_{g2}$
- **Maximize:** $F_1(p) = \frac{1}{f_2 - f_1}\int_{f_1}^{f_2} G(f)\,df$
- **Minimize:** $F_2(p) = \frac{1}{f_2 - f_1}\int_{f_1}^{f_2} (G(f) - \overline{G})^2\,df$
- **Maximize:** $F_3(p) = \Gamma_1 = \frac{\langle g\rangle_p}{\sigma_g^p}$



3D MHD Equilibria



Wendelstein 7-X
Max Planck Institute for Plasma Physics (IPP)
Greifswald, Germany   Completed 2015

Optimization of Stellarator Equilibria

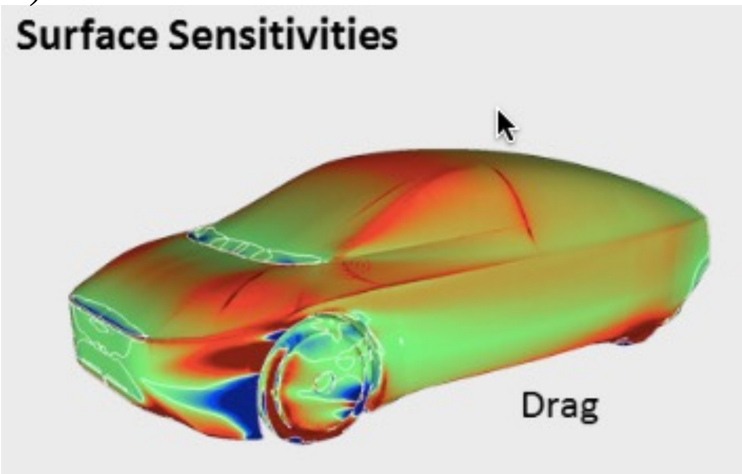
➔ **Figures of Merit (FoM) - Examples**
Plasma Beta, Rotational Transform, Quasi-symmetry

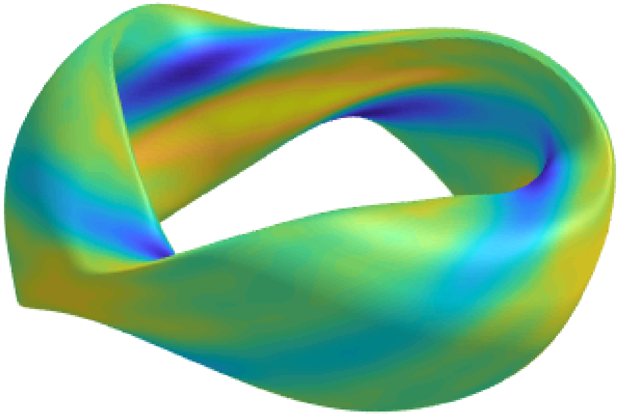
➔ **FoMs** depend on **boundary or coil shapes**

➔ **Shape Gradient Sensitivity Functions**
gradient based optimization
establish tolerances

Landreman and Paul, 2018 Nucl. Fusion 58 076023,
E. Paul, M. Landreman, TMA, J. Plasma Phys. (2019), vol. 85, 905850207,
J. Plasma Phys. (2021), vol. 87, 905870214

C. Othmer, J. Math. Industry 4, 6 (2014). **DRAG**





Rotational transform

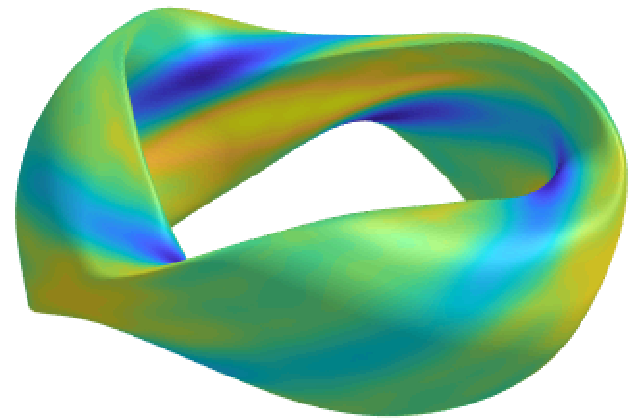
Adjoint Symmetry Simplifies Calculations

Adjoint Approach to gradient calculation

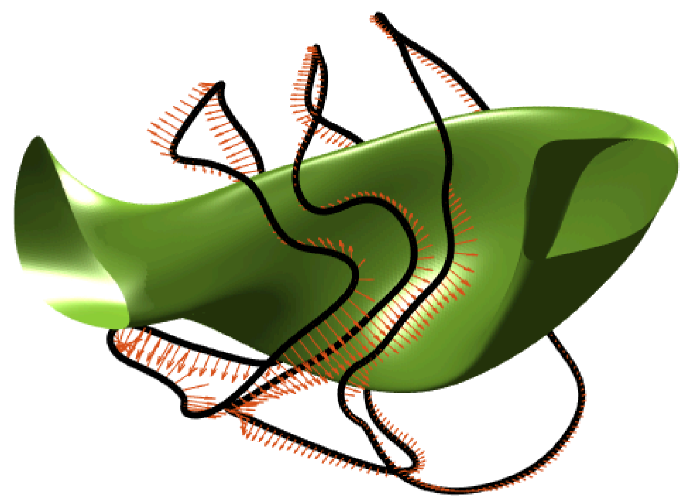
> 500 X Speed – Up over direct calculation

Uses VMEC & DIAGNO
Hirshman and Whitman, 1983 Phys. Fluids 25 3553
H.J. Gardner 1990 *Nucl. Fusion* 30 1417

Different Figures of Merit Possible
Plasma pressure – beta
Rotational transform
Toroidal current
Neoclassical radial transport -1/ν regime
Energetic particle drifts
Quasi-symmetry
Others

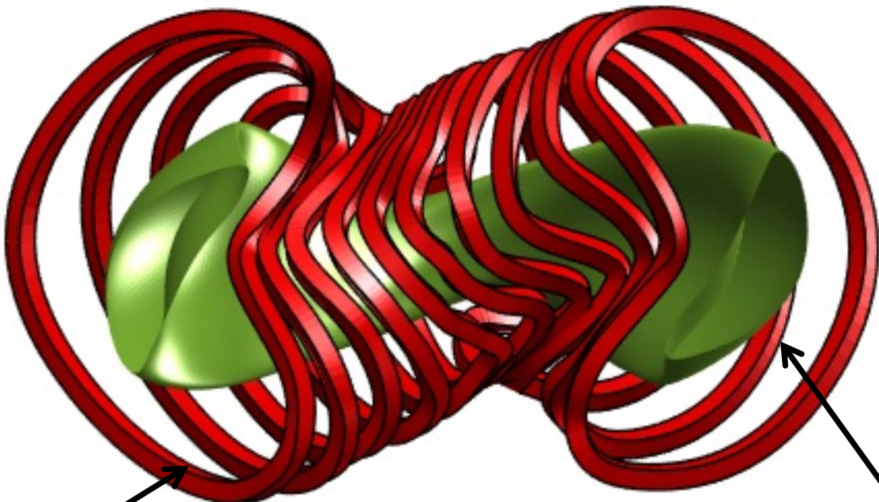


Surface shape sensitivity



Coil location sensitivity

3D MHD Toroidal Equilibrium



Coils

Assume good flux surfaces in plas

In plasma

$$-\nabla p + \frac{\mathbf{J}\times\mathbf{B}}{c} = 0$$

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J}$$

In vacuum

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J}_C$$

coil current

Poloidal flux

$$\mathbf{B} = \nabla\alpha\times\nabla\theta - \nabla\Phi_p(\alpha)\times\nabla\zeta$$

$$= \nabla\alpha\times\nabla\left(\theta - \iota(\alpha)\zeta\right)$$

Toroidal Flux

$$\iota(\alpha) = d\Phi_p(\alpha)/d\alpha$$

Rotational transform

Linear Perturbations to Equilibrium

<u>Generalized Forces:</u>

$$\mathbf{J}_C \Rightarrow \mathbf{J}_C + \delta\mathbf{J}_C$$

Changes in current/shape/location of coils

$$\nabla p \Rightarrow \nabla p + \nabla \cdot \delta \underline{\underline{\mathbf{P}}}$$

Added pressure tensor

$$\Phi_p(\alpha) \Rightarrow \Phi_p(\alpha) + \delta\Phi_p(\alpha)$$

Change in poloidal flux profile

$$\iota(\alpha) = d\Phi_p(\alpha) / d\alpha$$

<u>Generalized responses:</u>

$$\mathbf{A}_V \Rightarrow \mathbf{A}_V + \delta\mathbf{A}_V$$

Changes in vacuum fields

$$\mathbf{B} \Rightarrow \mathbf{B} + \nabla \times \left(\xi \times \mathbf{B} - \delta\Phi_P \nabla \zeta\right)$$

Changes in magnetic field

$$I_T \Rightarrow I_T + \delta I_T(\alpha)$$

Changes in toroidal current profile

Generalized Forces and Responses

Responses				**Forces:**
Vacuum fields	$\delta \mathbf{A}_V$		$\delta \mathbf{J}_C$	Coil currents
Plasma displacement	ξ	$= \underline{\underline{O}}$	$\nabla \cdot \delta \underline{\underline{P}}$	Pressure tensor
Toroidal current profile	$d\delta I_T / d\alpha$		$\delta \Phi_P$	Rotational transform

$$\begin{pmatrix} \delta \mathbf{A}_V \\ \xi \\ d\delta I_T / d\alpha \end{pmatrix} = \underline{\underline{O}} \begin{pmatrix} \delta \mathbf{J}_C \\ \nabla \cdot \delta \underline{\underline{P}} \\ \delta \Phi_P \end{pmatrix}$$

More generically, for two different perturbations

$$\delta x_i^{(1)} = \sum_j O_{ij}\, \delta F_j^{(1)} \qquad \delta x_i^{(2)} = \sum_j O_{ij}\, \delta F_j^{(2)}$$

Onsager Symmetry Gives

$$\sum_j \left\{ \delta x_i^{(1)} \delta F_i^{(2)} - \delta x_i^{(2)} \delta F_i^{(1)} \right\} = 0$$

Onsager Symmetry for 3D MHD Equilibria
Self-adjoint MHD Force Operator

$$\int_{VP} d^3x \left(\xi^{(1)} \cdot \nabla \cdot \delta \underline{\underline{\mathbf{P}}}_L^{(2)} - \xi^{(2)} \cdot \nabla \cdot \delta \underline{\underline{\mathbf{P}}}_L^{(1)} \right)$$

True
Adjoint

Pressure - Displacement

$$- \frac{2\pi}{c} \int_{VP} d\alpha \left(\delta I_T^{(2)} \frac{d}{d\alpha} \delta \Phi_p^{(1)} - \delta I_T^{(1)} \frac{d}{d\alpha} \delta \Phi_p^{(2)} \right)$$

Rotational transform –Toroidal current

$$+ \frac{1}{4\pi} \int_S d^2x \, \mathbf{n} \cdot \left(\xi^{(1)} \delta \mathbf{B}^{(2)} \cdot \mathbf{B} - \xi^{(2)} \delta \mathbf{B}^{(1)} \cdot \mathbf{B} \right) = 0$$

Surface displacement

Specify BC's & Constraints

Make this appear to be change in FoM

True

Adjoint

$$\int_{VP} d^3x \left(\xi^{(1)} \cdot \nabla \cdot \delta \underline{\underline{\mathbf{P}}}_L^{(2)} - \boxed{\xi^{(2)} \cdot \nabla \cdot \delta \underline{\underline{\mathbf{P}}}_L^{(1)}} \right)$$

Pressure - Displacement

$$-\frac{2\pi}{c} \int_{VP} d\alpha \left(\delta I_T^{(2)} \frac{d}{d\alpha} \delta \Phi_p^{(1)} - \delta I_T^{(1)} \frac{d}{d\alpha} \delta \Phi_p^{(2)} \right)$$

Rotational transform –Toroidal current

$$+\frac{1}{4\pi} \int_S d^2x \, \mathbf{n} \cdot \left(\xi^{(1)} \delta \mathbf{B}^{(2)} \cdot \mathbf{B} - \boxed{\xi^{(2)} \delta \mathbf{B}^{(1)} \cdot \mathbf{B}} \right) = 0$$

Sensitivity function

Surface displacement

Specify BC's & Constraints

Make this appear to be change in FoM

$$\int_{VP} d^3x \Big(\qquad - \boxed{\xi^{(2)} \cdot \nabla \cdot \delta \underline{\underline{\mathbf{P}}}_L^{(1)} \Big)}$$

True

Adjoint

Pressure - Displacement

$$+\frac{1}{4\pi}\int_S d^2x \, \mathbf{n} \cdot \Big(\qquad \boxed{-\xi^{(2)} \delta \mathbf{B}^{(1)} \cdot \mathbf{B} \Big)} = 0$$

Sensitivity function

Surface displacement

J. Plasma Phys. (2021), *vol.* 87, 905870214 © The Author(s), 2021.
Published by Cambridge University Press
doi:10.1017/S0022377821000283

Gradient-based optimization of 3D MHD equilibria

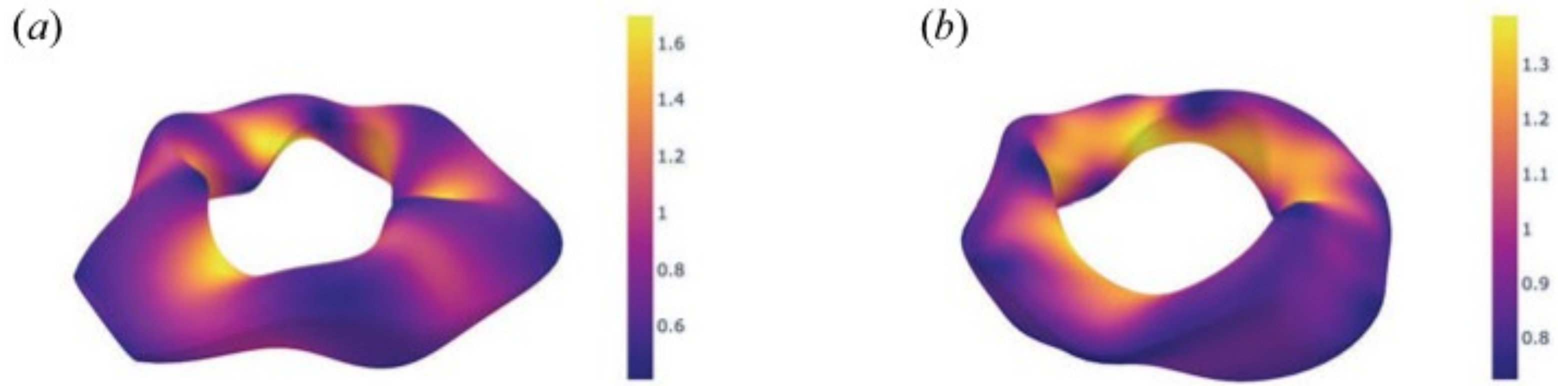
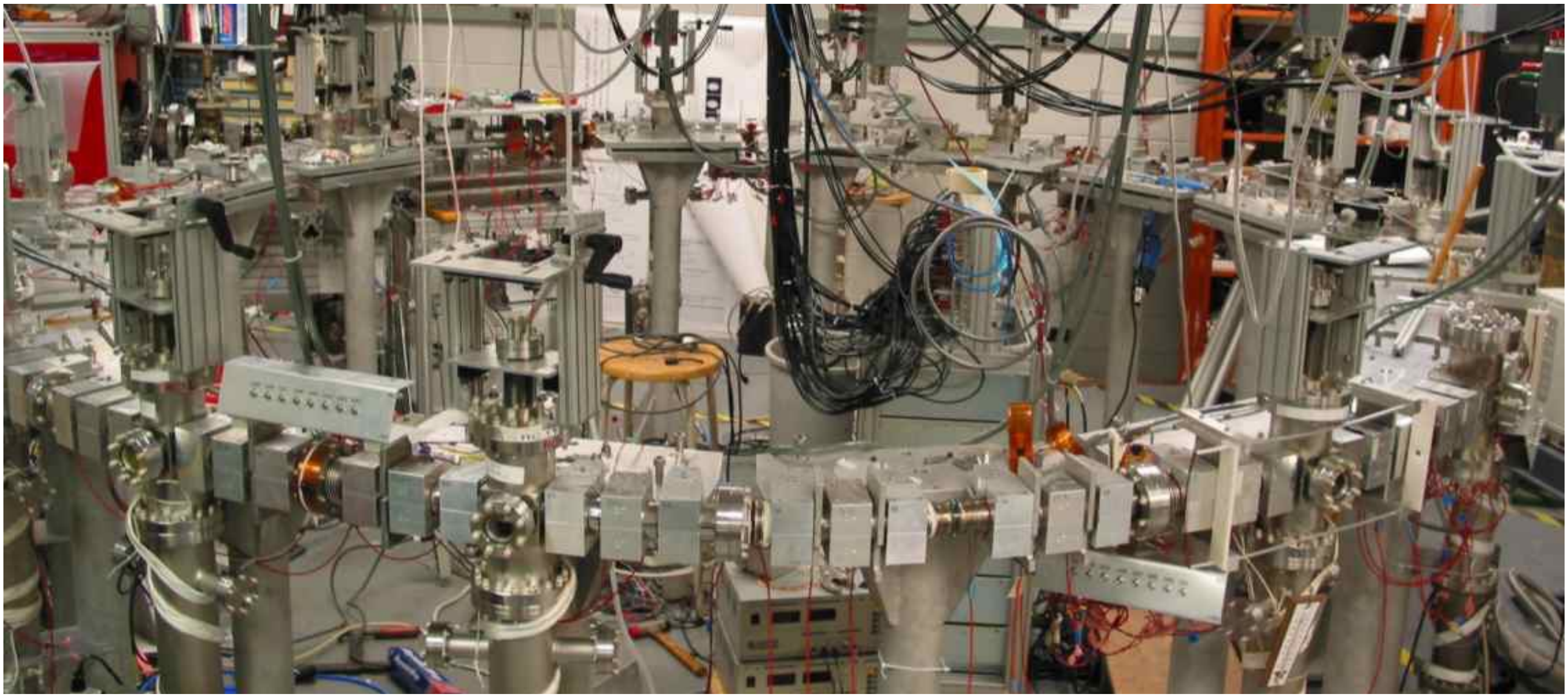


FIGURE 11. The magnetic field strength on (*a*) the initial boundary (4.14) and (*b*) the boundary optimized for quasi-symmetry on the axis (4.13).

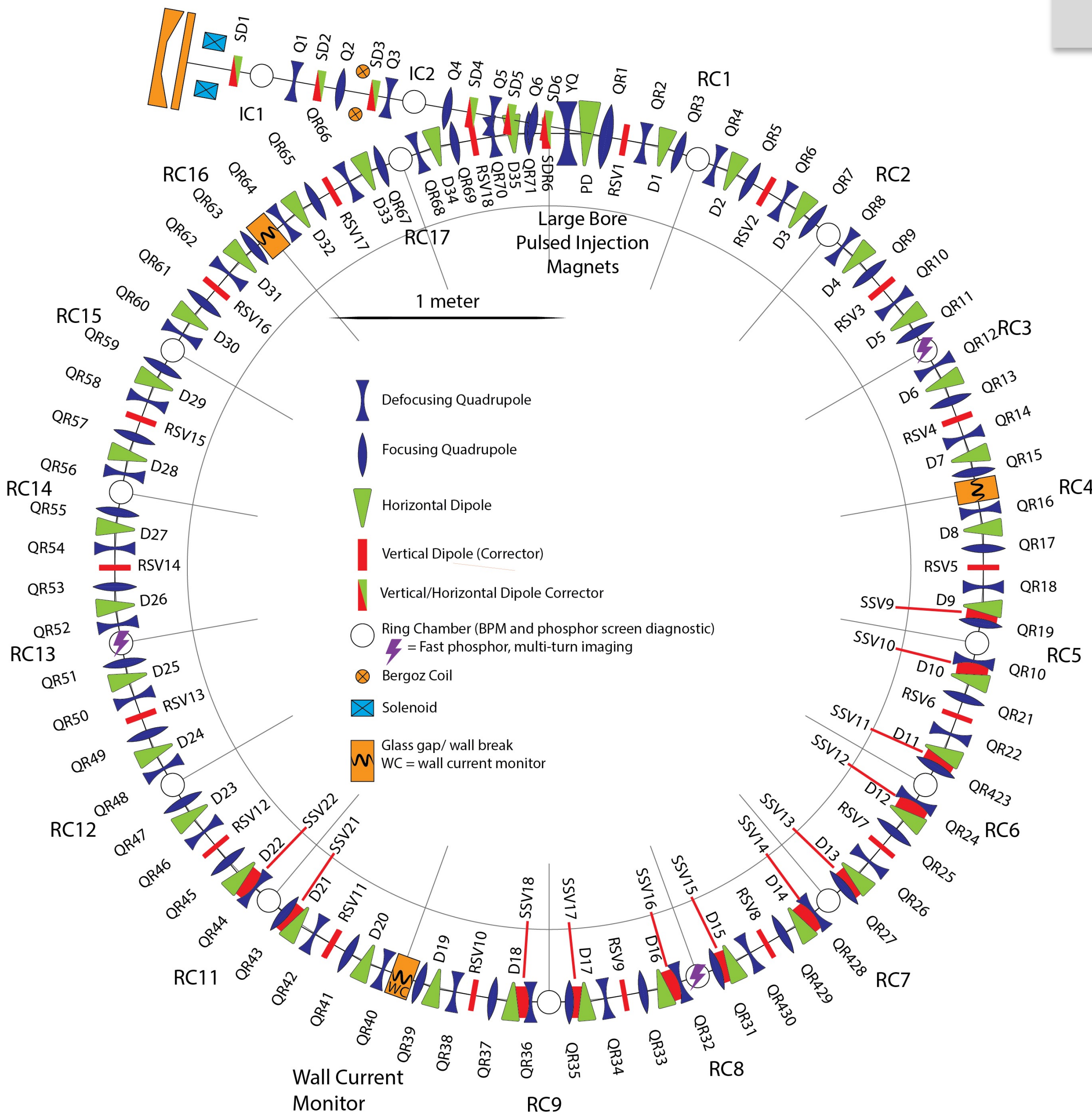
Optimization of Focusing Magnets in Accelerator Lattices

The University of Maryland Electron Ring



UMER is a fully functional electron storage ring

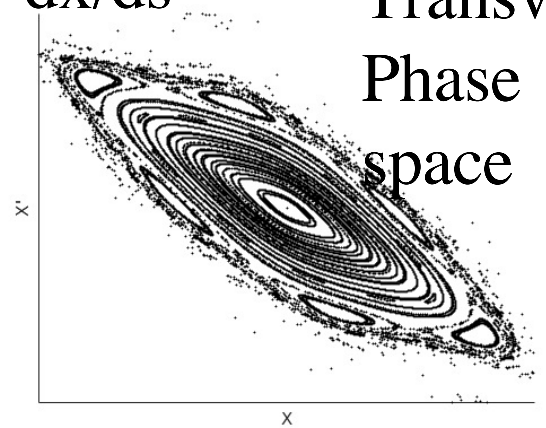
UMER Systems and Layout



- 167 Magnets, power supplies & controls.

Transverse Phase space

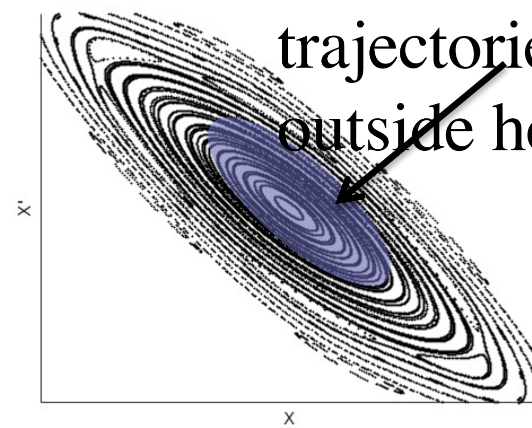
x'=dx/ds



x

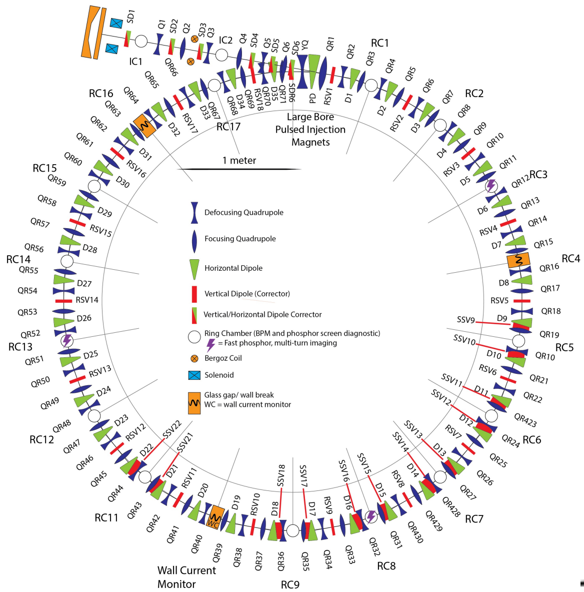
x'=dx/ds

Want to avoid trajectories outside here

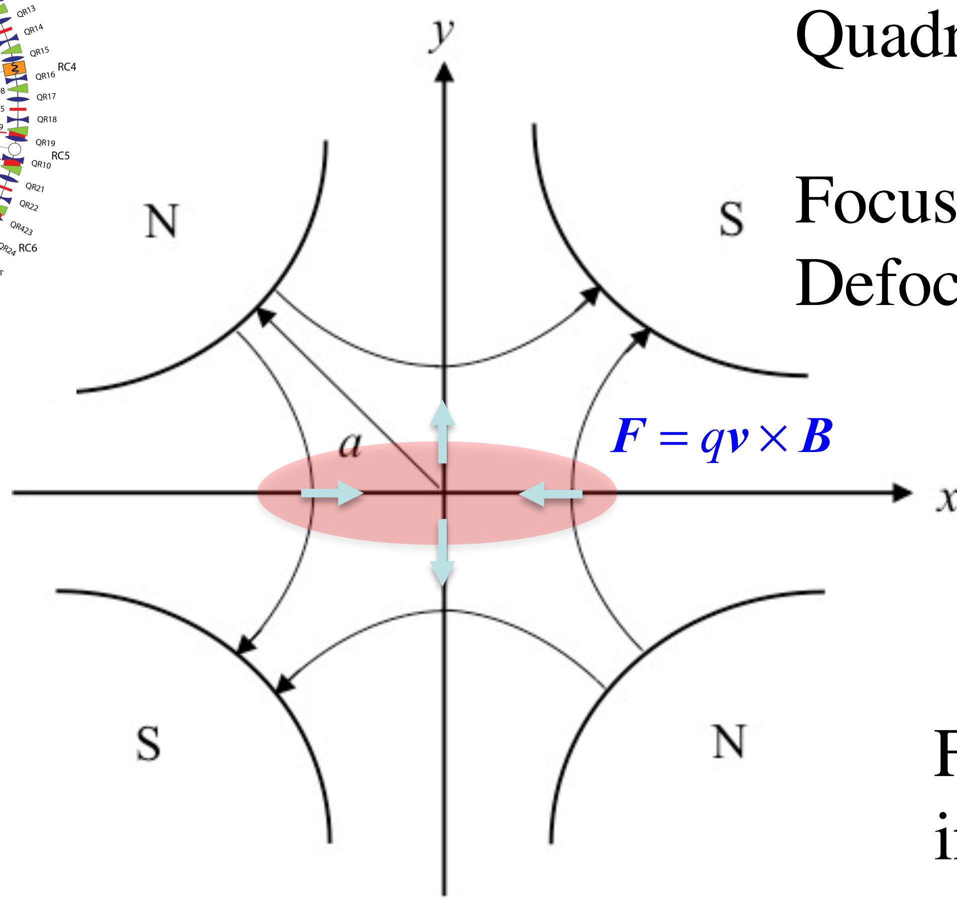


x

Focusing Basics



32 Quadrupole magnets



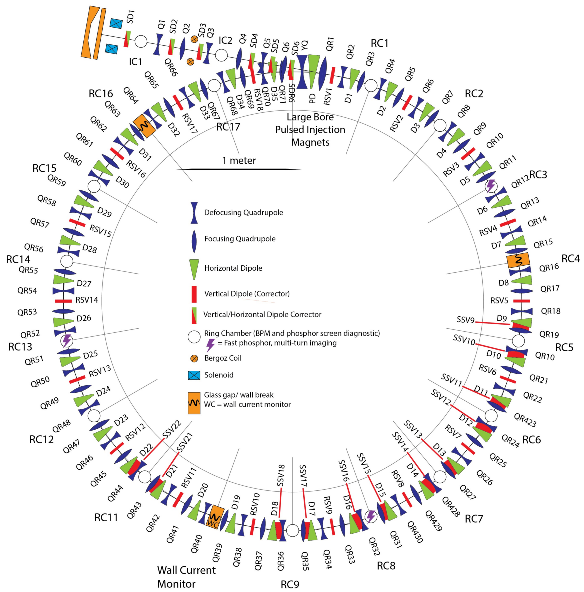
$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$

Quadrupole Magnetic Field

Focusing in x-direction
Defocusing in y-direction

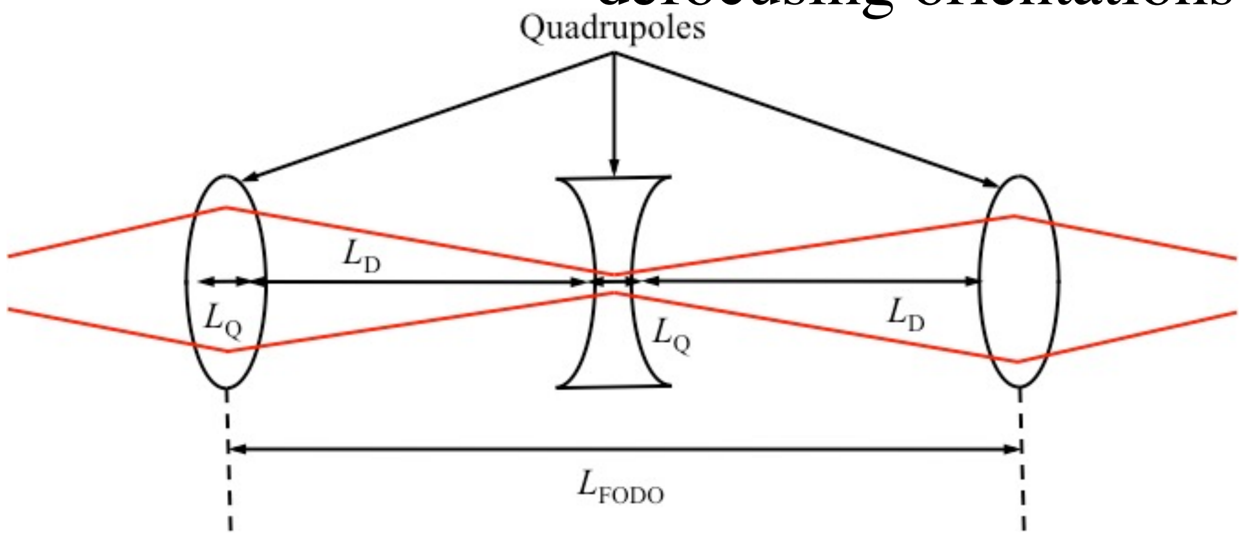
$q\,\boldsymbol{v}$ – out of page

Field strength increases linearly with distance from the axis



FODO Lattice

Alternate focusing and defocusing orientations



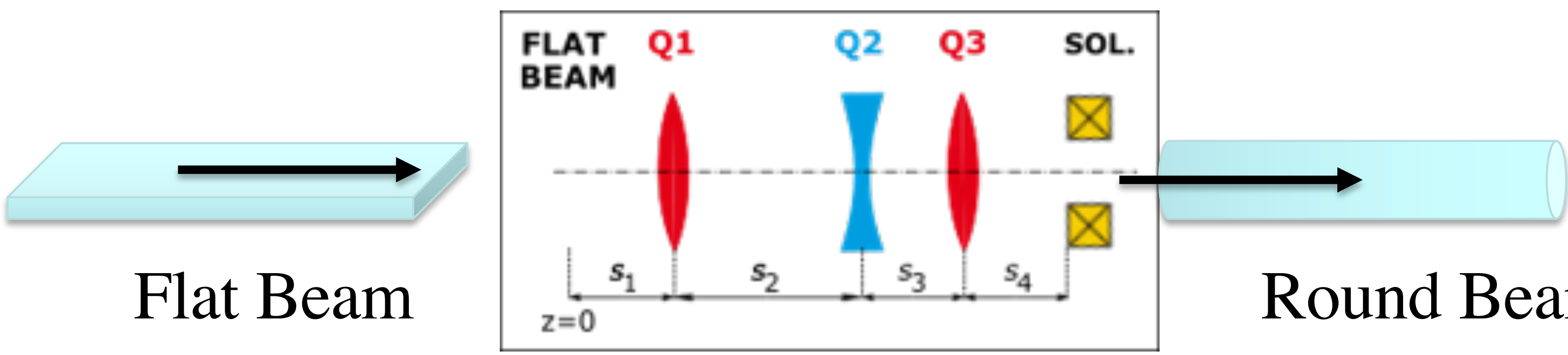
32 Quadrupole magnets

Qualitative explanation: beam passes through focusing quad when it is big (strong fields). Passes through defocusing quad when it is small (weak fields). Net effect is focusing.

Beam distribution depends on many parameters   How to opti

Optimization of Flat to Round Transformers Using Adjoint Techniques*

L. Dovlatyan, B. Beaudoin, S. Bernal, I. Haber, D. Sutter and T. M. Antonsen Jr , Phys Rev Accel and Beams V25, 044002 (2022).



Flat to Round and Round to Flat transformers are proposed for cooling of hadron beams. Ya. Derbenev, Adapting optics for high-energy electron cooling, Tech. Rep. UM-HE-98-04-A (1998).

Cylindrical hadron beam cools via collisions when co-propagated with electron beam.

Steps

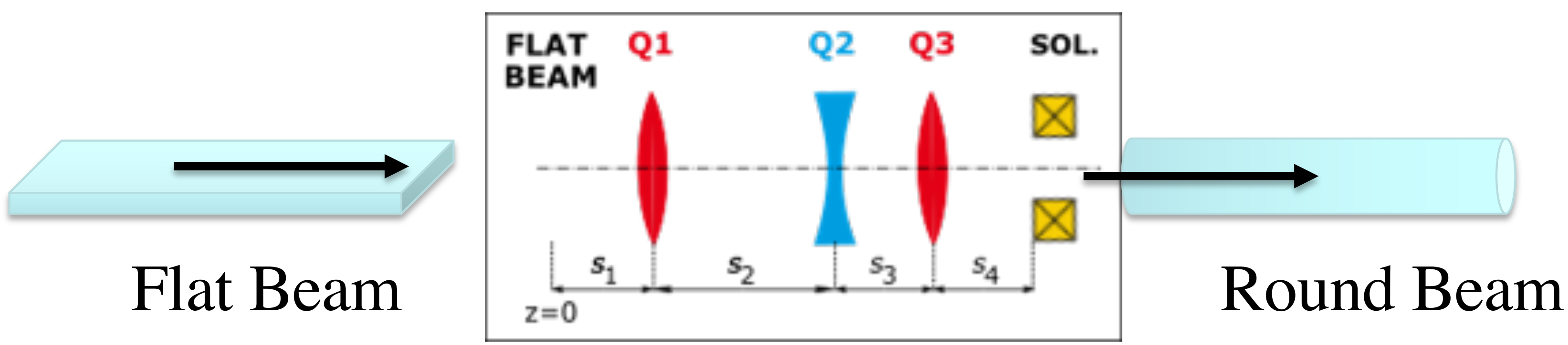
1. Derive system of moment equations (include self fields)

2. Linearize (to compute parameter gradient)

3. Find adjoint system

4. Decide on Figures of Merit

5. Optimize by Gradient Descent



Moment Equations

$$\frac{d}{dz}\mathbf{Q} = \mathbf{P}$$
$$\frac{d}{dz}\mathbf{P} = \mathbf{E} + \mathbf{O}\cdot\mathbf{Q}$$
$$\frac{d}{dz}\mathbf{E} = \mathbf{O}\cdot\mathbf{P} + \mathbf{N}L$$
$$\frac{d}{dz}L = -\mathbf{N}^{\dagger}\cdot\mathbf{Q}$$

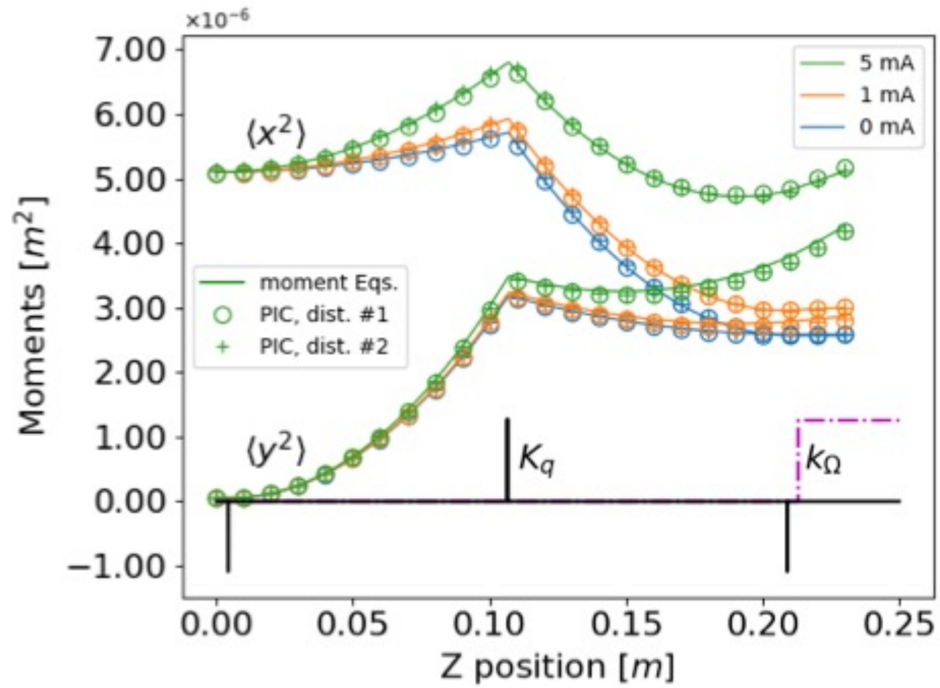
$$\mathbf{O} = \mathbf{O}_{Bz} + \mathbf{O}_{Quads} + \mathbf{O}_{\substack{space\\charge}}$$
$$\mathbf{N} = \mathbf{N}_{Quads} + \mathbf{N}_{\substack{space\\charge}}$$

↑ Depend on magnet parameters

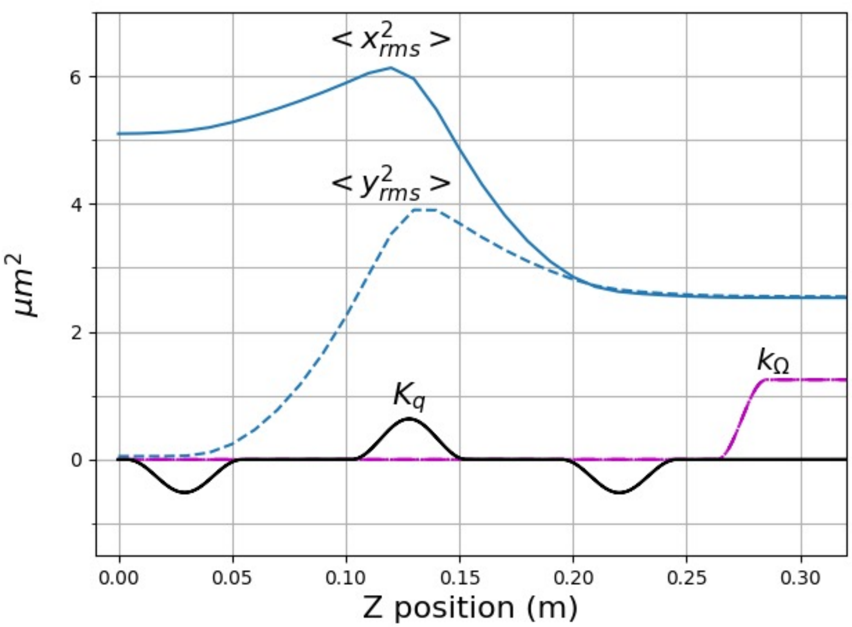
Symbols –PIC
Lines – Moment Eqs.

$$\mathbf{Q} = \begin{pmatrix} \langle x^2 + y^2\rangle/2 \\ \langle x^2 - y^2\rangle/2 \\ \langle xy\rangle \end{pmatrix} \qquad L = \langle xy' - yx'\rangle$$

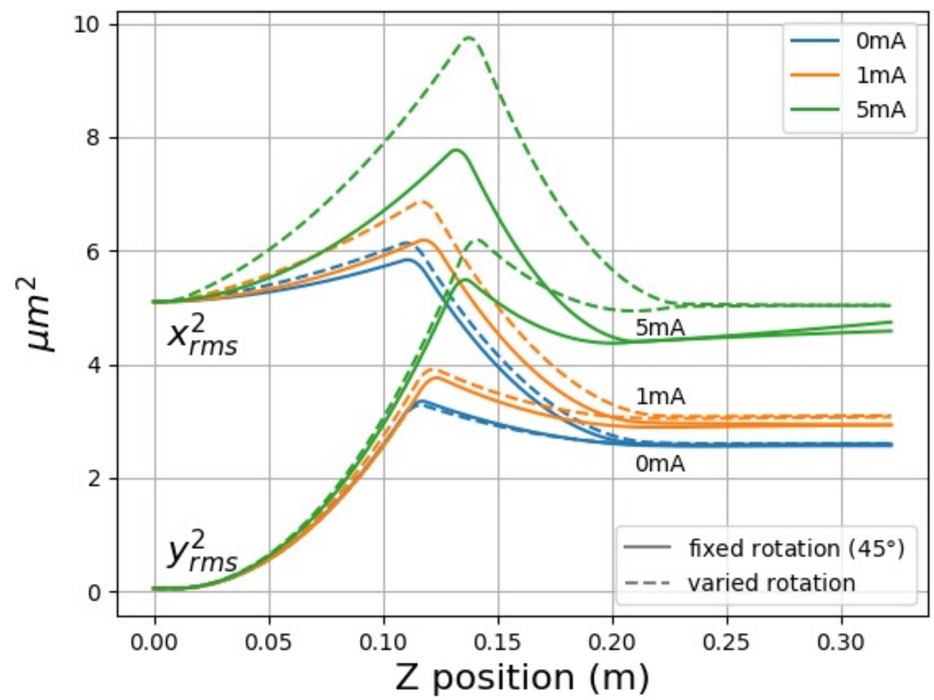
$$\mathbf{P} = \begin{pmatrix} \langle xx' + yy'\rangle \\ \langle xx' - yy'\rangle \\ \langle yx' + xy'\rangle \end{pmatrix} \qquad \mathbf{E} = \begin{pmatrix} \langle x'^2 + y'^2\rangle \\ \langle x'^2 - y'^2\rangle \\ 2\langle y'x'\rangle \end{pmatrix}$$



More Optimization



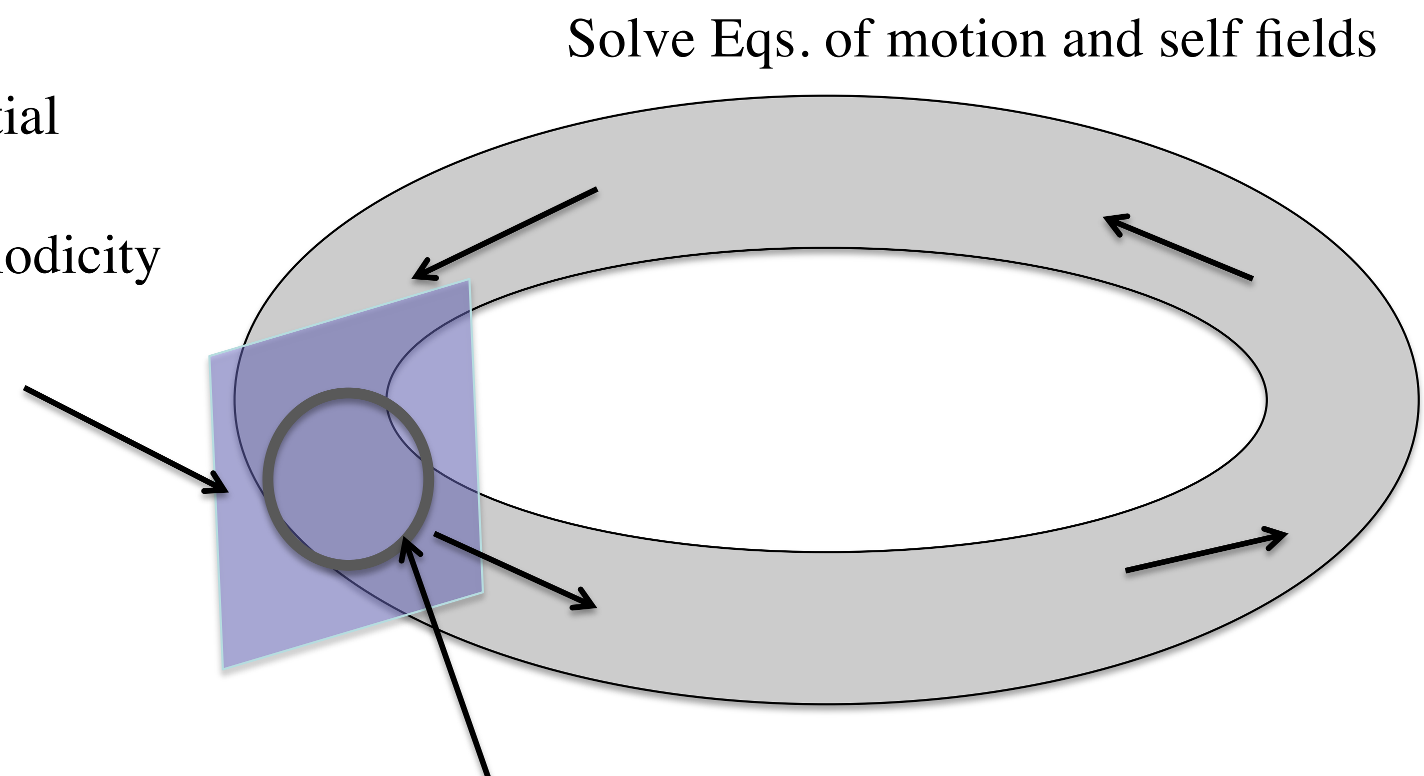
Continuous magnetic field profiles



Variable magnet orientations

Circular Accelerators-Periodicity

Particles return to initial plane.
Need to maintain periodicity of distribution, not individual orbits



Big problems:
Do periodic distributions exist? Most likely no.
How to relaunch particles to optimize?

Particles + Periodicity = Problems

Conclusion

Adjoint methods are a powerful way to evaluate parameter dependences in many systems involving charged particle dynamics.

Thank you.

Moment Equations

Transverse phase space:

$$x, x' = \frac{dx}{dz}, y, y' = \frac{dy}{dz}$$

The second order moments are averages of the 4 by 4 elements of the Sigma matrix.

$$\underline{\underline{\Sigma}} = \begin{bmatrix} xx & xx' & xy & xy' \\ x'x & x'x' & x'y & x'y' \\ yx & yx' & yy & yy' \\ y'x & y'x' & y'y & y'y' \end{bmatrix}$$

Moments: $\mathbf{Q}, \mathbf{P}, \mathbf{E}, L$

$$\mathbf{Q} = \begin{pmatrix} Q_+ \\ Q_- \\ Q_x \end{pmatrix} = \begin{pmatrix} \langle x^2 + y^2 \rangle / 2 \\ \langle x^2 - y^2 \rangle / 2 \\ \langle xy \rangle \end{pmatrix} \quad \mathbf{P} = \frac{d}{dz}\mathbf{Q} = \begin{pmatrix} P_+ \\ P_- \\ P_x \end{pmatrix} = \begin{pmatrix} \langle xx' + yy' \rangle \\ \langle xx' - yy' \rangle \\ \langle yx' + xy' \rangle \end{pmatrix}$$

$$\mathbf{E} = \begin{pmatrix} E_+ \\ E_- \\ E_x \end{pmatrix} = \begin{pmatrix} \langle x'^2 + y'^2 \rangle \\ \langle x'^2 - y'^2 \rangle \\ 2\langle y'x' \rangle \end{pmatrix}$$

Angular momentum $\quad L = \langle xy' - yx' \rangle$

Linearized System

Base case

$$\frac{d}{dz}\mathbf{Q} = \mathbf{P}$$

$$\frac{d}{dz}\mathbf{P} = \mathbf{E} + \mathbf{O}\cdot\mathbf{Q}$$

$$\frac{d}{dz}\mathbf{E} = \mathbf{O}\cdot\mathbf{P} + \mathbf{N}L$$

$$\frac{d}{dz}L = -\mathbf{N}^{\dagger}\cdot\mathbf{Q}$$

Linear perturbation due to true change in parameters

$$\frac{d}{dz}\delta\mathbf{Q}^{(X)} = \delta\mathbf{P}^{(X)}$$

$$\frac{d}{dz}\delta\mathbf{P}^{(X)} = \delta\mathbf{E}^{(X)} + \mathbf{O}\cdot\delta\mathbf{Q}^{(X)} + \delta\mathbf{O}^{(X)}\cdot\mathbf{Q}$$

$$\frac{d}{dz}\delta\mathbf{E}^{(X)} = \mathbf{O}\cdot\delta\mathbf{P}^{(X)} + \mathbf{N}\delta L^{(X)} + \delta\mathbf{O}^{(X)}\cdot\mathbf{P} + \delta\mathbf{N}^{(X)}L$$

$$\frac{d}{dz}\delta L^{(X)} = -\mathbf{N}^{\dagger}\cdot\delta\mathbf{Q}^{(X)} - \delta\mathbf{N}^{\dagger(X)}\cdot\mathbf{Q}$$

Adjoint system

$$\frac{d}{dz}\delta\mathbf{Q}^{(Y)} = \delta\mathbf{P}^{(Y)}$$

$$\frac{d}{dz}\delta\mathbf{P}^{(Y)} = \delta\mathbf{E}^{(Y)} + \mathbf{O}\cdot\delta\mathbf{Q}^{(Y)}$$

$$\frac{d}{dz}\delta\mathbf{E}^{(Y)} = \mathbf{O}\cdot\delta\mathbf{P}^{(Y)} + \mathbf{N}\delta L^{(Y)} + \delta\dot{\mathbf{E}}^{(Y)}$$

$$\frac{d}{dz}\delta L^{(Y)} = -\mathbf{N}^{\dagger}\cdot\delta\mathbf{Q}^{(Y)}$$

Sensitivity functions

$$\delta FoM = \int_{z_i}^{z_f} dz\left\{\delta\mathbf{P}^{(Y)}\cdot\delta\mathbf{O}_{Q,B}{}^{(X)}\cdot\mathbf{Q} + \delta L^{(Y)}\mathbf{Q}\cdot\delta\mathbf{N}_{Q,B}{}^{(X)} - \delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{O}_{Q,B}{}^{(X)}\cdot\mathbf{P} - \delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{N}_{Q,B}{}^{(X)}L\right\}$$

Change in magnet parameters

Figure of Merit and Gradient

Constant radius, Round

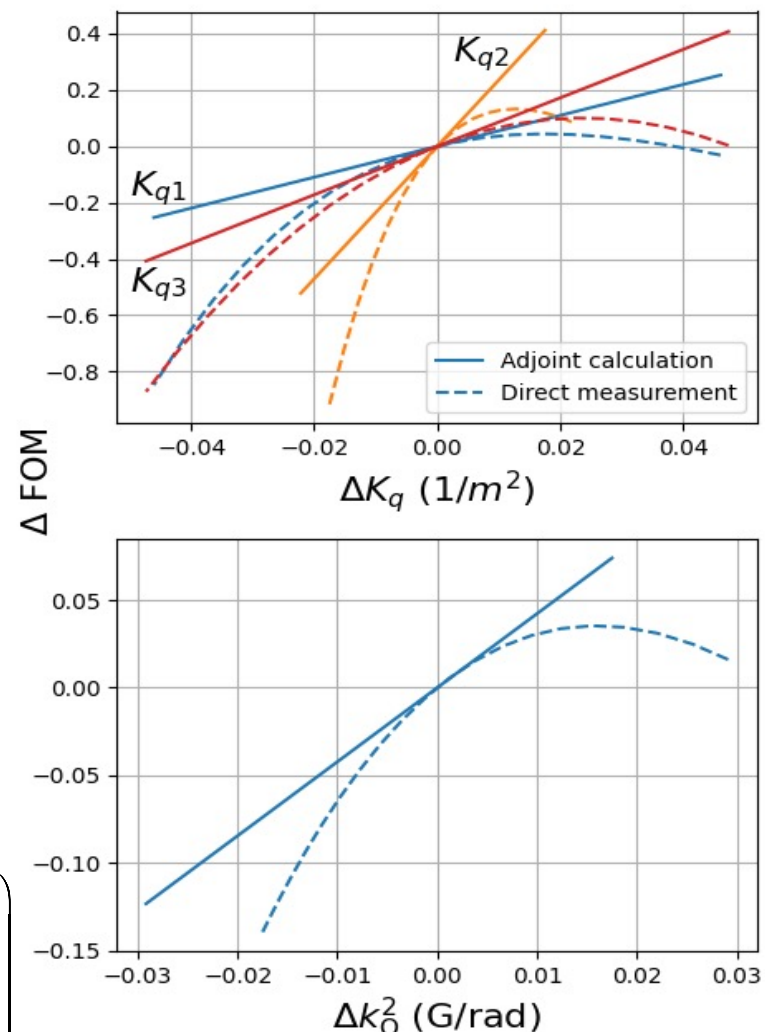
$$F=\frac{1}{2}\left[\left|\mathbf{P}\right|^{2}+k_{0}^{2}\left(Q_{-}^{2}+Q_{x}^{2}\right)+k_{0}^{-2}\left(E_{-}^{2}+E_{x}^{2}\right)\right]$$

$$+\frac{1}{2}\left[k_{0}^{-2}\left(E_{+}-\frac{1}{2}k_{\Omega}^{2}Q_{+}+\Lambda\right)^{2}+\left(2E_{+}Q_{+}-L^{2}\right)^{2}\right]$$

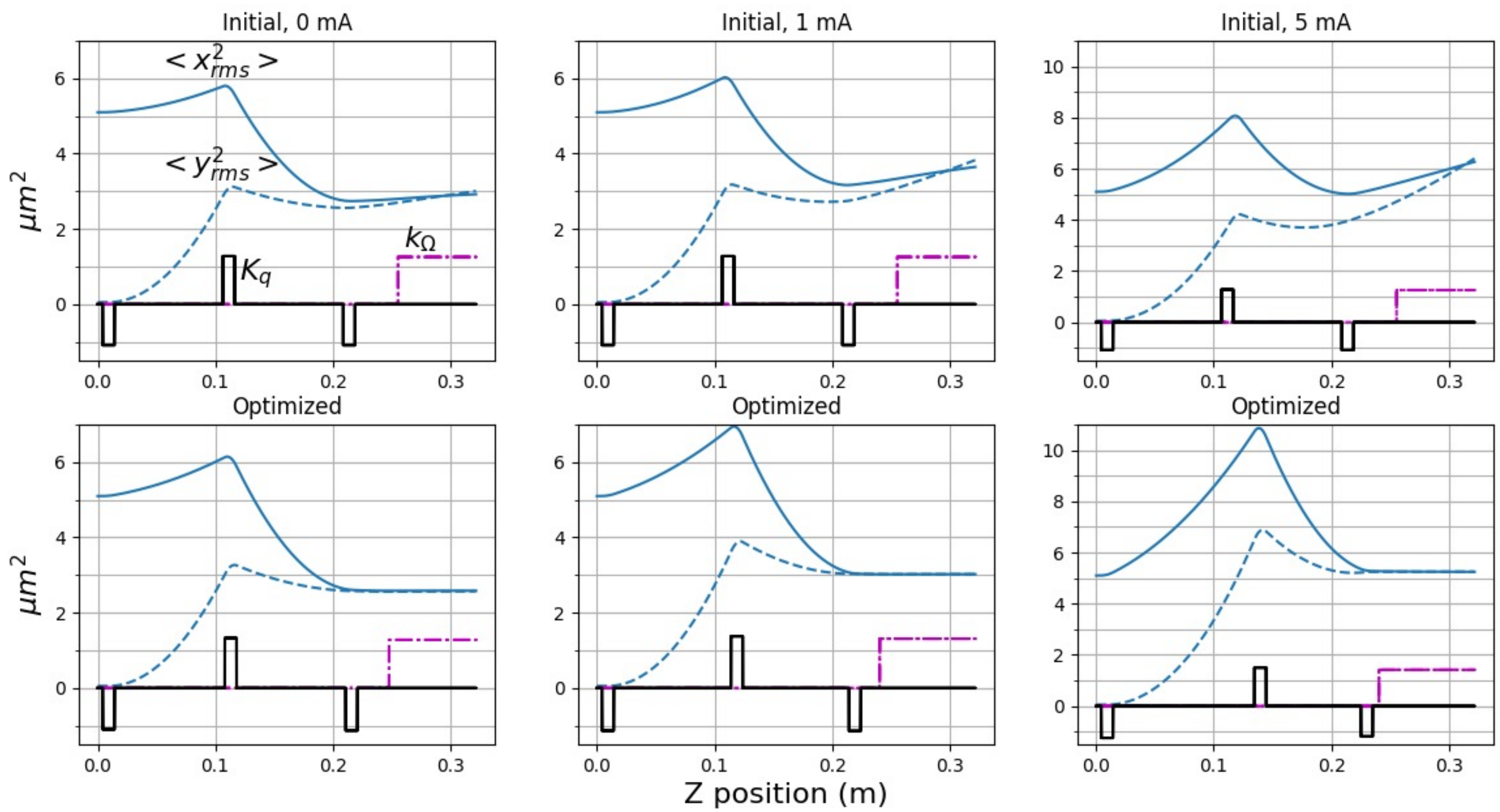
Radial force balance, Rigid rotation

$$L=\left\langle xy'-yx'\right\rangle$$

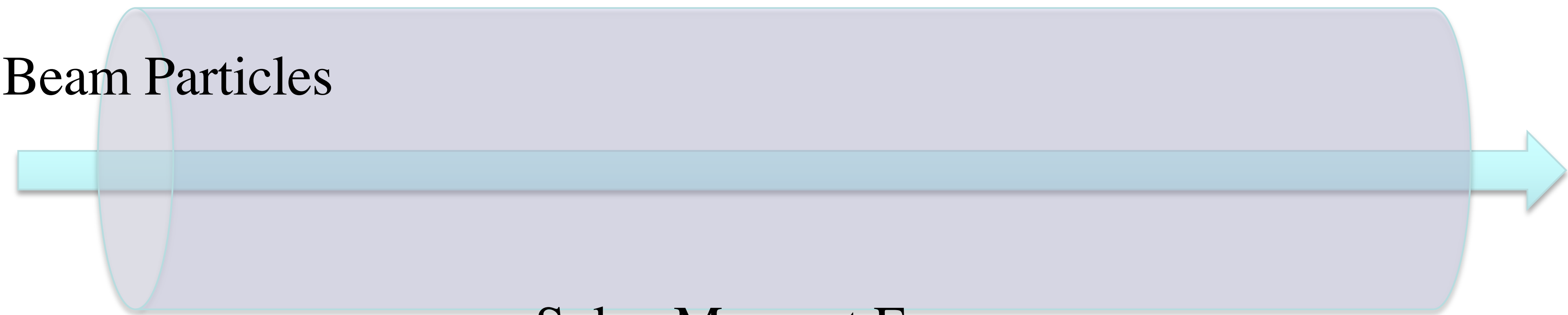
$$\mathbf{Q}=\begin{pmatrix}\frac{1}{2}\left\langle x^{2}+y^{2}\right\rangle\\ \frac{1}{2}\left\langle x^{2}-y^{2}\right\rangle\\ \left\langle xy\right\rangle\end{pmatrix}\quad \mathbf{P}=\begin{pmatrix}\left\langle xx'+yy'\right\rangle\\ \left\langle xx'-yy'\right\rangle\\ \left\langle yx'+xy'\right\rangle\end{pmatrix}\quad \mathbf{E}=\begin{pmatrix}\left\langle x'^{2}+y'^{2}\right\rangle\\ \left\langle x'^{2}-y'^{2}\right\rangle\\ 2\left\langle y'x'\right\rangle\end{pmatrix}$$



Optimization – Space Charge Compensation



Next Step – Circular Accelerators



Solve Moment Eqs.

$Z=Z_i$ $\qquad$ $Z=Z_f$

$$\boldsymbol{X}=\left(k_0\boldsymbol{Q}(z_i),\boldsymbol{P}(z_i),L(z_i),k_0^{-1}\boldsymbol{E}(z_i)\right) \qquad \boldsymbol{X}_f(X,a)=\left(k_0\boldsymbol{Q}(z_f),\boldsymbol{P}(z_f),L(z_f),k_0^{-1}\boldsymbol{E}(z_f)\right)$$

Periodicity: Enforce $\quad W(\boldsymbol{X},\boldsymbol{a})=\frac{1}{2}\left|\boldsymbol{X}_f(\boldsymbol{X},\boldsymbol{a})-\boldsymbol{X}\right|^2=0$

Optimize: Minimize $\quad$ Figure of Merit $F(\mathbf{X},\mathbf{a})$ $\qquad$ $\boldsymbol{a}=$ parameter list

Constrained Optimization
by Multiple Relaxation

Adjoint with a Chaser !

Next Step – Circular Accelerators

$Z=Z_i$ $Z=Z_f$

$$\boldsymbol{X} = \left(k_0 \boldsymbol{Q}(z_i), \boldsymbol{P}(z_i), L(z_i), k_0^{-1}\boldsymbol{E}(z_i)\right)$$

$$\boldsymbol{X}_f(X,a) = \left(k_0 \boldsymbol{Q}(z_f), \boldsymbol{P}(z_f), L(z_f), k_0^{-1}\boldsymbol{E}(z_f)\right)$$

Solve Moment Eqs.

Periodicity: Minimize

$$W(\boldsymbol{X},\boldsymbol{a}) = \frac{1}{2}\left|\boldsymbol{X}_f(\boldsymbol{X},\boldsymbol{a}) - \boldsymbol{X}\right|^2$$

$$\frac{d\mathbf{a}}{d\tau} = -\underline{\underline{\mathbf{a}}}^2 \cdot \frac{d}{d\mathbf{a}} F = -\underline{\underline{\mathbf{a}}}^2 \cdot \left(\frac{\partial}{\partial \mathbf{a}} F \bigg|_{W'=0} + \frac{\partial}{\partial \mathbf{a}} F \bigg|_X \right)$$

Minimize

Figure of Merit $F(\mathbf{X},\mathbf{a})$

This is the tricky term

The Tricky Term

$$\frac{d\mathbf{a}}{d\tau} = -\underline{\underline{\mathbf{a}}}^2 \cdot \frac{d}{d\mathbf{a}} F = -\underline{\underline{\mathbf{a}}}^2 \cdot \left(\frac{\partial}{\partial \mathbf{a}} F \Big|_{W'=0} + \frac{\partial}{\partial \mathbf{a}} F \Big|_{X} \right)$$

This is the tricky term

$$\frac{\partial}{\partial \mathbf{a}} F \Big|_{W'=0} = \frac{\partial}{\partial \mathbf{X}} F \Big|_{\mathbf{a}} \cdot \frac{\partial \mathbf{X}}{\partial \mathbf{a}} \Big|_{W'=0}$$

Expand $W'(\mathbf{X},\mathbf{a})$ to first order

$$\frac{\partial}{\partial \mathbf{X}} W(\mathbf{X},\mathbf{a}) = \delta\mathbf{X} \frac{\partial^2}{\partial \mathbf{X} \partial \mathbf{X}} W(\mathbf{X},\mathbf{a}) + \delta\mathbf{a} \frac{\partial^2}{\partial \mathbf{a} \partial \mathbf{X}} W(\mathbf{X},\mathbf{a}) = 0$$

$$\frac{\delta \mathbf{X}}{\delta \mathbf{a}} \Big|_{W'=0} = -\frac{\partial^2}{\partial \mathbf{a} \partial \mathbf{X}} W(\mathbf{X},\mathbf{a}) \left(\frac{\partial^2}{\partial \mathbf{X} \partial \mathbf{X}} W(\mathbf{X},\mathbf{a}) \right)^{-1}$$

Requires evaluating and inverting large matrices

Constrained Optimization by Multiple Relaxation

or

Adjoint with a Chaser !

$$\frac{d}{d\tau}\boldsymbol{X} = -\frac{\partial}{\partial \boldsymbol{X}} W(\boldsymbol{X},\boldsymbol{a})\Big|_{a}$$ The equilibrium

$$\frac{d}{d\tau}\boldsymbol{Y} = -\frac{\partial}{\partial \boldsymbol{Y}}\left[W(\boldsymbol{Y},\boldsymbol{a}) + \lambda F(\boldsymbol{Y},\boldsymbol{a})\right]$$ The chaser

Parameter evolution

$$\mu\frac{d}{d\tau}\boldsymbol{a} = -\underline{\underline{\boldsymbol{a}}}^2 \cdot \frac{\partial}{\partial \boldsymbol{a}}\left[\frac{W(\boldsymbol{Y},\boldsymbol{a}) - W(\boldsymbol{X},\boldsymbol{a})}{\lambda}\right] - \underline{\underline{\boldsymbol{a}}}^2 \cdot \frac{\partial}{\partial \boldsymbol{a}} F(X,a)$$

$$\frac{\partial}{\partial \boldsymbol{a}} F(X,a)\Big|_{\partial W/\partial X = 0}$$

Only first derivatives required (6)

How is it Supposed to Work

$\mu > 1$ So that "**a**" evolves "slowly".

$$\boldsymbol{X} \to \boldsymbol{X}_e(\boldsymbol{a}), \qquad 0 = -\frac{\partial}{\partial \boldsymbol{X}} W(\boldsymbol{X},\boldsymbol{a})\Big|_{X_e,\boldsymbol{a}}$$

X tends to local equilibrium

$$\boldsymbol{Y} \to \boldsymbol{X}_e + \delta \boldsymbol{Y} \qquad 0 = -\delta \boldsymbol{Y} \cdot \frac{\partial^2 W(\boldsymbol{Y},\boldsymbol{a})}{\partial \boldsymbol{X}\,\partial \boldsymbol{X}}\Big|_{X_e,\boldsymbol{a}} - \lambda \frac{\partial}{\partial \boldsymbol{X}} F(\boldsymbol{X},\boldsymbol{a})\Big|_{X_e,\boldsymbol{a}}$$

The chaser tends to a slightly different equilibrium

Allow ***a*** to change: $\boldsymbol{a} \to \boldsymbol{a} + \delta \boldsymbol{a}$

$$0 = -\delta X_e \cdot \frac{\partial^2 W(\boldsymbol{Y},\boldsymbol{a})}{\partial \boldsymbol{X}\,\partial \boldsymbol{X}}\Big|_{X_e,\boldsymbol{a}} - \delta \boldsymbol{a} \cdot \frac{\partial^2 W(\boldsymbol{Y},\boldsymbol{a})}{\partial \boldsymbol{a}\,\partial \boldsymbol{X}}\Big|_{X_e,\boldsymbol{a}}$$

The ***X*** equilibrium changes

$$\lambda \delta F = \lambda \delta \boldsymbol{X}_e \frac{\partial}{\partial \boldsymbol{X}} F(\boldsymbol{X},\boldsymbol{a})\Big|_{W'=0} = \delta \boldsymbol{a} \delta \boldsymbol{Y} : \frac{\partial^2}{\partial \boldsymbol{a}\,\partial \boldsymbol{X}} W(\boldsymbol{X},\boldsymbol{a})\Big|_{x_e,a} \simeq \delta \boldsymbol{a} \cdot \frac{\partial}{\partial \boldsymbol{a}} \left[W(\boldsymbol{Y},\boldsymbol{a}) - W(\boldsymbol{X},\boldsymbol{a}) \right]_{x_e}$$

Constrained Optimization by Multiple Relaxation

or

Adjoint with a Chaser !

$$\frac{d}{d\tau}\boldsymbol{X} = -\frac{\partial}{\partial \boldsymbol{X}} W(\boldsymbol{X},\boldsymbol{a})\Big|_{a}$$ The equilibrium

$$\frac{d}{d\tau}\boldsymbol{Y} = -\frac{\partial}{\partial \boldsymbol{Y}}\left[W(\boldsymbol{Y},\boldsymbol{a}) + \lambda F(\boldsymbol{Y},\boldsymbol{a})\right]$$ The chaser

Parameter evolution

$$\mu\frac{d}{d\tau}\boldsymbol{a} = -\underline{\underline{\boldsymbol{a}}}^2 \cdot \frac{\partial}{\partial \boldsymbol{a}}\left[\frac{W(\boldsymbol{Y},\boldsymbol{a}) - W(\boldsymbol{X},\boldsymbol{a})}{\lambda}\right] - \underline{\underline{\boldsymbol{a}}}^2 \cdot \frac{\partial}{\partial \boldsymbol{a}} F(X,a)$$

$$\frac{\partial}{\partial \boldsymbol{a}} F(X,a)\Big|_{\partial W/\partial X=0}$$

Only first derivatives required (4)

Moment Eqs. – Circular Accelerators

$Z=Z_i$ $Z=Z_f$

$$\boldsymbol{X} = \left(k_0\boldsymbol{Q}(z_i), \boldsymbol{P}(z_i), L(z_i), k_0^{-1}\boldsymbol{E}(z_i)\right)$$

Solve Moment Eqs. →

$$\boldsymbol{X}_f(X,a) = \left(k_0\boldsymbol{Q}(z_f), \boldsymbol{P}(z_f), L(z_f), k_0^{-1}\boldsymbol{E}(z_f)\right)$$

Periodicity: Minimize $W(\boldsymbol{X},\boldsymbol{a}) = \frac{1}{2}\left|\boldsymbol{X}_f(\boldsymbol{X},\boldsymbol{a}) - \boldsymbol{X}\right|^2$

Solve Adjoint Eqs. ←

Figure of Merit $F(\mathbf{X},\mathbf{a})$

$$\frac{\partial}{\partial \boldsymbol{a}} W(\boldsymbol{X},\boldsymbol{a})\Big|_X \qquad \frac{\partial}{\partial \boldsymbol{X}} W(\boldsymbol{X},\boldsymbol{a})\Big|_a$$

$$\frac{\partial}{\partial \boldsymbol{a}} F(\boldsymbol{X},\boldsymbol{a})\Big|_X \qquad \frac{\partial}{\partial \boldsymbol{X}} F(\boldsymbol{X},\boldsymbol{a})\Big|_a$$

Can be evaluated with 6 solutions of adjoint Eqs.

Conclusion

Adjoint method allows for optimization of Round to Flat and Flat to Round transformers.

Periodic lattices may (?) be handled by “Adjoint with a Chaser”. We are currently testing using moment equations

Formalism extended to treat particle description - done

“Adjoint with a Chaser” may be extended to Stellarator optimization, with a side of ALPO©.

Conclusion: Next Steps

Add Magnetic field

sensitivity function

$$\sum_j I_j \left(\delta\hat{\mathbf{p}}_j \cdot \delta\mathbf{x}_j - \delta\mathbf{p}_j \cdot \delta\hat{\mathbf{x}}_j \right)\Big|_{T_j} = -q\varepsilon_0 \int_S da\, \delta\Phi_A\, n \cdot \nabla \delta\hat{\Phi}_S - \mu_0 \int d^3x \delta\mathbf{j}_m \cdot q\delta\hat{\mathbf{A}}_s$$

Change in magnetization current

Add time dependence

Implement in an optimization routine

Basic Formulation – Linear Algebra

We wish to solve : $\underline{\underline{A}} \cdot \underline{x} = \underline{B}$ for many *B*'s.

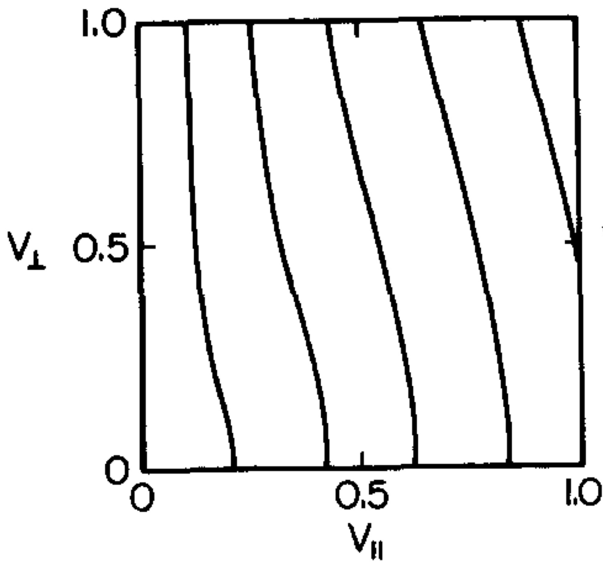
And then evaluate for each B: $D = \underline{C} \cdot \underline{x}^{\dagger}$ D(B) is the answer.

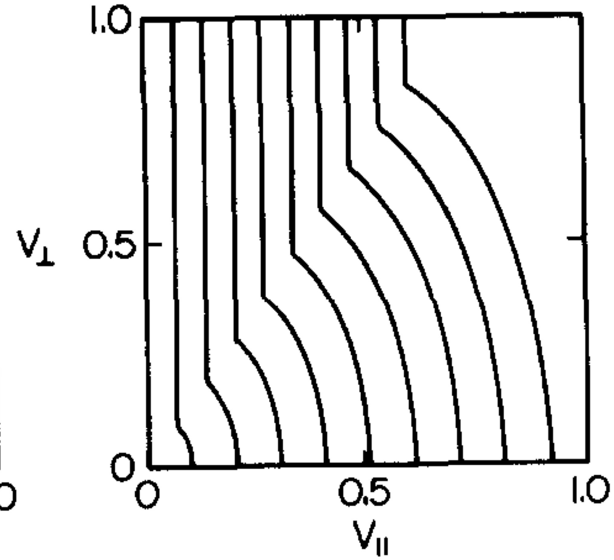
Instead solve for $\underline{y}$ *once*: $\underline{\underline{A}}^{\dagger} \cdot \underline{y} = \underline{C}$

Then: $D = \underline{B}^{\dagger} \cdot \underline{y}$

Radial Flux driven by RF

$$\left\langle \int d^3v \, f \, v_d \cdot \nabla \psi \right\rangle = \left\langle \int d^3v \, \Gamma \cdot \frac{\partial}{\partial \mathbf{v}} \, g \Big/ f_M \right\rangle$$





RF Induced Transport

Perturbed neoclassical DF

TMA and K. Yoshioka, PoF 29, (1986)

$$v_{\parallel}\mathbf{b}\cdot\nabla f+\frac{\partial}{\partial\mathbf{v}}\cdot\Gamma=C(f)$$

Response to a radial gradient

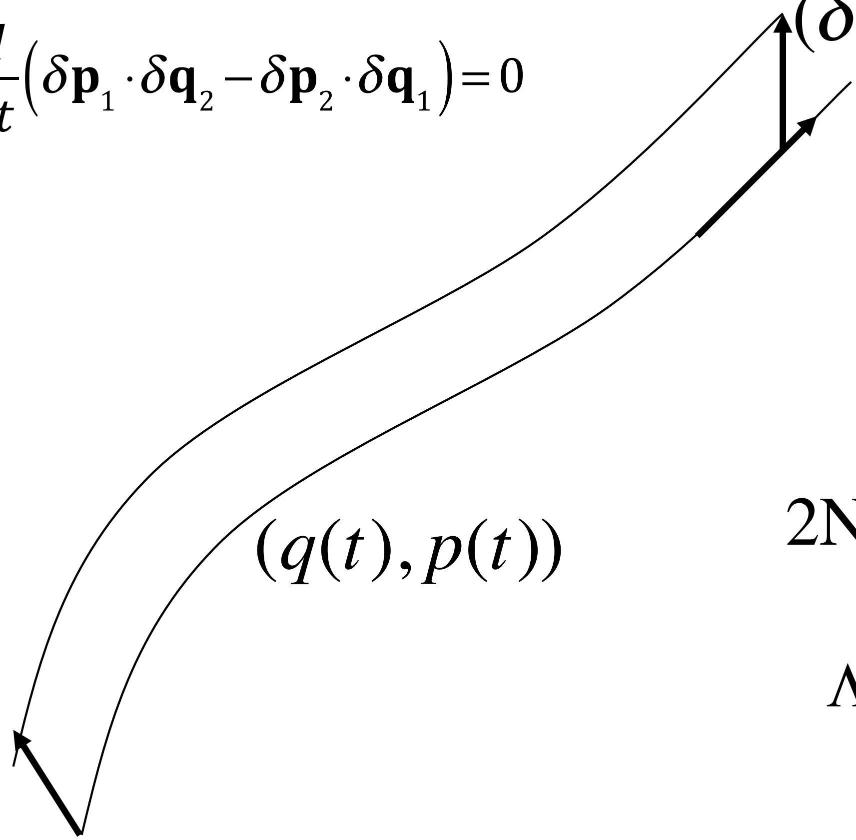
$$v_{\parallel}\mathbf{b}\cdot\nabla g+v_{d}\cdot\nabla f_{M}=C(g)$$

Fluctuation induced radial flux

$$\left\langle\int d^{3}v\; f\, v_{d}\cdot\nabla\psi\right\rangle=\left\langle\int d^{3}v\;\Gamma\cdot\frac{\partial}{\partial\mathbf{v}}\, g\big/f_{M}\right\rangle$$

Jacobian Matrix – M(t)

$$\frac{d}{dt}\left(\delta\mathbf{p}_1\cdot\delta\mathbf{q}_2-\delta\mathbf{p}_2\cdot\delta\mathbf{q}_1\right)=0$$



$$\begin{pmatrix}\delta\mathbf{q}(t)\\ \delta\mathbf{p}(t)\end{pmatrix}=\underline{\underline{M}}(t)\cdot\begin{pmatrix}\delta\mathbf{q}(0)\\ \delta\mathbf{p}(0)\end{pmatrix}$$

2Nx2N

2N Eiegenvectors and Eigenvalues of M

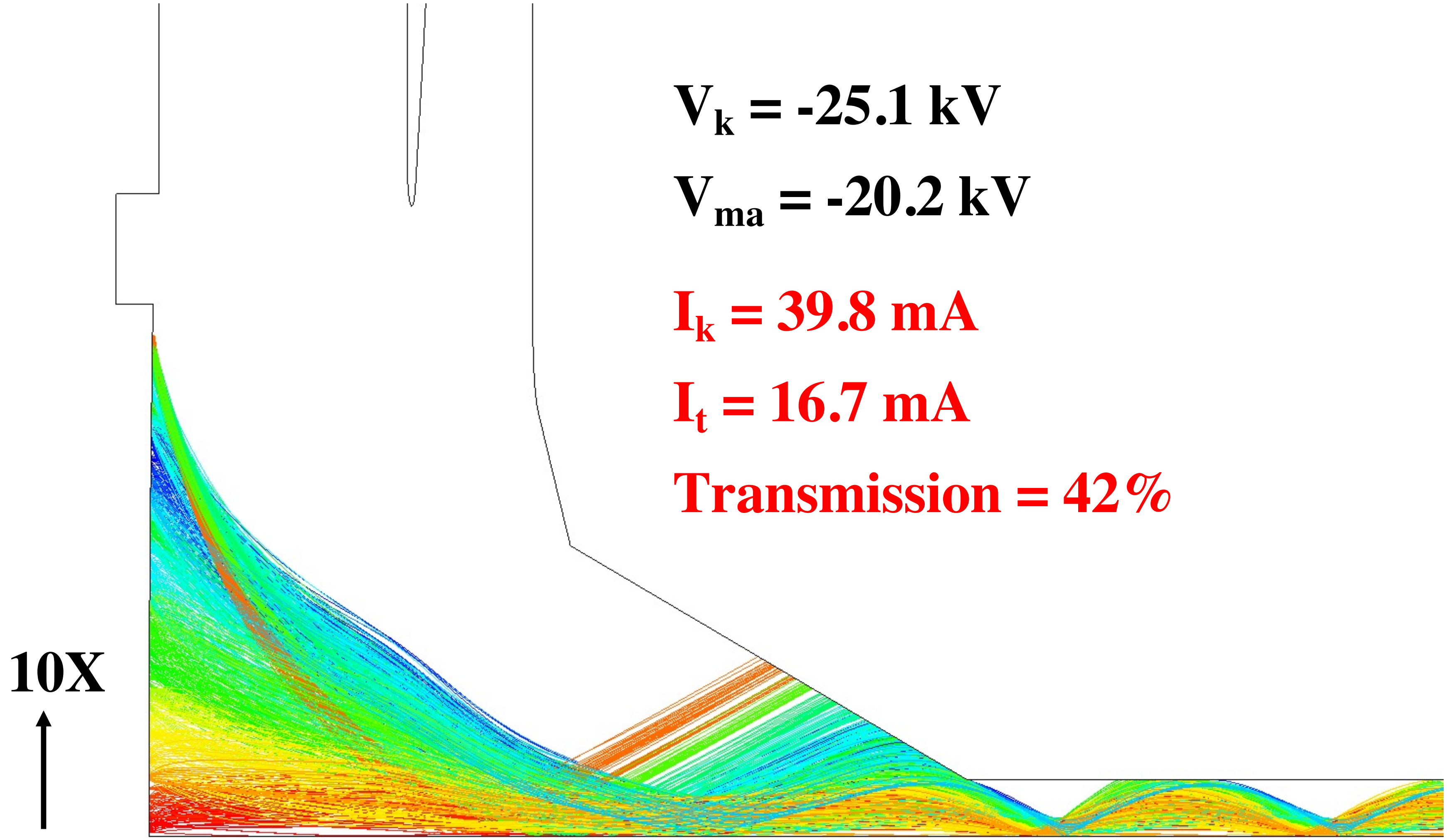
$$\Lambda(t)\begin{pmatrix}\delta\mathbf{q}(0)\\ \delta\mathbf{p}(0)\end{pmatrix}=\underline{\underline{M}}(t)\cdot\begin{pmatrix}\delta\mathbf{q}(0)\\ \delta\mathbf{p}(0)\end{pmatrix}$$

Solutions come in N pairs- $\Lambda_1\Lambda_2=1$

Eigenvectors from different pairs orthogonal

$$\left(\delta\mathbf{p}_1\cdot\delta\mathbf{q}_2-\delta\mathbf{p}_2\cdot\delta\mathbf{q}_1\right)=0$$

Beamstick: *Gun Baseline Geometry*
Particle Trajectories at Actual Voltages



V_k = -25.1 kV

V_{ma} = -20.2 kV

I_k = 39.8 mA

I_t = 16.7 mA

Transmission = 42%

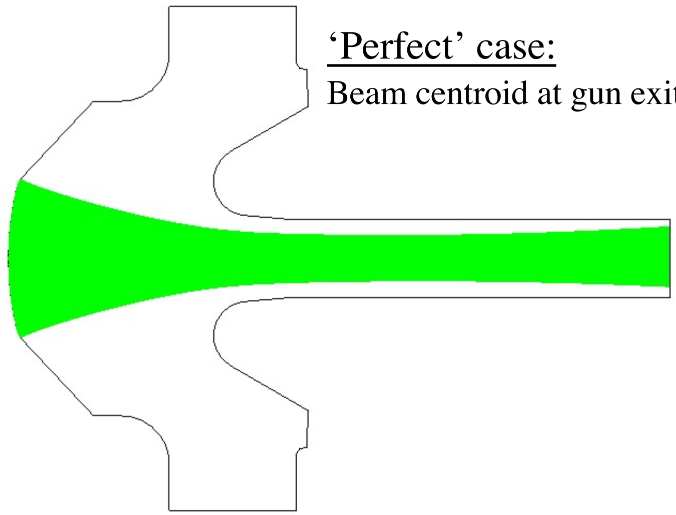
Approved for public release; Distribution unlimited

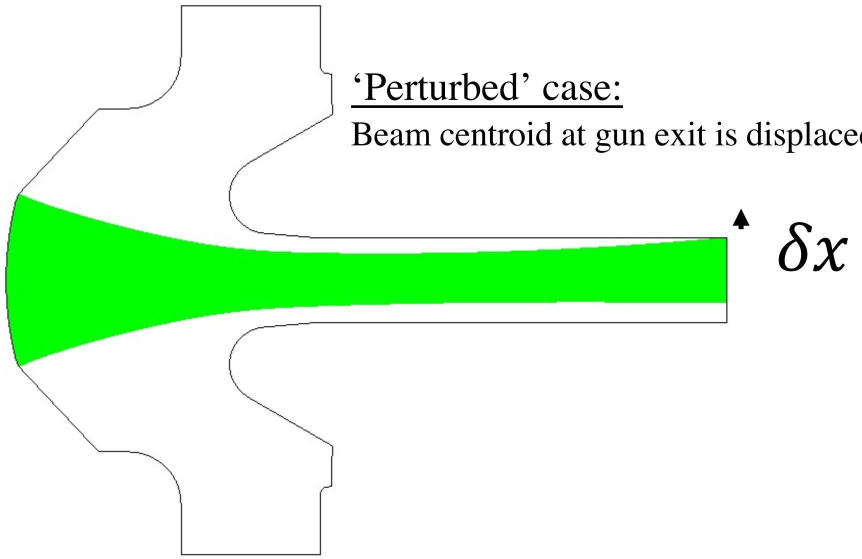
Theoretical Study of Statistical Variations

Example of the Adjoint Method in Action

Problem: Compute the displacement of the beam in a sheet beam gun due to a small change in anode potential or a small displacement of the anode:

MICHELLE Simulations of Sheet Beam Gun

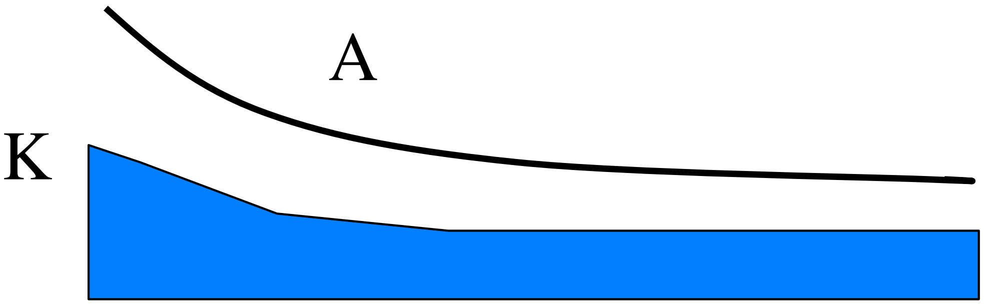




The adjoint method gives us a way to compute the displacement of the beam *without* re-running MICHELLE:

$$\delta x = -\frac{q}{4\pi\lambda I}\int_S da\,\mathbf{n}\cdot\delta\Phi\nabla\delta\hat{\Phi}$$

δx = Beam centroid displacement at gun exit
$\delta\Phi$ = Small change or error in anode or other electrode potential
$-\boldsymbol{n}\cdot\nabla\hat{\Phi}$= Sensitivity (Green's) function



Code (Michelle) solves the following equations:

Hamilton's Equations for N particles j=1,N

$$\frac{d\boldsymbol{x}_j}{dt} = \frac{\partial H}{\partial \boldsymbol{p}} \qquad \frac{d\boldsymbol{p}_j}{dt} = -\frac{\partial H}{\partial \boldsymbol{x}}$$

Accumulates a charge density

$$\rho(\boldsymbol{x}) = \sum_j I_j \int_0^{T_j} dt\ \delta(\boldsymbol{x} - \boldsymbol{x}_j(t))$$

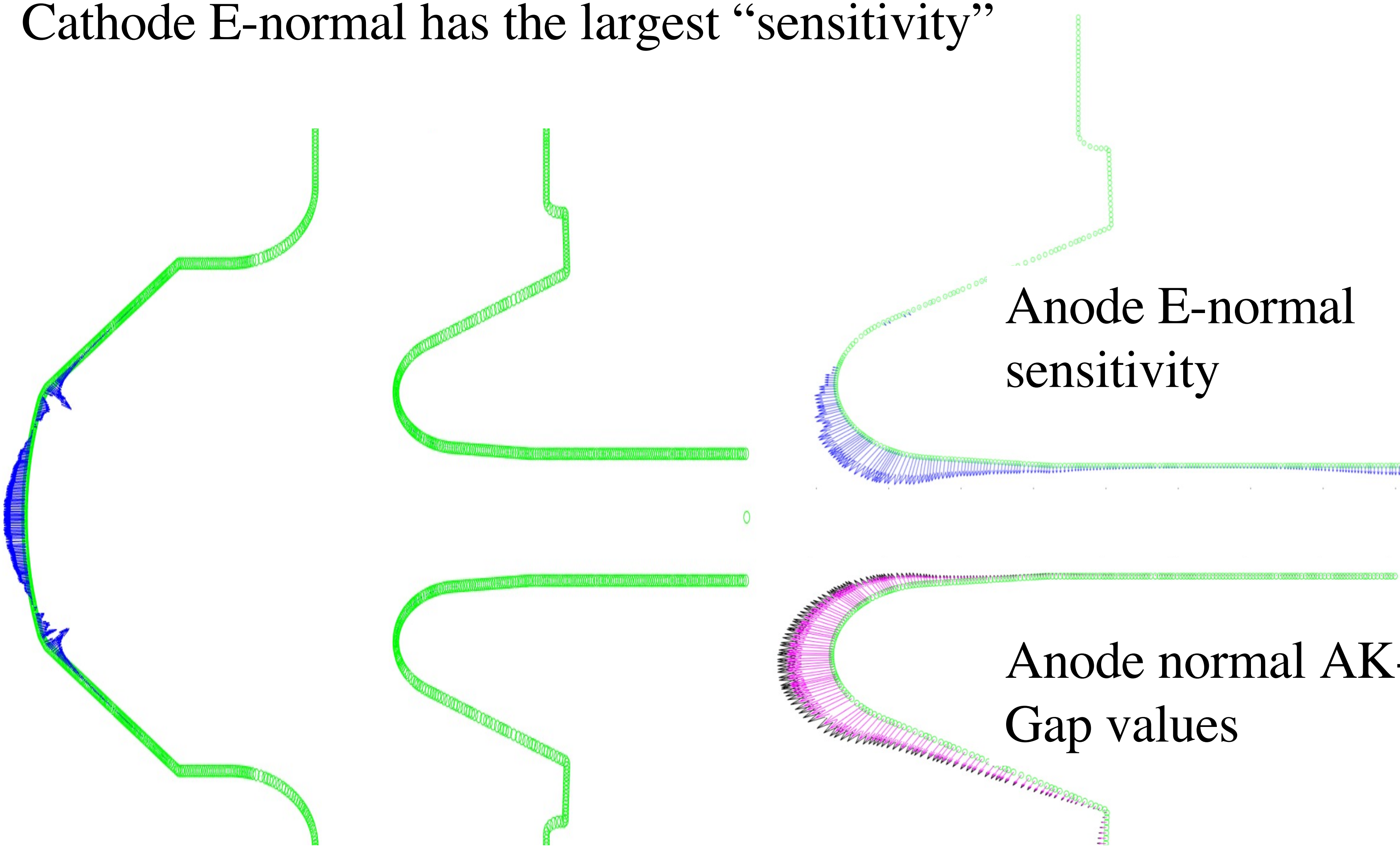
Solves Poisson Equation

$$-\nabla^2 \Phi = \rho / \varepsilon_0$$

Iterates until converged

RMS radius sensitivity

Cathode E-normal has the largest "sensitivity"



Anode E-normal sensitivity

Anode normal AK-Gap values

$$\lambda I R_{RMS} \delta R_{RMS} = \lambda \sum_j I_j \left(\mathbf{x}_j \cdot \delta \mathbf{x}_j \right) \Big|_{T_j} = -q \varepsilon_0 \int_S da\, \delta \Phi \left(\mathbf{n} \cdot \nabla \delta \hat{\Phi} \right)$$

Adjoint Equations

Base case

$$\frac{d}{dz}\mathbf{Q} = \mathbf{P}$$

$$\frac{d}{dz}\mathbf{P} = \mathbf{E} + \mathbf{O}\cdot\mathbf{Q}$$

$$\frac{d}{dz}\mathbf{E} = \mathbf{O}\cdot\mathbf{P} + \mathbf{N}L$$

$$\frac{d}{dz}L = -\mathbf{N}^{\dagger}\cdot\mathbf{Q}$$

Linear perturbation due to change in parameters

$$\frac{d}{dz}\delta\mathbf{Q}^{(X)} = \delta\mathbf{P}^{(X)}$$

$$\frac{d}{dz}\delta\mathbf{P}^{(X)} = \delta\mathbf{E}^{(X)} + \mathbf{O}\cdot\delta\mathbf{Q}^{(X)} + \delta\mathbf{O}^{(X)}\cdot\mathbf{Q}$$

$$\frac{d}{dz}\delta\mathbf{E}^{(X)} = \mathbf{O}\cdot\delta\mathbf{P}^{(X)} + \mathbf{N}\delta L^{(X)} + \delta\mathbf{O}^{(X)}\cdot\mathbf{P} + \delta\mathbf{N}^{(X)}L$$

$$\frac{d}{dz}\delta L^{(X)} = -\mathbf{N}^{\dagger}\cdot\delta\mathbf{Q}^{(X)} - \delta\mathbf{N}^{\dagger(X)}\cdot\mathbf{Q}$$

Adjoint system

$$\frac{d}{dz}\delta\mathbf{Q}^{(Y)} = \delta\mathbf{P}^{(Y)}$$

$$\frac{d}{dz}\delta\mathbf{P}^{(Y)} = \delta\mathbf{E}^{(Y)} + \mathbf{O}\cdot\delta\mathbf{Q}^{(Y)}$$

$$\frac{d}{dz}\delta\mathbf{E}^{(Y)} = \mathbf{O}\cdot\delta\mathbf{P}^{(Y)} + \mathbf{N}\delta L^{(Y)} + \delta\dot{\mathbf{E}}^{(Y)}$$

$$\frac{d}{dz}\delta L^{(Y)} = -\mathbf{N}^{\dagger}\cdot\delta\mathbf{Q}^{(Y)}$$

Can show

$$\left(\delta\mathbf{P}^{(Y)}\cdot\delta\mathbf{P}^{(X)} - \delta\mathbf{Q}^{(X)}\cdot\delta\mathbf{E}^{(Y)} - \delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{E}^{(X)} - \delta L^{(Y)}\delta L^{(X)}\right)\Big|_{z=z_i}^{z=z_f}$$

$$= \int_{z_i}^{z_f} dz\left\{\delta\mathbf{P}^{(Y)}\cdot\delta\mathbf{O}_{Q,B}{}^{(X)}\cdot\mathbf{Q} + \delta L^{(Y)}\mathbf{Q}\cdot\delta\mathbf{N}_{Q,B}{}^{(X)} - \delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{O}_{Q,B}{}^{(X)}\cdot\mathbf{P} - \delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{N}_{Q,B}{}^{(X)}L\right\}$$

Arbitrary Changes in focusing magnets

Adjoint sensitivity

Change in Figure of Merit

Figure of Merit $F(\mathbf{P},\mathbf{Q},\mathbf{E},L)$ $\quad \delta F = \delta\mathbf{P}^{(X)}\cdot\frac{\partial F}{\partial \mathbf{P}} + \delta\mathbf{Q}^{(X)}\cdot\frac{\partial F}{\partial \mathbf{Q}} + \delta\mathbf{E}^{(X)}\cdot\frac{\partial F}{\partial \mathbf{E}} + \delta L^{(X)}\frac{\partial F}{\partial L}\Big|$

$$\left(\delta\mathbf{P}^{(Y)}\cdot\delta\mathbf{P}^{(X)} - \delta\mathbf{Q}^{(X)}\cdot\delta\mathbf{E}^{(Y)} - \delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{E}^{(X)} - \delta L^{(Y)}\delta L^{(X)}\right)\Big|_{z=z_i}^{z=z_f}$$

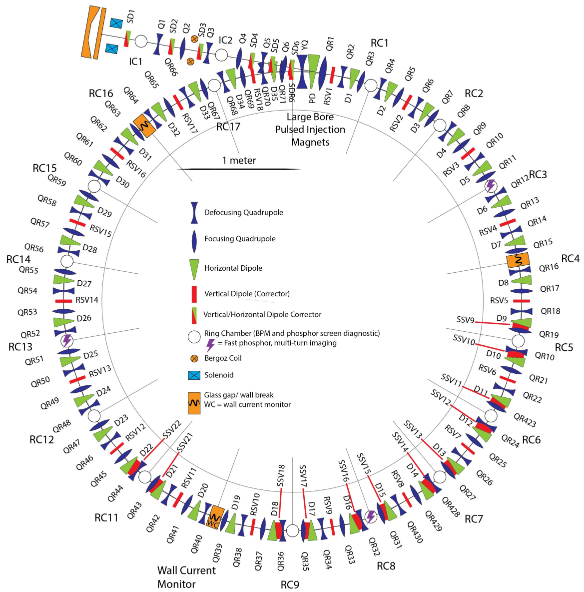
$$= \int_{z_i}^{z_f} dz\left\{\delta\mathbf{P}^{(Y)}\cdot\delta\mathbf{O}_{Q,B}{}^{(X)}\cdot\mathbf{Q} + \delta L^{(Y)}\mathbf{Q}\cdot\delta\mathbf{N}_{Q,B}{}^{(X)} - \delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{O}_{Q,B}{}^{(X)}\cdot\mathbf{P} - \delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{N}_{Q,B}{}^{(X)}L\right\}$$

$$\left(\delta\mathbf{P}^{(Y)}\cdot\delta\mathbf{P}^{(X)} - \delta\mathbf{Q}^{(X)}\cdot\delta\mathbf{E}^{(Y)} - \delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{E}^{(X)} - \delta L^{(Y)}\delta L^{(X)}\right)\Big|_{z=z_i}^{z=z_f}$$

$$= \int_{z_i}^{z_f} dz\left\{\delta\mathbf{P}^{(Y)}\cdot\delta\mathbf{O}_{Q,B}{}^{(X)}\cdot\mathbf{Q} + \delta L^{(Y)}\mathbf{Q}\cdot\delta\mathbf{N}_{Q,B}{}^{(X)} - \delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{O}_{Q,B}{}^{(X)}\cdot\mathbf{P} - \delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{N}_{Q,B}{}^{(X)}L\right\}$$

Can show

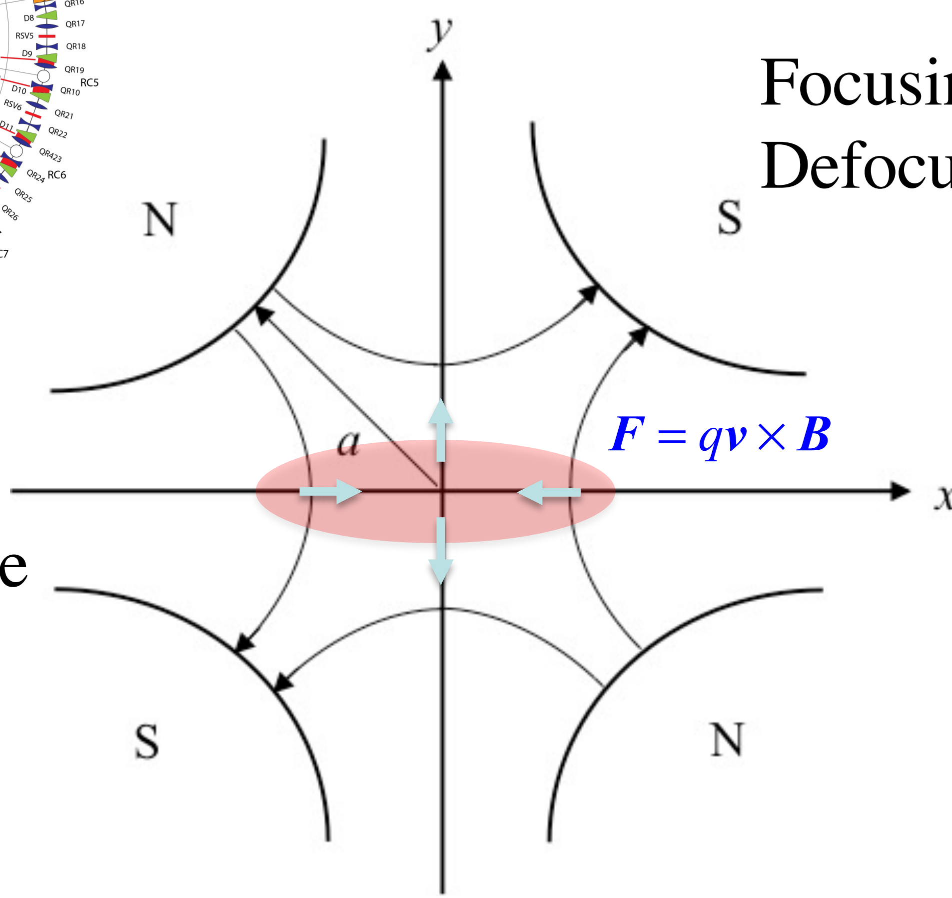
Adjoint sensitivity



Focusing Basics

Quadrupole Magnetic Field

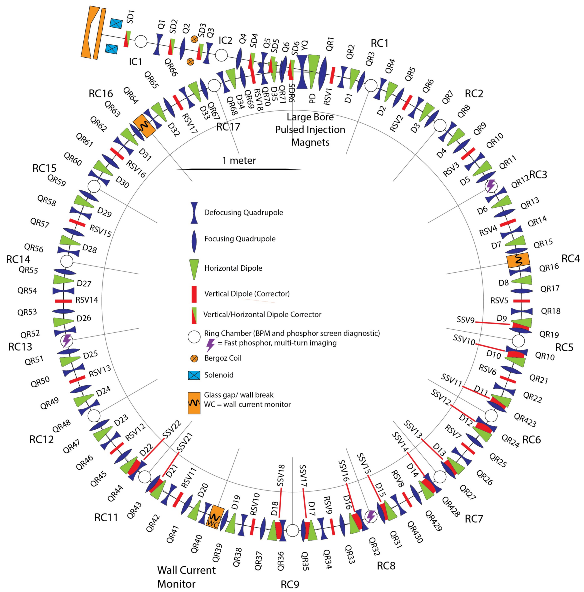
Focusing in x-direction
Defocusing in y-direction



$q\,\boldsymbol{v}$ – out of page

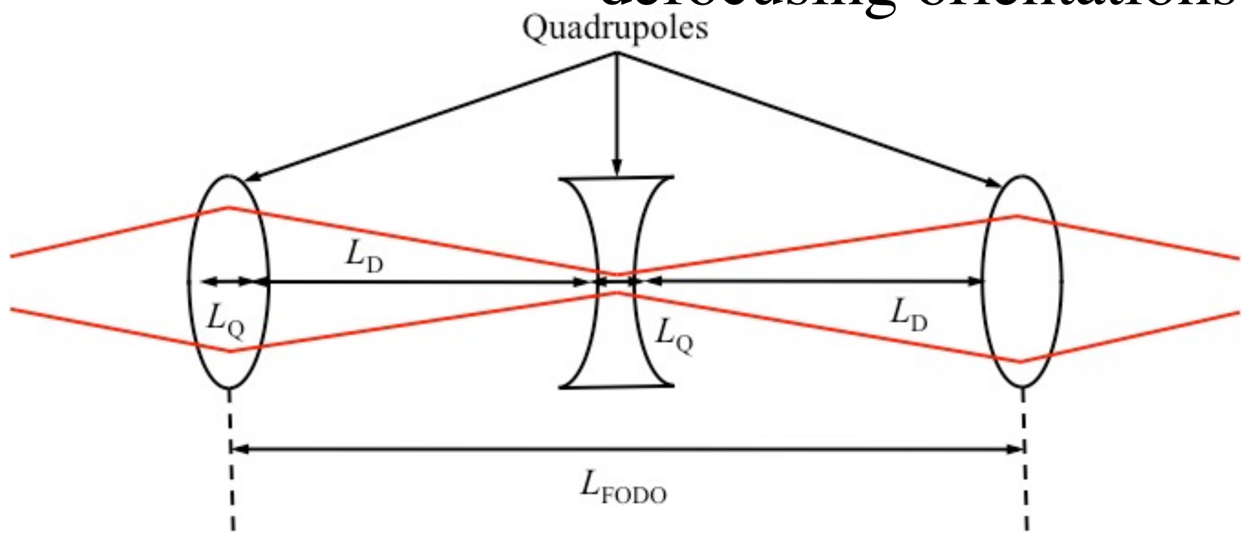
32 Quadrupole magnets

Field strength increases linearly with distance from the axis



FODO Lattice

Alternate focusing and defocusing orientations



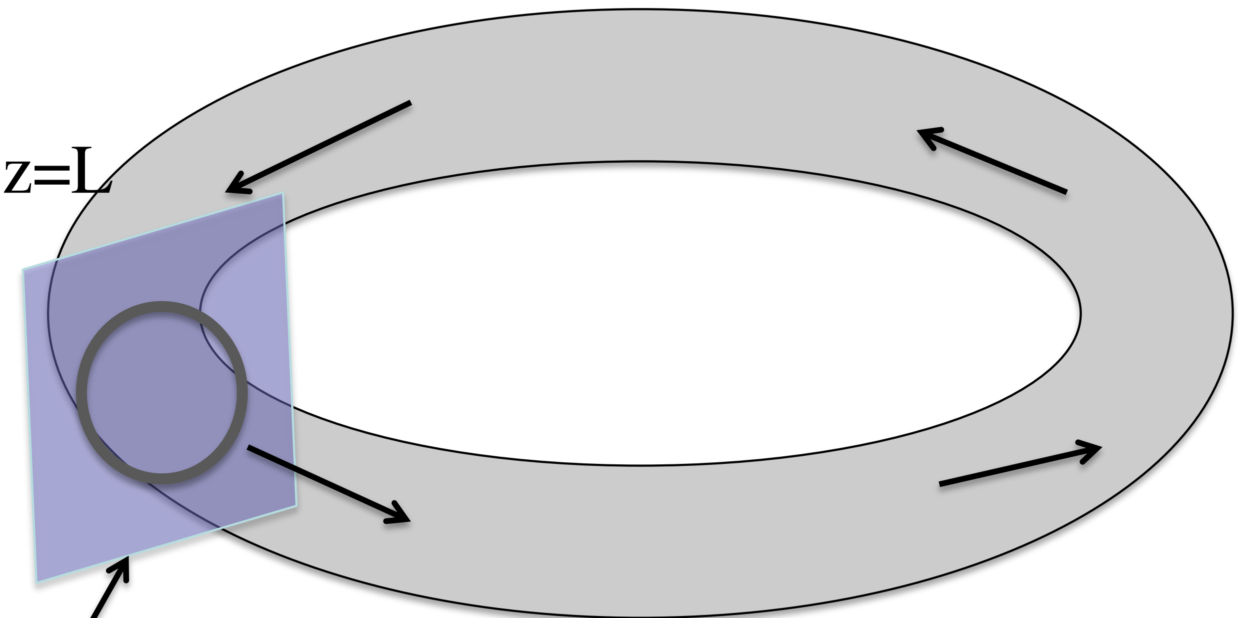
32 Quadrupole magnets

Qualitative explanation: beam passes through focusing quad when it is big (strong fields). Passes through defocusing quad when it is small (weak fields). Net effect is focusing.

Beam distribution depends on many parameters   How to optimize?

Circular Accelerators-Periodicity

Need to maintain periodicity of distribution, not individual orbits



Pick adjoint coordinate perturbations to realize desired FoM

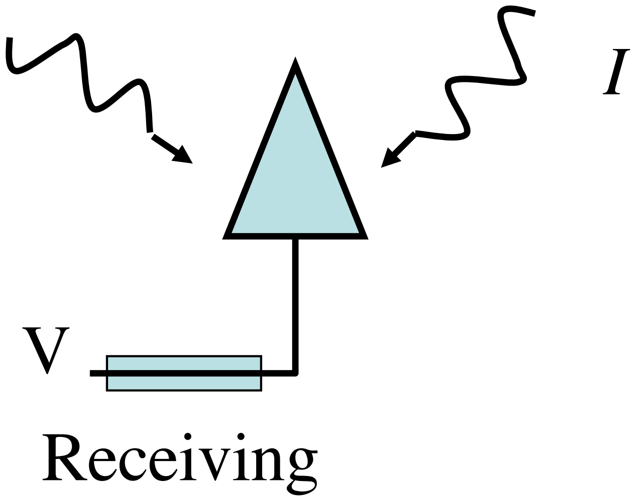
$$\sum_j \frac{I_j}{I}\left(\delta\mathbf{p}_j^{(X)}\cdot\delta\mathbf{x}_j^{(Y)} - \delta\mathbf{p}_j^{(Y)}\cdot\delta\mathbf{x}_j^{(X)}\right)\Big|_0^L =$$

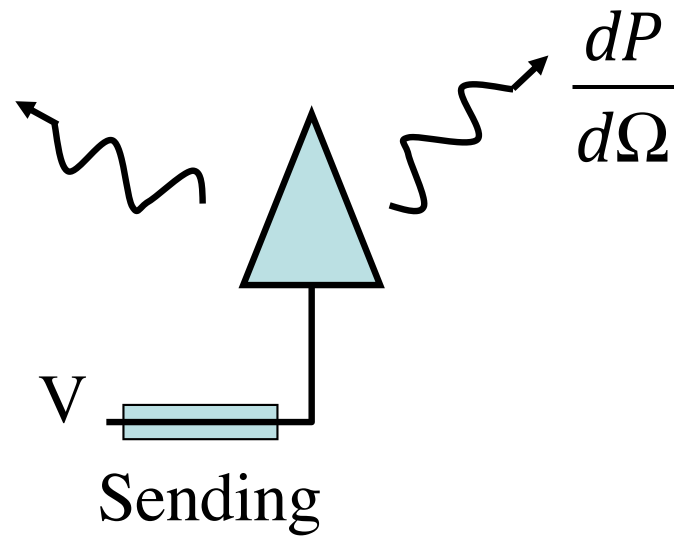
Sensitivity to changes in magnet parameters

Too hard!

$$-\frac{q}{4\pi I}\int_B d^2x\,\delta\phi^{(X)}\mathbf{n}\cdot\nabla\delta\phi^{(Y)} + q\boxed{\int d^3x\,\delta\mathbf{j}_m^{(X)}\cdot\delta\mathbf{A}^{(Y)}} + \left[(X)\leftrightarrow(Y)\right]$$

Effective Area – Antenna Gain





Power received → $P_R = A_e(\Omega) I$ ← Incident intensity

Effective area → $A_e(\Omega) = \frac{\lambda^2 G(\Omega)}{4\pi}$

↔ gain

$G(\Omega) = \frac{dP}{d\Omega} / P_T$ ← Power per unit solid angle

$P_T = \int \frac{dP}{d\Omega} d\Omega$